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Some Results on the Gamma Function for Negative Integers

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Abstract: The Gamma function $\Gamma^{(s)}(-r)$ is defined by $\Gamma^{(s)}(-r) = N - lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}\ln^s t\, e^{-t}\,dt$ for $r, s = 0, 1, 2, \dots,$ where N is the neutrix having domain $N' = \{\epsilon : \ 0 < \epsilon < \infty\}$ with negligible functions finite linear sums of the functions $\epsilon^\lambda \ln^{s-1}\epsilon,\ \ln^s\epsilon : \lambda < 0,\ s = 1, 2, \dots$ and all functions which converge to zero in the normal sense as ϵ tends to zero. In the classical sense $Gamma$ functions is not defined for the negative integer. In this study, it is proved that $\Gamma(-r) = \frac{(-1)^r}{r!}\phi(r) - \frac{(-1)^r}{r!}\gamma$ for $r = 1, 2, \dots,$ where $\phi(r) = \sum_{i=1}^r \frac{1}{i}$. Further results are also proved.

Keywords: Gamma function, neutrix, neutrix limit.

1. Introduction

In mathematics, there are several special functions that have particular significance and many applications. One of the well known such function is the Gamma functions, see for example, [3]. The gamma function $\Gamma(x)$ is considered as a generalization of the factorial and $\Gamma(x)$ is usually defined for $x > 0$ by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

In the classical sense since $\Gamma(0) = \dfrac{\Gamma(1)}{0}$, then it follows that $\Gamma(n)$ is not defined for integers $n \le 0$. However the extension formula gives finite values for $\Gamma(z)$, for $\Re(z) \le 0$ since $\Gamma(z)$ is analytic everywhere except at $z = 0, -1, -2, \dots,$ and the residue at $z = k$ is given by

$$\mathrm{Res}_{z=k}\Gamma(z) = \frac{(-1)^k}{k!}.$$

Now if we consider $x > 0$, then it follows that

$$\Gamma(x+1) = x\Gamma(x). \tag{1}$$

Now the equation (1) can then be used to define $\Gamma(x)$ for $x < 0$ and $x \ne -1, -2, \dots$ and further this is one of the most important formulas that was satisfied by the Gamma function.

It follows easily by induction that if $-n < x < -n + 1$ then

$$\Gamma(x) = \int_0^\infty t^{x-1}\left[e^{-t} - \sum_{i=0}^{n-1}\frac{(-t)^i}{i!}\right]dt.$$

Note that in the classical sense $Gamma$ functions is not defined for the negative integers. It was then proved in [2] that

$$\Gamma^{(s)}(x) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{x-1}\ln^r t\, e^{-t}\,dt$$

for $s = 0, 1, 2, \dots$ and $x \ne 0, -1, -2, \dots$. This suggested that $\Gamma^{(s)}(-r)$ could be defined by

$$\Gamma^{(s)}(-r) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}\ln^s t\, e^{-t}\,dt \tag{2}$$

for $r, s = 0, 1, 2, \dots,$ where N is the neutrix, see [1], having domain $N' = \{\epsilon : \ 0 < \epsilon < \infty\}$ with negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{s-1}\epsilon,\ \ \ln^s\epsilon :\ \ \lambda < 0,\ \ s = 1, 2, \dots$$

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and all functions which converge to zero in the normal sense as ϵ tends to zero. It was proved that the neutrix limit in equation (1) existed for $r,s = 0,1,2,\ldots$, see [11,13].

We note that, Jack Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to the quantum field theories, in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations, see [4] and [5].

Now by using the equation (2) as a definition, the following theorem was proved in [2], but we give here a simpler proof.

Theorem 1. The $\Gamma^{(s)}(0)$ exists and given by

$$\Gamma^{(s)}(0) = \frac{\Gamma^{(s+1)}(1)}{s+1} \tag{3}$$

for $s = 0,1,2,\ldots$. In particular,

$$\Gamma(0) = \Gamma'(1) = -\gamma, \tag{4}$$

where γ denotes Euler's constant which is defined as

$$\gamma = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k} - \ln(n)\right).$$

Proof. We have

$$\int_\epsilon^\infty t^{-1}\ln^s t e^{-t}\,dt = \frac{1}{s+1}\int_\epsilon^\infty e^{-t}\,d\ln^{s+1} t$$

$$= \frac{e^{-\epsilon}\ln^{s+1}\epsilon}{s+1} + \frac{1}{s+1}\int_\epsilon^\infty \ln^{s+1} te^{-t}\,dt$$

and so

$$\Gamma^{(s)}(0) = \mathrm{N-}\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-1}\ln^s te^{-t}\,dt$$

$$= \frac{1}{s+1}\int_0^\infty \ln^{s+1} te^{-t}\,dt = \frac{\Gamma^{(s+1)}(1)}{s+1},$$

proving equation (3). Since $\Gamma^{(s+1)}(1)$ is defined in the normal sense, $\Gamma^{(s)}(0)$ is therefore defined for $s = 0,1,2,\ldots$. The equation (4) follows on noting that $\Gamma'(1) = -\gamma$.

The following theorem was also proved in [2], but we again give here a simpler proof.

Theorem 2. For $r = 1,2,\ldots$, $\Gamma(-r)$ is given by

$$\Gamma(-r) = \frac{(-1)^r}{rr!} - \frac{1}{r}\Gamma(-r+1). \tag{5}$$

Proof. We have

$$\int_\epsilon^\infty t^{-r-1}e^{-t}\,dt = -\frac{1}{r}\int_\epsilon^\infty e^{-t}\,dt^{-r}$$

$$= \frac{\epsilon^{-r}e^{-\epsilon}}{r} - \frac{1}{r}\int_\epsilon^\infty t^{-r}e^{-t}\,dt$$

and so

$$\begin{aligned}\Gamma(-r) &= \mathrm{N-}\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}e^{-t}\,dt\\ &= \frac{(-1)^r}{rr!} - \frac{1}{r}\int_0^\infty t^{-r}e^{-t}\,dt\\ &= \frac{(-1)^r}{rr!} - \frac{1}{r}\Gamma(-r+1),\end{aligned}$$

proving equation (4) for $r = 1,2,\ldots$.

2. Main Results

We now prove some further results for the Gamma function.

Theorem 3. For $r = 1,2,\ldots$, $\Gamma(-r)$ exists and given by

$$\begin{aligned}\Gamma(-r) &= \frac{(-1)^r}{r!}\phi(r) + \frac{(-1)^r}{r!}\Gamma(0)\\ &= \frac{(-1)^r}{r!}\phi(r) - \frac{(-1)^r}{r!}\gamma\end{aligned} \tag{6}$$

where $\phi(r) = \sum_{i=1}^r \frac{1}{i}$.

Proof. When $r = 1$, equation (6) reduces to equation (5) and so equation (6) holds when $r = 1$. Now assume that equation (6) holds for some r. Then using equation (5) and our assumption, we have

$$\Gamma(-r-1) = \frac{(-1)^{r+1}}{(r+1)(r+1)!} - \frac{1}{(r+1)}\Gamma(-r)$$

$$= \frac{(-1)^{r+1}}{(r+1)(r+1)!} + \frac{(-1)^{r+1}}{(r+1)!}\phi(r) - \frac{(-1)^{r+1}}{(r+1)!}\gamma$$

$$= \frac{(-1)^{r+1}}{(r+1)!}\phi(r+1) - \frac{(-1)^{r+1}}{(r+1)!}\gamma$$

and so equation (6) is true for $r+1$. Equation (6) now follows by induction for $r = 1,2,\ldots$.

In the next we prove the existence of the derivative for $\Gamma(-r)$.

Theorem 4. The derivative is given by

$$\Gamma'(-r) = \sum_{i=1}^{r}\frac{(-1)^r}{ir!}\phi(i) - \frac{(-1)^r}{r!}\phi(r)\gamma \tag{7}$$

for $r = 1,2,\ldots$.

Proof. We have

$$\int_\epsilon^\infty t^{-r-1}\ln t\, e^{-t}\,dt = -\frac{1}{r}\int_\epsilon^\infty \ln t\, e^{-t}\,dt^{-r}$$

$$= \frac{1}{r}\ln\epsilon\, e^{-\epsilon}\epsilon^{-r} - \frac{1}{r}\int_\epsilon^\infty (t^{-r}\ln t\, e^{-t} - t^{-r-1}\, e^{-t})\,dt$$

and it follows that

$$\Gamma'(-r) = \text{N}-\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}\ln t e^{-t}\,dt$$
$$= 0 - \frac{1}{r}\Gamma'(-r+1) + \frac{1}{r}\Gamma(-r) \quad (8)$$

for $r = 1, 2, \ldots$.

Now assume that equation (7) holds for some r. Then from our assumption and equations (6) and (8), we have

$$\Gamma'(-r-1) = -\frac{1}{r+1}\Gamma'(-r) + \frac{1}{r+1}\Gamma(-r-1)$$

$$= \sum_{i=1}^{r}\frac{(-1)^{r+1}}{i(r+1)!}\phi(i) - \frac{(-1)^{r+1}}{(r+1)!}\phi(r)\gamma$$
$$+\frac{(-1)^{r+1}}{(r+1)(r+1)!}\phi(r+1) - \frac{(-1)^{r+1}}{(r+1)(r+1)!}\gamma$$
$$= \sum_{i=1}^{r+1}\frac{(-1)^{r+1}}{i(r+1)!}\phi(i) - \frac{(-1)^{r+1}}{(r+1)!}\phi(r+1)\gamma$$

and so equation (7) holds for $r+1$. Equation (7) now follows by induction.

Theorem 5. For $s = 1, 2, \ldots$,

$$\Gamma^{(s)}(-1) = \sum_{i=1}^{s}\frac{s!}{(i+1)!}\Gamma^{(i+1)}(1) + s!(\gamma - 1). \quad (9)$$

Proof. We have

$$\int_\epsilon^\infty t^{-2}\ln^s t\, e^{-t}\,dt = -\int_\epsilon^\infty \ln^s t\, e^{-t}\,dt^{-1}$$

$$= \ln^s \epsilon\, e^{-\epsilon}\epsilon^{-1} - \int_\epsilon^\infty (t^{-1}\ln^s t\, e^{-t} - st^{-2}\ln^{s-1} t\, e^{-t})\,dt$$

and it follows that

$$\Gamma^{(s)}(-1) = \text{N}-\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-2}\ln^s t e^{-t}\,dt$$
$$= 0 - \Gamma^{(s)}(0) + s\Gamma^{(s-1)}(-1), \quad (10)$$

for $s = 1, 2, \ldots$.

Now assume that equation (9) holds for some s. Then from our assumption and equations (3) and (10), we have

$$\Gamma^{(s+1)}(-1) = -\Gamma^{(s+1)}(0) + (s+1)\Gamma^{(s)}(-1)$$

$$= -\frac{1}{s+2}\Gamma^{(s+2)}(1) + \sum_{i=1}^{s}\frac{(s+1)!}{(i+1)!}\Gamma^{(i+1)}(1)$$
$$+(s+1)!(\gamma - 1)$$
$$= \sum_{i=1}^{s+1}\frac{(s+1)!}{(i+1)!}\Gamma^{(i+1)}(1) + (s+1)!(\gamma - 1)$$

and so equation (9) holds for $s+1$. Equation (9) now follows by induction. More generally we have the following theorem.

Theorem 6. For $s = 1, 2, \ldots$,

$$\Gamma^{(s)}(-r) + \frac{1}{r}\Gamma^{(s)}(-r+1) = \frac{s}{r}\Gamma^{(s-1)}(-r). \quad (11)$$

Proof. We have

$$\int_\epsilon^\infty t^{-r-1}\ln^s t\, e^{-t}\,dt = -\frac{1}{r}\int_\epsilon^\infty \ln^s t\, e^{-t}\,dt^{-r}$$

$$= -\frac{1}{r}\int_\epsilon^\infty (t^{-r}\ln^s t\, e^{-t} - st^{-r-1}\ln^{s-1} t\, e^{-t})\,dt$$
$$+\frac{1}{r}\ln^s \epsilon\, e^{-\epsilon}\epsilon^{-r}$$

and it follows that

$$\Gamma^{(s)}(-r) = \text{N}-\lim_{\epsilon\to 0}\int_\epsilon^\infty t^{-r-1}\ln^s te^{-t}\,dt$$
$$= 0 - \frac{1}{r}\Gamma^{(s)}(-r+1) + \frac{s}{r}\Gamma^{(s-1)}(-r),$$

proving equation (11).

Theorem 7.

$$\Gamma^{(s)}(-r) = \frac{s}{r!}\sum_{i=0}^{r-1}(-1)^i(r-i-1)!\Gamma^{(s-1)}(-r+i)$$
$$+\frac{(-1)^r}{r!}\Gamma^{(s)}(0)$$
$$= \frac{s}{r!}\sum_{i=0}^{r-1}(-1)^i(r-i-1)!\Gamma^{(s-1)}(-r+i) \quad (12)$$
$$+\frac{(-1)^r}{(s+1)r!}\Gamma^{(s+1)}(1),$$

for $r, s = 1, 2, \ldots$.

Proof. When $r = 1$, equation (12) reduces to

$$\Gamma^{(s)}(-1) = s\Gamma^{(s-1)}(-1) - \Gamma^{(s)}(0),$$

and so equation (12) holds by equation (11) when $r = 1$ for $s = 1, 2, \ldots$.

Now assume that equation (12) holds for some r and $s = 1, 2, \ldots$. Then using equation (11) and our assumption, we have

$$\Gamma^{(s)}(-r-1) = \frac{s}{r+1}\Gamma^{(s-1)}(-r-1) - \frac{1}{r+1}\Gamma^{(s)}(-r)$$

$$= \frac{s}{r+1}\Gamma^{(s-1)}(-r-1)$$
$$-\frac{s}{(r+1)!}\sum_{i=0}^{r-1}(-1)^i(r-i-1)!\Gamma^{(s-1)}(-r+i)$$

$$
\begin{aligned}
&-\frac{(-1)^r}{(r+1)!}\Gamma^{(s)}(0) \\
&= \frac{s}{r+1}\Gamma^{(s-1)}(-r-1) \\
&\quad -\frac{s}{(r+1)!}\sum_{i=1}^{r}(-1)^i(r-i)!\Gamma^{(s-1)}(-r+i-1) \\
&\quad -\frac{(-1)^r}{(r+1)!}\Gamma^{(s)}(0) \\
&= \frac{s}{(r+1)!}\sum_{i=0}^{r}(-1)^i(r-i)!\Gamma^{(s-1)}(-r+i-1) \\
&\quad +\frac{(-1)^{r+1}}{(r+1)!}\Gamma^{(s)}(0).
\end{aligned}
$$

and so equation (12) holds for $r+1$ and $s = 1, 2, \ldots$. Equation (12) now follows by induction.

Further for similar results on the neutrix products of distributions, see [6], [7], [9], [12] and [13]. In particular for the composition of singular distributions, see [16].

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