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A New Reduction Algorithm for Differential-Algebraic Systems with Power Series Coefficients

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Abstract: A new reduction algorithm for differential-algebraic systems with power series coefficients has been presented in this paper. In this algorithm, the given system of differential-algebraic equations is transformed into another simpler system having same properties. Maple implementation of the proposed algorithm is discussed and sample computations are presented to illustrate the proposed algorithm.

Keywords: Differential-algebraic systems, Reduction algorithms, Null space, Elementary row-column operations.

1 Introduction

A first-order matrix differential system can be represented as

$$\mathscr{A}(z)Du(z)+\mathscr{B}(z)u(z)=f(z), \tag{1}$$

where z is a complex variable, $\mathscr{A}(z),\mathscr{B}(z)$ are $m\times n$ matrices of analytic functions, $f(z)$ is an m-dimensional vector of analytic functions, $u(z)$ is an n-dimensional unknown vector to be determined and $D=\frac{d}{dz}$ is a differential operator. In operator notations, the first-order matrix differential system (1) can be represented by an equation of the form

$$Lu=f,$$

where $L=\mathscr{A}D+\mathscr{B}$ is a matrix differential operator. If $m=n$ and $\mathscr{A}$ is regular (i.e., $\det(\mathscr{A})\neq 0$), then the system (1) is called a *system of linear ordinary differential equations* or linear differential system (LDS). If $\mathscr{A}\equiv 0$, then the system (1) becomes a purely algebraic system and there are several methods available in the literature to find all possible solutions. If $m\neq n$ or $\mathscr{A}$ is singular matrix, then the system (1) turns out to be a *system of differential-algebraic equations* or simply, differential-algebraic system (DAS). Differential-algebraic system is a composed system of ordinary differential equations coupled with purely algebraic equations, hence DAS differ from LDS in many aspects. This paper mainly focused on DAS with some necessary conditions.

DAS of the form (1) arise naturally in many applications of science and engineering, for example in mechanics, control theory. Many scientists and engineers have studied the DAS intensively from a numerical point of view and developed new approaches to solve DAS, see, for example, [2–4, 6–11, 26, 28, 29]. Generally speaking, most of authors handled the DAS to separate the ordinary differential system from the given DAS which is a first-order system of ODEs expressing u' in terms of u and z, computed by differentiating (1) successively and then using basic algebraic techniques. The number of differentiations of the initial DAE required to generate the underlying ODE is called the *differential index*. There are other alternative index definitions available, see for example, [4, 9, 10]. Some authors used the notion of differential index and some authors solve the DAS using the techniques of reduction algorithms [2, 6, 8, 9]. W. A. Harris et. al. developed an algebraic algorithm in [30] to reduce the given DAS of the form (1) into a similar system which produces first-order systems of ODEs and algebraic systems of lower sizes with some necessary conditions on the right-hand side. They handled DAS with coefficient matrices holomorphic at $z=0$ and discussed about the existence of solutions and the number

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of solutions which are holomorphic at $z = 0$. Carole El Bacha et. al. discussed in [2] the importance of such systems in science and engineering, and proposed new algorithms for decoupling them into a purely differential part and a purely algebraic one. The authors recalled the first and basic reduction algorithm for differential-algebraic systems developed by W. A. Harris et. al. [30], and proposed an alternate algorithm to reduce the given system [5]. The aim of this paper is to develop a new algebraic reduction algorithm which reduces the given DAS to another simpler and equivalent system where we can easily apply the classical theory of differential equations. Various symbolic algorithms are available in [11–25] for solving system of differential equations and system of DAEs

The rest of paper is organized as follows: Section 2 recalls the basic concepts of the reduction algorithms and Section 2.1 presents a new reduction algorithm for DAS and Certain examples are solved in Section 2.2 to illustrate the algorithm. Section 3 discusses the Maple implementation of the proposed algorithm with sample computations.

2 A New Reduction Algorithm

Let $\mathbb{K}$ be a subfield of the field of complex numbers $\mathbb{C}$. Note that $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. We denote the ring of formal power series by $\mathbb{K}[[z]]$ in the variable z and $\mathbb{K}((z))$ denote its quotient field, i.e., $\mathbb{K}((z)) = \mathbb{K}[[z]][z^{-1}]$. The ring of differential operators is denoted by $\mathbb{K}[[z]][D]$ with coefficients in $\mathbb{K}[[z]]$, i.e., the set of finite sums $\sum a_i D^i$ with $a_i \in \mathbb{K}[[z]]$ is equipped with the addition and the multiplication defined by

$$D^i D^j = D^{i+j}, \quad i, j \in \mathbb{N},$$
$$Df = fD + \frac{df}{dz},$$

where $f \in \mathbb{K}[[z]]$. In this paper, we consider a system of the following form

$$\mathscr{A}(z)Du(z) + \mathscr{B}(z)u(z) = f(z), \tag{2}$$

where $\mathscr{A}(z), \mathscr{B}(z) \in \mathbb{K}[[z]]^{n\times n}$, $f(z) \in \mathbb{K}[[z]]^n$ and $u(z) \in \mathbb{K}[[z]]^n$. The corresponding matrix differential operator of the system (2) is $L = \mathscr{A}D + \mathscr{B} \in \mathbb{K}[[z]][D]^{n\times n}$. We recall the basic concepts of the matrix differential operators, see [1, 2, 5, 27, 30] for further details.

Definition 1. *1. A matrix differential operator $T \in \mathbb{K}[[z]][D]^{n\times n}$ is said to be* unimodular matrix *if there exists a $V \in \mathbb{K}[[z]][D]^{n\times n}$ such that $VT = TV = I_n$. In other words, it is two-sided invertible matrix in $\mathbb{K}[[z]][D]^{n\times n}$ i.e. $det(T) =$ constant $\neq 0$ in $\mathbb{K}$.*

2. Let $L = \mathscr{A}D + \mathscr{B} \in \mathbb{K}[[z]][D]^{n\times n}$ be a matrix differential operator. Then the rank of L defined to be the rank of leading coefficient matrix of A i.e., rank (L) = rank ($\mathscr{A}$).

3. Two matrix differential operators $L, \widetilde{L} \in \mathbb{K}[[z]][D]^{n\times n}$ are said to be equivalent *if there exist two unimodular matrices $S, T \in \mathbb{K}[[z]][D]^{n\times n}$ such that $\widetilde{L} = SLT$.*

Example 1. Consider the following two matrices in z.

$$\mathscr{A} = \begin{pmatrix} z^2 - 2z + 1 & -2z^5 + 2z^4 + 3z^3 - 4z^2 + z \\ -2z^3 + 5z^2 - 4z + 1 & 2z^6 - 8z^4 + 7z^3 - z \end{pmatrix}$$

and

$$\mathscr{B} = \begin{pmatrix} z^2 - 2z + 1 & 0 \\ 0 & z^6 - 3z^5 + 2z^4 + 2z^3 - 3z^2 + z \end{pmatrix}.$$

There are two unimodular matrices S and T such that $\mathscr{B} = S\mathscr{A}T$, where

$$S = \begin{pmatrix} 1 & 0 \\ -z^4 + z^3 + z^2 - 2z + \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

and

$$T = \begin{pmatrix} 2z^3 + 2z^2 - z + 1 & 2z^3 + 2z^2 - z \\ 1 & 1 \end{pmatrix}$$

with $\det(S) = -\frac{1}{2}$ and $\det(T) = 1$.

Remark. Let $L \in \mathbb{K}[[z]][D]^{n\times n}$ be a matrix differential operator and $T \in \mathbb{K}[[z]][D]^{n\times n}$ be an unimodular matrix differential operator. Then Rank of L = Rank of LT = Rank of TL.

Definition 2. *Let $S \subseteq \mathbb{C}$ be a subset of the set of complex numbers. Let $z \in S$. Then z is an* isolated point *of S if and only if there exists a neighborhood of z in $\mathbb{C}$ which contains no points of S except z: $\exists\, \varepsilon \in R_{>0}\ :\ N_\varepsilon(z) \cap S = \{z\}$.*

2.1 Reduction Algorithm for Differential-Algebraic Systems

In order to develop a reduction algorithm for a given differential operator $L \in \mathbb{K}[[z]][D]^{n\times n}$, the following lemma (see, for example, [2, 5, 30]) is one of the essential steps for the algorithm. The lemma shows that any matrix of formal power series centered at origin (i.e., the entries of $\mathscr{A}$ are formal power series at $z = 0$) can be transformed into a block matrix.

Lemma 1. *Let $\mathscr{A} \in \mathbb{K}[[z]]^{n\times n}$ be the matrix whose entries are holomorphic (analytic) power series at non-isolated point $z = 0$. Then there exist two unimodular matrices $S, T \in \mathbb{K}[[z]]^{n\times n}$ such that*

$$S\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix}, \quad \mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \end{pmatrix}, \quad S\mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathscr{A}^{11} \in \mathbb{K}[[z]]^{r\times r}$ is a block matrix and r being the rank of the matrix $\mathscr{A}$.

For the sake of completeness, we include a sketch of the proof similar to [30, Lemma 1].

Proof. Suppose the rank of $\mathscr{A}$ is r. For a given $\mathscr{A} \in \mathbb{K}[[z]]^{n\times n}$, the rank of $\mathscr{A}$ can change only at isolated points. Hence there exists a $r\times r$ submatrix with non-zero determinant. Denote this submatrix $\mathscr{A}^{11}$. Assume that $\mathscr{A}$ has the block partition form

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ \mathscr{A}^{21} \end{pmatrix} \text{ or } \mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} & \mathscr{A}^{12} \end{pmatrix} \text{ or } \mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} & \mathscr{A}^{12} \\ \mathscr{A}^{21} & \mathscr{A}^{22} \end{pmatrix}.$$

We should show that there are two unimodular matrices S and T such that

$$S\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix}, \quad \mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \end{pmatrix}, \quad S\mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose $S = \begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}$, then $S\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix}$ gives that $S = \begin{pmatrix} I & 0 \\ -z^{\alpha}\mathscr{A}^{21}(\mathscr{A}^{11})^{-1} & z^{\alpha}I \end{pmatrix}$, similarly if $T = \begin{pmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{pmatrix}$, then $\mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \end{pmatrix}$ gives that $T = \begin{pmatrix} I & -z^{\alpha}(\mathscr{A}^{11})^{-1}\mathscr{A}^{12} \\ 0 & z^{\alpha}I \end{pmatrix}$. If we denote $G = -z^{\alpha}(\mathscr{A}^{11})^{-1} \in \mathbb{K}[[z]]^{r\times r}$, then we have

$$S = \begin{pmatrix} I & 0 \\ \mathscr{A}^{21}G & z^{\alpha}I \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} I & G\mathscr{A}^{12} \\ 0 & z^{\alpha}I \end{pmatrix},$$

such that

$$S\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix}, \quad \mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \end{pmatrix} \quad \text{and} \quad S\mathscr{A}T = \begin{pmatrix} \mathscr{A}^{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathscr{A}^{11} \in \mathbb{K}[[z]]^{r\times r}$ and r being the rank of the matrix $\mathscr{A}$.

Remark. The matrices $S\mathscr{A}$ and $\mathscr{A}T$ in Lemma 1, may also have the form

$$S\mathscr{A} = \begin{pmatrix} 0 \\ \mathscr{A}^{21} \end{pmatrix}, \quad \mathscr{A}T = \begin{pmatrix} 0 & \mathscr{A}^{12} \end{pmatrix}.$$

Remark. Recall that f is said to be *holomorphic* at the point z_0 if f is complex differentiable on some neighborhood of z_0. If the elements of $\mathscr{A}$ are holomorphic, then rank of $\mathscr{A}(z)$ can change only at isolated points. Hence the rank of $\mathscr{A}(z)$ is constant in a deleted neighborhood of $z = 0$.

Remark. If there is no isolated point in the domain of the functions in the matrix $\mathscr{A}$ of Lemma 1, then $G = -(\mathscr{A}^{11})^{-1} \in \mathbb{K}[[z]]^{r\times r}$, and hence

$$S = \begin{pmatrix} I & 0 \\ \mathscr{A}^{21}G & I \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} I & G\mathscr{A}^{12} \\ 0 & I \end{pmatrix}.$$

Example 2. Consider

$$\mathscr{A} = \begin{pmatrix} 1-z & 2-z & 1 & 0 \\ 2z & 0 & 1 & z \\ 2 & 4-2z & 3 & z \\ 1-5z & 2-z & -1 & -2z \end{pmatrix}.$$

Following Lemma 1, one can construct two unimodular matrices S and T as follows.

Suppose $\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} & \mathscr{A}^{12} \\ \mathscr{A}^{21} & \mathscr{A}^{22} \end{pmatrix}$. Then $G = \begin{pmatrix} 0 & -\frac{1}{2z} \\ \frac{1}{z-2} & \frac{z-1}{2z(z-2)} \end{pmatrix} \in \mathbb{K}[[z]]^{2\times 2}$, and hence

$$S = \begin{pmatrix} I & 0 \\ \mathscr{A}^{21}G & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$T = \begin{pmatrix} I & G\mathscr{A}^{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2z} & -\frac{1}{2} \\ 0 & 1 & \frac{3z-1}{2z(z-2)} & \frac{z-1}{2(z-2)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

such that

$$S\mathscr{A} = \begin{pmatrix} 1-z & 2-z & 1 & 0 \\ 2z & 0 & 1 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathscr{A}T = \begin{pmatrix} 1-z & 2-z & 0 & 0 \\ 2z & 0 & 0 & 0 \\ 2 & 4-2z & 0 & 0 \\ 1-5z & 2-z & 0 & 0 \end{pmatrix} \text{ and}$$

$$S\mathscr{A}T = \begin{pmatrix} 1-z & 2-z & 0 & 0 \\ 2z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\mathscr{A}^{11} = \begin{pmatrix} 1-z & 2-z \\ 2z & 0 \end{pmatrix} \in \mathbb{K}[[z]]^{2\times 2}$ and 2 being the rank of the matrix $\mathscr{A}$.

Suppose $\operatorname{rank}(\mathscr{A}) = r$ and $\operatorname{rank}(\mathscr{B}) = k$. By applying Lemma 1 to the matrix $\mathscr{A}$ of the matrix differential operator L, we can construct an unimodular matrix $S_{\mathscr{A}}$ such that

$$S_{\mathscr{A}}L = \mathscr{A}_1 D + \mathscr{B}_1, \tag{3}$$

where $\mathscr{A}_1 = S_{\mathscr{A}}\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix}$, $\mathscr{B}_1 = S_{\mathscr{A}}\mathscr{B} = \begin{pmatrix} \mathscr{B}^{11} \\ \mathscr{B}^{21} \end{pmatrix}$. Now using Lemma 1 to the matrix $\mathscr{B}_1$ of the matrix differential operator $S_{\mathscr{A}}L$ in equation (3), we have an unimodular matrix $S_{\mathscr{B}}$ such that

$$S_{\mathscr{B}}S_{\mathscr{A}}L = \mathscr{A}_2 D + \mathscr{B}_2, \tag{4}$$

where $\mathscr{A}_2 = S_{\mathscr{B}}S_{\mathscr{A}}\mathscr{A} = \begin{pmatrix} \widehat{\mathscr{A}}^{11} \\ 0 \end{pmatrix}$ and $\mathscr{B}_2 = S_{\mathscr{B}}S_{\mathscr{A}}\mathscr{B} = \begin{pmatrix} 0 \\ \widehat{\mathscr{B}}^{21} \end{pmatrix}$. If we denote $S^{\star} = S_{\mathscr{B}}S_{\mathscr{A}}$ and $\widehat{L} = S^{\star}L$, then we have, from equation (4), that $\widehat{L} = \mathscr{A}_2 D + \mathscr{B}_2$.

Again by applying Lemma 1 to the matrix $\mathscr{A}_2$ of the matrix differential operator $\widehat{L}$, we can construct an unimodular matrix $T_{\mathscr{A}}$ such that

$$\widehat{L}T_{\mathscr{A}} = \mathscr{A}_3 D + \mathscr{B}_3, \tag{5}$$

where $\mathscr{A}_3 = \mathscr{A}_2 T_{\mathscr{A}} = \begin{pmatrix}\mathscr{A}_{11} & 0\end{pmatrix}$ and $\mathscr{B}_3 = \mathscr{B}_2 T_{\mathscr{A}} = \begin{pmatrix}\mathscr{B}_{11} & \mathscr{B}_{12}\end{pmatrix}$. Now using Lemma 1 to the matrix $\mathscr{B}_3$ of the matrix differential operator $\widehat{L}T_{\mathscr{A}}$ in equation (5), one can construct an unimodular matrix $T_{\mathscr{B}}$ such that

$$\widehat{L}T_{\mathscr{A}}T_{\mathscr{B}} = \mathscr{A}_4 D + \mathscr{B}_4, \tag{6}$$

where $\mathscr{A}_4 = \mathscr{A}_2 T_{\mathscr{A}} T_{\mathscr{B}} = \begin{pmatrix}\widehat{\mathscr{A}_{11}} & 0\end{pmatrix}$ and $\mathscr{B}_4 = \mathscr{B}_2 T_{\mathscr{A}} T_{\mathscr{B}} = \begin{pmatrix}0 & \widehat{\mathscr{B}}_{21}\end{pmatrix}$. If we denote $T^\star = T_{\mathscr{A}} T_{\mathscr{B}}$ and $L^\star = \widehat{L}T^\star$, then we have, from equation (6),

$$L^\star = \mathscr{A}^\star D + \mathscr{B}^\star, \tag{7}$$

where $L^\star = S^\star L T^\star$, $\mathscr{A}^\star = S^\star \mathscr{A} T^\star$, $\mathscr{B}^\star = S^\star \mathscr{B} T^\star$, and $S^\star = S_{\mathscr{B}} S_{\mathscr{A}}$, $T^\star = T_{\mathscr{A}} T_{\mathscr{B}}$. If rank$(L)$, number of non-zero rows in $L^\star$ and number of non-zero columns in $L^\star$ coincide, then the given matrix differential operator is in reduced form. If do not coincide, then we multiply operator $L^\star$ on the left and right by two suitable matrices, say P and Q of $GL_n(\mathbb{K}[[z]])$ to obtain the reduced form. The reduced form of the given DAS is in the following form

$$\widetilde{L} = \widetilde{\mathscr{A}} D + \widetilde{\mathscr{B}}, \tag{8}$$

where $\widetilde{L} = SLT$, $\widetilde{\mathscr{A}} = S\mathscr{A}T$, $\widetilde{\mathscr{B}} = S\mathscr{B}T$, and $S = PS_{\mathscr{B}} S_{\mathscr{A}}$, $T = T_{\mathscr{A}} T_{\mathscr{B}} Q$. One can easily check that $S, T \in GL_n(\mathbb{K}[[z]])$. We can decompose the reduced system (8) into two systems, one is in purely differential system and other one is in purely algebraic system, with some necessary conditions, based on the ranks of $\mathscr{A}$ and $\mathscr{B}$ as in the following theorem.

Theorem 1. *Let $L = \mathscr{A}D + \mathscr{B} \in \mathbb{K}[[z]][D]^{n\times n}$ be a matrix differential operator with rank$(\mathscr{A}) = r$ and rank$(\mathscr{B}) = k$. Then we can construct two unimodular matrices $S, T \in GL_n(\mathbb{K}[[z]])$ such that the system (2), can be decomposed*

(i) *for $r + k = n$, as*

$$\widetilde{\mathscr{A}}^{11} v_1' = \widetilde{f}_1,$$
$$\widetilde{\mathscr{B}}^{22} v_2 = \widetilde{f}_2,$$

where $\widetilde{\mathscr{A}}^{11} \in \mathbb{K}[[z]]^{r\times r}$, $\widetilde{\mathscr{B}}^{22} \in \mathbb{K}[[z]]^{k\times k}$ are invertible matrices, $v = T^{-1}u = \begin{pmatrix}v_1\\ v_2\end{pmatrix}$, $\widetilde{f} = Sf = \begin{pmatrix}\widetilde{f}_1\\ \widetilde{f}_2\end{pmatrix}$.

(ii) *for $r + k < n$, as*

$$\widetilde{\mathscr{A}}^{11} v_1' = \widetilde{f}_1,$$
$$\widetilde{\mathscr{B}}^{22} v_3 = \widetilde{f}_3,$$

where $\widetilde{\mathscr{A}}^{11}, \widetilde{\mathscr{B}}^{22}$ are invertible matrices over $\mathbb{K}[[z]]$ of order r, k respectively, $v = T^{-1}u = \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}$, $\widetilde{f} = Sf = \begin{pmatrix}\widetilde{f}_1\\ \widetilde{f}_2\\ \widetilde{f}_3\end{pmatrix}$ with some necessary conditions on right hand side expressed by $\widetilde{f}_2 = 0$.

Proof. Applying Lemma 1 to the matrix differential operator L, one can construct two unimodular matrices $S, T \in GL_n(\mathbb{K}[[z]])$. By substituting $u(z) = Tv(z)$ in equation (2) and left multiplying the resultant equation with S, we have the following reduced form

$$SLTv(z) = S\mathscr{A}TDv(z) + S\mathscr{B}Tv(z) = Sf(z) \quad \text{or} \tag{9}$$
$$\widetilde{L}v = \widetilde{\mathscr{A}}Dv + \widetilde{\mathscr{B}}v = \widetilde{f},$$

where $\widetilde{L} = SLT \in \mathbb{K}[[z]][D]^{n\times n}$, $\widetilde{\mathscr{A}} = S\mathscr{A}T \in \mathbb{K}[[z]]^{n\times n}$, $\widetilde{\mathscr{B}} = S\mathscr{B}T \in \mathbb{K}[[z]]^{n\times n}$ and $\widetilde{f} = Sf \in \mathbb{K}[[z]]^{n\times 1}$.

(i) If $r + k = n$, then the reduced DAS (9) has the form

$$\widetilde{L}v = \widetilde{\mathscr{A}}Dv + \widetilde{\mathscr{B}}v = \widetilde{f},$$

where $\widetilde{L} = \begin{pmatrix}\widetilde{\mathscr{A}}^{11}D & 0\\ 0 & \widetilde{\mathscr{B}}^{22}\end{pmatrix}$, $\widetilde{\mathscr{A}} = \begin{pmatrix}\widetilde{\mathscr{A}}^{11} & 0\\ 0 & 0\end{pmatrix}$, $\widetilde{\mathscr{B}} = \begin{pmatrix}0 & 0\\ 0 & \widetilde{\mathscr{B}}^{22}\end{pmatrix}$, $v = \begin{pmatrix}v_1\\ v_2\end{pmatrix}$, $\widetilde{f} = \begin{pmatrix}\widetilde{f}_1\\ \widetilde{f}_2\end{pmatrix}$, and $\widetilde{\mathscr{A}}^{11} \in \mathbb{K}[[z]]^{r\times r}$, $\widetilde{\mathscr{B}}^{22} \in \mathbb{K}[[z]]^{k\times k}$. Hence, the DAS in equation (2) is decomposed into two systems, one is purely differential system and second is purely algebraic system given by

$$\widetilde{\mathscr{A}}^{11} v_1' = \widetilde{f}_1,$$
$$\widetilde{\mathscr{B}}^{22} v_2 = \widetilde{f}_2.$$

In particular, if $r = k$, then $\widetilde{\mathscr{A}}^{11}, \widetilde{\mathscr{B}}^{22} \in \mathbb{K}[[z]]^{\frac{n}{2}\times\frac{n}{2}}$.

(ii) If $r + k < n$, then the reduced DAS (9) has the form

$$\widetilde{L}v = \widetilde{\mathscr{A}}Dv + \widetilde{\mathscr{B}}v = \widetilde{f},$$

where $\widetilde{L} = \begin{pmatrix}\widetilde{\mathscr{A}}^{11}D & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \widetilde{\mathscr{B}}^{22}\end{pmatrix}$, $\widetilde{\mathscr{A}} = \begin{pmatrix}\widetilde{\mathscr{A}}^{11} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\widetilde{\mathscr{B}} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \widetilde{\mathscr{B}}^{22}\end{pmatrix}$, $v = \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix}$, $\widetilde{f} = \begin{pmatrix}\widetilde{f}_1\\ \widetilde{f}_2\\ \widetilde{f}_3\end{pmatrix}$, and $\widetilde{\mathscr{A}}^{11} \in \mathbb{K}[[z]]^{r\times r}$, $\widetilde{\mathscr{B}}^{22} \in \mathbb{K}[[z]]^{k\times k}$. Hence, the DAS in equation (2) is decomposed into two systems, one is purely differential system and second is purely algebraic system given by

$$\widetilde{\mathscr{A}}^{11} v_1' = \widetilde{f}_1,$$
$$\widetilde{\mathscr{B}}^{22} v_3 = \widetilde{f}_3,$$

with some necessary conditions on the right hand side expressed by $\widetilde{f}_2 = 0$.

2.2 Examples

The following examples illustrate the proposed method, presented in Theorem 1.

*Example 3.*Consider a matrix differential operator of DAS

$$L = \mathscr{A}D + \mathscr{B}$$
$$= \begin{pmatrix} D+1 & (1-z)D+z & (2-z)D & D-1 & 1 \\ 1 & 2zD & 2 & D+1 & zD \\ 2D & 2D+z & (4-2z)D & 3D-2 & zD+1 \\ D+1 & (1-5z)D+\frac{1}{2}z & (2-z)D+1 & -D & \frac{1}{2}-2zD \\ D+5 & (1+z)D+2z & (2-z)D+6 & 2D+1 & zD+2 \end{pmatrix}, \tag{10}$$

where $\mathscr{A} = \begin{pmatrix} 1 & 1-z & 2-z & 1 & 0 \\ 0 & 2z & 0 & 1 & z \\ 2 & 2 & 4-2z & 3 & z \\ 1 & 1-5z & 2-z & -1 & -2z \\ 1 & 1+z & 2-z & 2 & z \end{pmatrix}$, and

$$\mathscr{B} = \begin{pmatrix} 1 & z & 0 & -1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & z & -2 & -2 & 1 \\ 1 & \frac{1}{2}z & 1 & 0 & \frac{1}{2} \\ 5 & 2z & 6 & 1 & 2 \end{pmatrix}.$$

Applying Lemma 1 to the matrix $\mathscr{A}$, we can construct an unimodular matrix $S_{\mathscr{A}}$. This can be achieved by finding a basis of left null space of $\mathscr{A}$,

$$\text{left null space of } \mathscr{A} = \begin{pmatrix} -2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

and the unimodular matrix $S_{\mathscr{A}}$ is

$$S_{\mathscr{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, multiplying operator L on the left by $S_{\mathscr{A}}$ yields the operator

$$S_{\mathscr{A}}L = \mathscr{A}_1 D + \mathscr{B}_1$$
$$= \begin{pmatrix} D+1 & (1-z)D+z & (2-z)D & D-1 & 1 \\ 1 & 2zD & 2 & D+1 & zD \\ -3 & -z & -4 & -1 & -1 \\ 2 & -\frac{1}{2}z & 5 & 3 & -\frac{1}{2} \\ 3 & z & 4 & 1 & 1 \end{pmatrix},$$

where

$$\mathscr{A}_1 = S_{\mathscr{A}}\mathscr{A} = \begin{pmatrix} \mathscr{A}^{11} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1-z & 2-z & 1 & 0 \\ 0 & 2z & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\mathscr{B}_1 = S_{\mathscr{A}}\mathscr{B} = \begin{pmatrix} \mathscr{B}^{11} \\ \mathscr{B}^{21} \end{pmatrix} = \begin{pmatrix} 1 & z & 0 & -1 & 1 \\ 1 & 0 & 2 & 1 & 0 \\ -3 & -z & -4 & -1 & -1 \\ 2 & -\frac{1}{2}z & 5 & 3 & -\frac{1}{2} \\ 3 & z & 4 & 1 & 1 \end{pmatrix}.$$

Applying Lemma 1 to the matrix $\mathscr{B}_1$, we can construct an unimodular matrix $S_{\mathscr{B}}$ (it is computed using a basis of left null space of $\mathscr{B}_1$) as follows

$$S_{\mathscr{B}} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{2} & 0 & 1 & 0 \\ -1 & -2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

such that

$$S_{\mathscr{B}}S_{\mathscr{A}}L = \mathscr{A}_2 D + \mathscr{B}_2$$
$$= \begin{pmatrix} \frac{1}{2}D & \frac{1}{2}(1-11z)D & (1-\frac{1}{2}z)D & -2D & -\frac{5}{2}zD \\ -D & -(1+3z)D & (z-2)D & -3D & -2zD \\ D & (1+3z)D & (2-z)D & 3D & 2zD \\ 2 & -\frac{1}{2}z & 5 & 3 & -\frac{1}{2} \\ 3 & z & 4 & 1 & 1 \end{pmatrix},$$

where

$$\mathscr{A}_2 = S_{\mathscr{B}}S_{\mathscr{A}}\mathscr{A} = \begin{pmatrix} \widehat{\mathscr{A}}^{11} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}(1-11z) & 1-\frac{1}{2}z & -2 & -\frac{5}{2}z \\ -1 & -1-3z & z-2 & -3 & -2z \\ 1 & 1+3z & 2-z & 3 & 2z \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\mathscr{B}_2 = S_{\mathscr{B}}S_{\mathscr{A}}\mathscr{B} = \begin{pmatrix} 0 \\ \widehat{\mathscr{B}}^{21} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -\frac{1}{2}z & 5 & 3 & -\frac{1}{2} \\ 3 & z & 4 & 1 & 1 \end{pmatrix}.$$

In the matrix differential operator $S_{\mathscr{B}}S_{\mathscr{A}}L$, third row is depending on second row, hence we can multiply left $S_{\mathscr{B}}S_{\mathscr{A}}L$ by the matrix (elementary row transformations matrix)

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

the resultant system is

$$PS_{\mathscr{B}}S_{\mathscr{A}}L = P\mathscr{A}_2 D + P\mathscr{B}_2$$
$$= \begin{pmatrix} \frac{1}{2}D & \frac{1}{2}(1-11z)D & (1-\frac{1}{2}z)D & -2D & -\frac{5}{2}zD \\ -D & -(1+3z)D & (z-2)D & -3D & -2zD \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -\frac{1}{2}z & 5 & 3 & -\frac{1}{2} \\ 3 & z & 4 & 1 & 1 \end{pmatrix}.$$

Thus, we have

$$S = PS_{\mathscr{B}}S_{\mathscr{A}} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 \\ -2 & -3 & 0 & 0 & 1 \\ -3 & -2 & 1 & 0 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Repeating the procedure, one can construct the unimodular matrix T similar to that of S, and T is given by

$$T = T_{\mathscr{A}}T_{\mathscr{B}}Q = \begin{pmatrix} 0 & -3 & \frac{z^3-4z^2+2}{2z^2-z-1} & \frac{1-3z}{2z} & \frac{1-z}{2} \\ -\frac{1}{z} & \frac{4}{z} & \frac{z}{1-z} & -\frac{1}{2z} & -\frac{1}{2} \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-z^3+2z}{2z^2-z-1} & 1 & 0 \\ 1 & 0 & \frac{5z^2+2z-2}{2z^2-z-1} & 0 & 1 \end{pmatrix}.$$

Now the reduced form of the given DAS (10) is

$$\widetilde{L} = SLT$$
$$= \begin{pmatrix} -\frac{1}{2z}(5z^2-11z+1)D & -\frac{1}{2z}(z^2+49z-4)D & 0 & 0 & 0 \\ -\frac{1}{z}(2z^2-3z-1)D & \frac{1}{z}(z^2-14z-4)D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4+z}{4z} & \frac{2-3z}{4} \\ 0 & 0 & 0 & \frac{3-8z}{2z} & \frac{5-4z}{2} \end{pmatrix}.$$

Example 4. Consider a matrix differential operator of DAS

$$\begin{aligned} L &= \mathscr{A}D + \mathscr{B} \\ &= \begin{pmatrix} 1-z & 2-z & 1 & 0 \\ 2z & 0 & 1 & z \\ 2 & 4-2z & 3 & z \\ 1-5z & 2-z & -1 & -2z \end{pmatrix} D + \begin{pmatrix} z & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ z & -2 & -2 & 1 \\ \frac{z}{2} & 1 & 0 & \frac{1}{2} \end{pmatrix} \end{aligned} \quad (11)$$

Following the Theorem 1 similar to Example 3, we can construct two unimodular matrices S and T as

$$S = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -1 & 1 & 1 & 0 \\ -2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -\frac{1}{z} & \frac{1}{z} & -\frac{1}{2z} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3z-1}{2z(z-2)} & \frac{z-1}{2(z-2)} \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Now the reduced form of the given DAS (11) is

$$\widetilde{L} = SLT$$
$$= \begin{pmatrix} -\frac{(5z^2-11z+1)D}{2z} & \frac{(z^2-32z+2)D}{4z} & 0 & 0 \\ \frac{(2z^2-3z-1)D}{z} & \frac{(z^2+10z+2)D}{2z} & 0 & 0 \\ 0 & 0 & -\frac{z^2+10z-4}{2z(z-2)} & \frac{z^2-8z+8}{2(z-2)} \\ 0 & 0 & \frac{13z^2+4z-10}{4z(z-2)} & \frac{z^2+6z-6}{4(z-2)} \end{pmatrix}.$$

*Example 5.*Consider a DAS similar to the Example 5.3.1 of [2, pp. 132].

$$\begin{aligned} L = &\begin{pmatrix} 1 & 1-z & 2-z & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ z & 1 & 1 & 0 & z & 1 \\ 1+z & 1-z & 3-z & -1 & z-1 & 2 \\ 2+2z & 4-2z & 6-2z & 0 & 2z-2 & 4 \\ 1 & -z & 2-z & -1 & -1 & 1 \end{pmatrix} D \\ &+ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & -2 & 0 \\ 1 & z & 2+z & 0 & -2 & 0 \\ 1 & 1 & 3 & 0 & -2 & 0 \end{pmatrix} \end{aligned} \quad (12)$$

Using the proposed algorithm in Theorem 1, we can construct two unimodular matrices S and T as

$$S = \begin{pmatrix} -\frac{1}{z} & \frac{1}{z} & \frac{1+z}{z} & \frac{1-z}{z} & -\frac{1}{z} & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 \\ -2 & 0 & -2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$T = \begin{pmatrix} -\frac{2}{z-1} & \frac{2(z^2+1)}{(z-1)^2} & -2 & -\frac{1}{(z-1)^2} & -\frac{z^2-2z-1}{(z-1)^2} & -\frac{1}{z-1} \\ -\frac{1}{z-1} & \frac{2z}{(z-1)^2} & -1 & -1 & 0 & 0 \\ \frac{1}{z-1} & -\frac{2z}{(z-1)^2} & 1 & \frac{z^2-z+1}{(z-1)^2} & -\frac{2z}{(z-1)^2} & \frac{1}{z-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The reduced form of the given DAS (12) is

$$\widetilde{L} = SLT$$
$$= \begin{pmatrix} \frac{2(2-z)}{z(z-1)}D & -\frac{2(z^2+1)}{z((z-1)^2)}D & \frac{2}{z}D & 0 & 0 & 0 \\ -D & -D & 0 & 0 & 0 & 0 \\ -\frac{(1+z)}{z-1}D & -\frac{z(3z^2-2z+3)}{(z-1)^2}D & -2zD & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2z^2-2z+1}{(z-1)^2} & -\frac{3z^2-2z+1}{(z-1)^2} & \frac{1}{z-1} \\ 0 & 0 & 0 & \frac{z^2+2z-1}{(z-1)^2} & -\frac{5z^2-2z+1}{(z-1)^2} & \frac{1+z}{z-1} \\ 0 & 0 & 0 & \frac{2z^2-z+1}{(z-1)^2} & -\frac{3z^2+1}{(z-1)^2} & \frac{2}{z-1} \end{pmatrix}.$$

Remark. In [2], C. E. Bachaa et. al. have presented a reduction algorithm for linear differential-algebraic equations of first-order. In the Example 5.3.1 of [2, pp. 132], authors have solved this type examples by reducing the differential operator L using a row-reduction algorithm and then again using a column-reduction algorithm to obtain the reduced matrix differential operator. However, in this paper, we use single algorithm to obtain the reduced algorithm as discussed above.

3 Maple Implementation

Now we discuss the Maple implementation of the proposed algorithm by creating different data types. The implemented Maple package, `DAS_Reduction`, of the proposed algorithm is provided with Maple worksheet at `www.srinivasaraothota.webs.com/research`. Using the Maple package, one can obtain the two unimodular matrices S, T and the reduced DAS of the given system. In Maple implementation, z is complex variable and $D = \frac{d}{dz}$ is the differential operator.

Pseudo-Code of Proposed Algorithm

$A, B \leftarrow coefficient\ matricies$
$n \leftarrow size\ of\ A$
$LNS\_A \leftarrow left\ null\ space\ of\ A$
$Sa \leftarrow identity\ matrix\ with\ LNS\_A$
$Aa \leftarrow Sa.A,\ Ba \leftarrow Sa.B$
$LNS\_Ba \leftarrow left\ null\ space\ of\ Ba$
$Sb \leftarrow identity\ matrix\ with\ LNS\_Ba$
$A1 \leftarrow Sb.Aa,\ B1 \leftarrow Sb.Ba,\ S1 \leftarrow Sb.Sa$
$RNS\_A1 \leftarrow right\ null\ space\ of\ A1$
$Ta \leftarrow identity\ matrix\ with\ RNS\_A1$
$A1a \leftarrow A1.Ta,\ B1a \leftarrow B1.Ta$
$RNS\_B1a \leftarrow right\ null\ space\ of\ B1a$
$Tb \leftarrow identity\ matrix\ with\ LNS\_B1a$
$A2 \leftarrow A1a.Tb,\ B2 \leftarrow B1a.Tb,\ T1 \leftarrow Ta.Tb$
$P \leftarrow left\ elementary\ matrix$
$Q \leftarrow right\ elementary\ matrix$
$S \leftarrow P.S1,\ T \leftarrow T1.Q$

*Example 6.*Consider the following matrix differential operator

$$L = \mathscr{A}D + \mathscr{B}$$
$$= \begin{pmatrix} D+1 & (1-z)D+z & (2-z)D & D-1 \\ \frac{1}{2} & 2zD+\frac{1}{2}z & 0 & D-\frac{1}{2} \\ 2D+2 & 2(D+z) & (4-2z)D & 3D-2 \\ D+3 & (1-5z)D+3z & (2-z)D & -(D+3) \end{pmatrix}, \tag{13}$$

where $\mathscr{A} = \begin{pmatrix} 1 & 1-z & 2-z & 1 \\ 0 & 2z & 0 & 1 \\ 2 & 2 & 4-2z & 3 \\ 1 & 1-5z & 2-z & -1 \end{pmatrix}$, $\mathscr{B} = \begin{pmatrix} 1 & z & 0 & -1 \\ \frac{1}{2} & \frac{z}{2} & 0 & -\frac{1}{2} \\ 2 & 2z & 0 & -2 \\ 3 & 3z & 0 & -3 \end{pmatrix}$.

Using Maple implementation, `DAS_Reduction`, of the proposed algorithm, we have

```
> A :=
Matrix([[1,1-x,2-x,1],[0,2x,0,1],
       [2,2,4-2x,3],[1,1-5x,2-x,-1]]);
> B :=
Matrix([[1,x,0,-1],[1/2,x/2,0,-1/2],
       [2,2x,0,-2],[3,3x,0,-3]]);
```

$$A := \begin{bmatrix} 1 & 1-x & 2-x & 1 \\ 0 & 2x & 0 & 1 \\ 2 & 2 & 4-2x & 3 \\ 1 & 1-5x & 2-x & -1 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & x & 0 & -1 \\ \frac{1}{2} & \frac{x}{2} & 0 & -\frac{1}{2} \\ 2 & 2x & 0 & -2 \\ 3 & 3x & 0 & -3 \end{bmatrix}$$

```
> S,T,Lreduced:=DAS_Reduction(A, B);
```

$$\begin{bmatrix} \frac{1}{6}\frac{18x-1}{6x-1} & -\frac{2}{3(6x-1)} & 0 & -\frac{1}{6} \\ \frac{1}{12x} & \frac{1}{3x} & 0 & -\frac{1}{12x} \\ -\frac{13}{6} & -\frac{2}{3} & 1 & \frac{1}{6} \\ -1 & 2 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2}\frac{3x-1}{x} \\ \frac{1}{2x} & \frac{1}{2x} & -\frac{1}{2x} & -\frac{1}{2x} \\ -\frac{1}{2}\frac{6x-1}{x(x-2)} & -\frac{1}{2}\frac{2x-1}{x(x-2)} & \frac{1}{2}\frac{6x-1}{x(x-2)} & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} D & \frac{2x-1}{6x-1}D & 0 & 0 \\ \frac{1}{2x}D & \frac{1}{2x}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\frac{6x-1}{x} \end{bmatrix},$$

From Maple implementation, we have two unimodular matrices, $S, T \in GL_n(\mathbb{K}[[z]])$,

$$S = \begin{pmatrix} \frac{1}{6}\frac{18z-1}{6z-1} & -\frac{2}{3(6z-1)} & 0 & -\frac{1}{6} \\ \frac{1}{12z} & \frac{1}{3z} & 0 & -\frac{1}{12z} \\ -\frac{13}{6} & -\frac{2}{3} & 1 & \frac{1}{6} \\ -1 & 2 & 0 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2}\frac{3x-1}{z} \\ \frac{1}{2z} & \frac{1}{2z} & -\frac{1}{2z} & -\frac{1}{2z} \\ -\frac{1}{2}\frac{6z-1}{z(z-2)} & -\frac{1}{2}\frac{2z-1}{z(z-2)} & \frac{1}{2}\frac{6z-1}{z(z-2)} & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

and the reduced matrix differential operator of the given DAS (13) is

$$\widetilde{L} = \begin{pmatrix} D & \frac{2z-1}{6z-1}D & 0 & 0 \\ \frac{1}{2z}D & \frac{1}{2z}D & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2}\frac{6z-1}{z} \end{pmatrix}$$

4 Conclusion

In this paper, we discussed a new reduction algorithm for differential-algebraic system with power series coefficients. Using the proposed algorithm, one can transform the given system of differential-algebraic equations into another simpler system having the same properties. Maple implementation of the proposed algorithm is also discussed and sample computations are presented to illustrate the algorithm. The implemented Maple package, `DAS_Reduction`, is provided at `www.srinivasaraothota.webs.com/research` with Maple worksheet. The proposed algorithm may be helpful to implement this method in commercial packages such as Mathematica, Matlab, Singular, Scilab etc.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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