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Modified Inverse Weibull Distribution

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Abstract: A generalized version of four parameter modified inverse weibull distribution (MIWD) is introduced in this paper. This distribution generalizes the following distributions: (1) Modified Inverse exponential distribution, (2) Modified Inverse Rayleigh distribution, (3) Inverse weibull distribution. We provide a comprehensive description of the mathematical properties of the modified inverse weibull distribution along with its reliability behaviour. We derive the moments, moment generating function and examine the order statistics. We propose the method of maximum likelihood for estimating the model parameters and obtain the observed information matrix.

Keywords: Reliability functions; moment estimation; moment generating function; least square estimation; order statistics; maximum likelihood estimation.

1 Introduction

The inverse weibull distribution is the life time probability distribution which is used in the reliability engineering discipline. The inverse weibull distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. Reliability and failure data both from life testing and in service records which is often modeled by the life time distributions such as the inverse exponential, inverse Rayleigh, inverse Weibull distributions. In this research we have developed a new reliability model called modified inverse weibull distribution. This paper focuses on all the properties of this model and presents the graphical analysis of modified inverse weibull reliability models. This paper present the relationship between shape parameter and other properties such as non- reliability function, reliability function, instantaneous failure rate, cumulative instantaneous failure rate models. This life time distribution is capable of modeling of various shapes of aging and failure criteria.The proposed model can be used as an alternative to inverse generalized exponential, inverse generalized Rayleigh, inverse generalized weibull distributions. The cumulative distribution function (CDF) of the Inverse weibull distribution is denoted by $F_{Inw}(t)$ and is defined as

$$F_{Inw}(t) = \exp\left(-\frac{1}{\alpha}\left(\frac{1}{t-\eta}\right)^{\beta}\right) \tag{1.1}$$

The CDF is given in equation (1.1) becomes identical with the CDF of Inverse Rayleigh distribution for $\beta = 2$, and for $\beta = 1$ it coincides with the Inverse Exponential distribution. In the probability theory of statistics the weibull and inverse weibull distributions are the family of continuous probability distributions which have the capability to develop many other life time distributions such as exponential, negative exponential, Rayleigh, inverse Rayleigh distributions and weibull families also known as type I, II and III extreme value distributions. Recently Ammar et al. [1] proposed a Modified Weibull distribution. Some works has already been done on of Inverse Weibull distribution by M. Shuaib Khan et al. [9-12]. These distributions have several attractive properties for more details we refer to [2]-[8], [13-17].In this paper we introduce new four parameter distribution called Modified Inverse Weibull distribution

$MIWD(\alpha,\beta,\gamma,\eta,t)$ with four parameters α,β,γ and η. Here we provide the statistical properties of this reliability model. The moment estimation, moment generating function and maximum likelihood estimates (MLE_S) of the unknown parameters are derived. The asymptotic confidence intervals of the parameters are discussed. The minimum and maximum order statistics models are derived. The joint density functions $g(t_1,t_n)$ of Modified Inverse Weibulldistribution $MIWD(\alpha,\beta,\gamma,\eta,t)$ are derived. The fisher Information matrix is also discussed.

2 Modified Inverse Weibull Distribution

The probability distribution of $MIWD(\alpha,\beta,\gamma,\eta,t)$ has four parameters α,β,γ and η. It can be used to represent the failure probability density function (PDF) is given by:

$$f_{MIW}(t)=\left(\alpha+\beta\gamma\left(\frac{1}{t-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right),\alpha>0,\ \beta>0,\ \gamma>0,\ ,\eta<t<\infty \qquad (2.1)$$

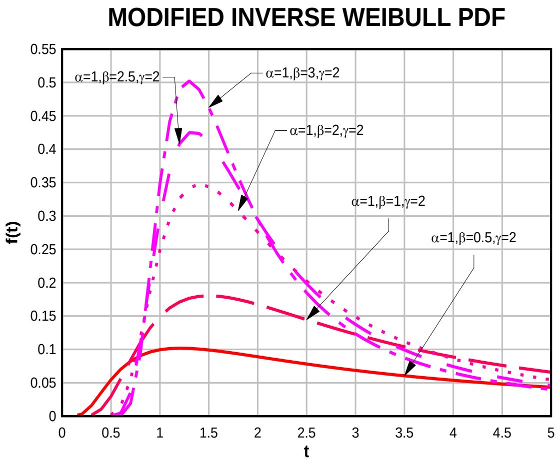

Fig 2.1 Modified Inverse Weibull PDF

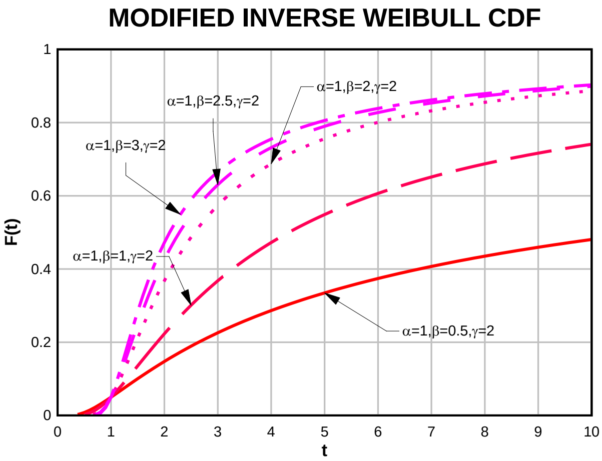

Fig 2.2 Modified Inverse WeibullCDF

Where β is the shape parameter representing the different patterns of the Modified Inverse Weibull probabilitydistribution. Here γ is a scale parameter representing the characteristic life and is also positive, η is a location parameter also called a guarantee time, failure-free time or minimum life. The Modified Inverse Weibulldistribution is said to be two-parameter when $\eta=0$. The pdf of the Modified Inverse Weibulldistribution is given in (2.1). Since the restrictions in (2.1) on the values of α,β,γ and η are always the same for the $MIWD(\alpha,\beta,\gamma,\eta,t)$. Fig. 2.1 shows the diverse shape of the Modified Inverse WeibullPDF with β (= 0.5, 1, 2, 2.5, 3), for $\alpha=1$, $\gamma=2$ and the value of $\eta=0$. It is important to note that all the figures based on the assumption that $\eta=0$.The cumulative distribution function (CDF) of the Modified Inverse Weibull distribution is denoted by $F_{MIW}(t)$ and is defined as

$$F_{MIW}(t)=\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right) \qquad (2.2)$$

When the CDF of the Modified Inverse Weibulldistribution has zero value then it represents no failure component by η. In the Modified Inverse WeibullCDF η is called minimum life. When $t=\eta+\gamma$ then $F_{MIW}(\eta+\gamma)=e^{-\frac{\alpha}{\gamma}-\gamma^{1-\beta}}$ for $\alpha=1$ and $\gamma=2$ it represents the characteristic life. Fig. 2.2 shows the special case of Modified Inverse WeibullCDF with $\eta=0$ and for the value of $\gamma=2$ and β (=0.5, 1, 2, 2.5, 3). It is clear from the Fig. 2.2 that all curves intersect at the point of (1, 0.04979), the characteristic point for the Modified Inverse WeibullCDF.

3Reliability Analysis

The Modified Inverse Weibulldistribution can be a useful characterization of life time data analysis. The reliability function (RF) of the Modified Inverse Weibull distribution is denoted by $R_{MIW}(t)$ also known as the survivor function and is defined as $1-F_{MIW}(t)$

$$R_{MIW}(t) = 1-\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right) \tag{3.1}$$

One of the characteristic in reliability analysis is the hazard rate function defined by

$$h_{MIW}(t) = \frac{f_{MIW}(t)}{1-F_{MIW}(t)}$$

The hazard function (HF) of the Modified Inverse Weibull distribution also known as instantaneous failure rate denoted by $h_{MIW}(t)$ and is defined as $f_{MIW}(t)/R_{MIW}(t)$

$$h_{MIW}(t) = \frac{\left(\alpha+\beta\gamma\left(\frac{1}{t-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right)}{1-\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right)} \tag{3.2}$$

It is important to note that the units for $h_{MIW}(t)$ is the probability of failure per unit of time, distance or cycles.

Theorem 3.1: The hazard rate function of a Modified Inverse Weibull distribution has the following properties:

(i) If $\beta = 2$, the failure rate is same as the $MIRD(\alpha,\gamma,\eta,t)$

(ii) If $\beta = 1$, the failure rate is same as the $MIED(\alpha,\gamma,\eta,t)$

(iii) If $\alpha = 0$, the failure rate is same as the $IWD(\beta,\gamma,\eta,t)$.

Proof.

(i) If $\beta = 2$, the failure rate is same as the $MIRD(\alpha,\gamma,\eta,t)$

$$h_{MIR}(t) = \frac{\left(\alpha+2\gamma\left(\frac{1}{t-\eta}\right)\right)\left(\frac{1}{t-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{2}\right)}{1-\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)^{2}\right)} \tag{3.3}$$

(ii) If $\beta = 1$, the failure rate is same as the $MIED(\alpha,\gamma,\eta,t)$

$$h_{MIE}(t) = \frac{(\alpha+\gamma)\left(\frac{1}{t-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)\right)}{1-\exp\left(-\frac{\alpha}{t-\eta}-\gamma\left(\frac{1}{t-\eta}\right)\right)} \tag{3.4}$$

(iii) If $\alpha = 0$, the failure rate is same as the $IWD(\beta,\gamma,\eta,t)$

$$h_{IW}(t)=\frac{\beta\gamma\left(\frac{1}{t-\eta}\right)^{\beta+1}\exp\left(-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right)}{1-\exp\left(-\gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right)} \tag{3.5}$$

Figure 3.1 illustrates the reliability pattern of a Modified Inverse Weibull distribution as the value of the shape parameters.

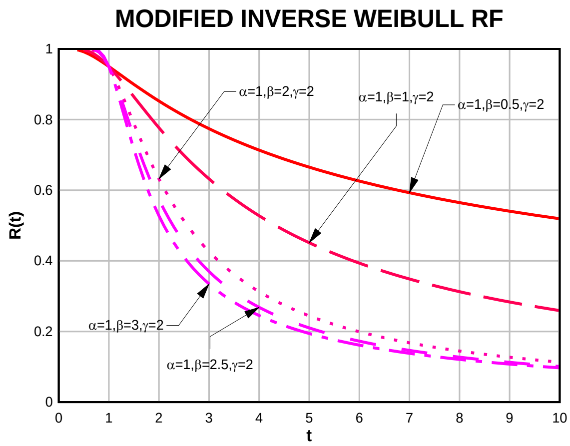

Fig 3.1 Modified Inverse Weibull PDF

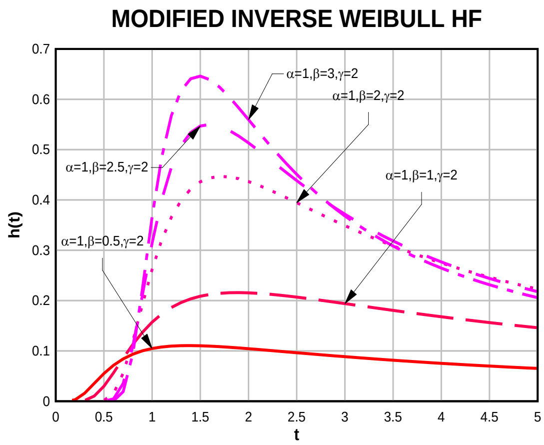

Fig3.2 Modified Inverse WeibullCDF

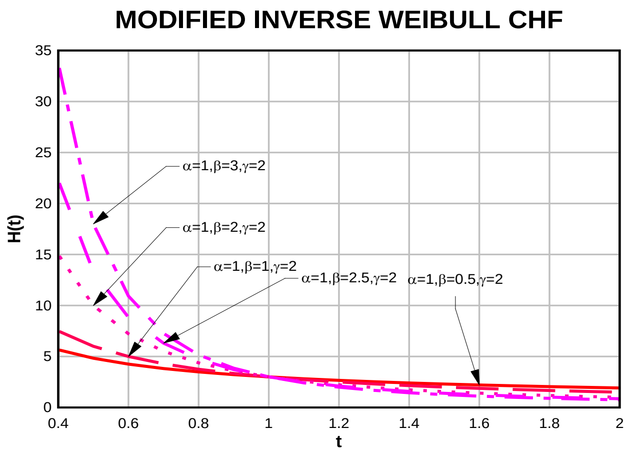

Fig 3.3 Modified Inverse WeibullCDF

It is important to note that $R_{MIW}(t)+F_{MIW}(t)=1$. Fig. 3.1 shows the Modified Inverse WeibullRF with $\alpha=1$, $\gamma=2$ and β (=0.5, 1, 2, 2.5, 3). It is clear that all curves intersect at the point of (1, 0.9502) the characteristic point for the Modified Inverse WeibullRF. When β = 0.5, the distribution has the decreasing HR. When β = 1, the HR is steadily decreasing which represents early failures. When $\beta>1$, the HF is continually increasing between $0.1\le t\le 1.5$ and then decreasing instantaneous failure rate between $1.6\le t\le 10$ which represents wear-out failures. The HR of the MIWD as given in equation (3.2) becomes identical with the HR of Modified Inverse Rayleigh distribution for $\beta=2$, and for $\beta=1$ it coincides with the Modified Inverse Exponential distribution. So the Modified Inverse Weibulldistribution is a very flexible distribution. Fig. 3.2 shows the Modified Inverse Weibull HF with $\alpha=1$, $\gamma=2$ and β (=0.5, 1, 2, 2.5, 3).

The Cumulative hazard function (CHF) of the Modified Inverse Weibulldistribution is denoted by $H_{MIW}(t)$ and is defined as

$$H_{MIW}(t) = -\ln\left|\exp\left(-\frac{\alpha}{t-\eta} - \gamma\left(\frac{1}{t-\eta}\right)^{\beta}\right)\right| \tag{3.6}$$

It is important to note that the units for $H_{MIW}(t)$ are the cumulative probability of failure per unit of time, distance or cycles. Fig. 3.3 shows the Modified Inverse WeibullCHF with $\alpha = 1$, $\gamma = 2$ and β (=0.5, 1, 2, 2.5, 3). It is important to note that as β increases the pattern of CHF strictly decreasing.

4Statistical properties

This section explain the statistical properties of the $MIWD(\alpha, \beta, \gamma, \eta, t)$.

4.1Quantile and median

The quantile t_q of the $MIWD(\alpha, \beta, \gamma, t)$ is the real solution of the following equation

$$\gamma\left(\frac{1}{t_q}\right)^{\beta} + \frac{\alpha}{t_q} + \ln|1-q| = 0 \tag{4.1}$$

The above equation has no closed form solution in t_q, so we have different cases by substituting the parametric values in the above quantile equation (4.1). So the derived special cases are

1. The q-thquantile of the $MIRD(\alpha, \gamma, t)$ by substituting $\beta = 2$

$$t_q = \frac{2\gamma}{-\alpha + \sqrt{\alpha^2 - 4\gamma\ln|1-q|}}$$

2. The q-thquantile of the $IWD(\beta, \gamma, t)$ by substituting $\alpha = 0$

$$t_q = \left(\frac{-\gamma}{\ln|1-q|}\right)^{\frac{1}{\beta}}$$

3. The q-thquantile of the $IRD(\gamma, t)$ by substituting $\alpha = 0$, $\beta = 2$

$$t_q = \sqrt{\frac{-\gamma}{\ln|1-q|}}$$

4. The q-thquantile of the $IED(\gamma, t)$ by substituting $\alpha = 0$, $\beta = 1$

$$t_q = \frac{-\alpha}{\ln|1-q|}$$

or by substituting $\beta = 1$

$$t_q = \frac{-(\alpha+\gamma)}{\ln|1-q|}$$

By putting $q = 0.5$ in equation (4.1) we can get the median of $MIWD(\alpha, \beta, \gamma, \eta, t)$

4.2Mode

The mode of the $MIWD(\alpha,\beta,\gamma,\eta,t)$ can be obtained as a solution of the following non-linear equation with respect to t

$$\left(\frac{\alpha}{t^2}+\beta\gamma\left(\frac{1}{t}\right)^{\beta+1}\right)^2-\left(\frac{2\alpha}{t^3}+\beta(\beta+1)\gamma\left(\frac{1}{t}\right)^{\beta+2}\right)=0 \tag{4.2}$$

The above equation (4.2) has not an unambiguous solution in the general form. The general form has the following special cases

(1) If we put $\beta=2$ and $\alpha=0$ then we have $IRD(\gamma,t)$ case, in this case equation (4.2) takes the following form

$$\left(2\gamma\left(\frac{1}{t}\right)^3\right)^2-6\gamma\left(\frac{1}{t}\right)^4=0$$

Solving this equation in t, we get the mode as $Mod(t)=\sqrt{\frac{2}{3}\gamma}$

(2) If we put $\beta=1$ then we have $MIED(\alpha,\gamma,t)$ case, in this case equation (4.2) takes the following form

$$\left(\frac{\alpha}{t^2}+\gamma\left(\frac{1}{t}\right)^2\right)^2-\left(\frac{2\alpha}{t^3}+2\gamma\left(\frac{1}{t}\right)^3\right)=0$$

Solving this equation in t, we get the mode as $Mod(t)=\frac{\alpha+\gamma}{2}$

(3) If we put $\alpha=0$ then we have $IWD(\beta,\gamma,t)$ case, in this case equation (4.2) takes the following form

$$\left(\beta\gamma\left(\frac{1}{t}\right)^{\beta+1}\right)^2-\beta(\beta+1)\gamma\left(\frac{1}{t}\right)^{\beta+2}=0$$

Solving this equation in t, we get the mode as $Mod(t)=\left(\frac{\beta+1}{\beta\gamma}\right)^{\frac{-1}{\beta}}$

Such that $\beta>0$, it is known that $RD(\beta)$ can be derived from $IWD(\beta,\gamma,t)$ when $\beta=2$, therefore the $RD(\beta)$ becomes $Mod(t)=\sqrt{\frac{2\gamma}{3}}$

4.3 Moments

The following theorem gives the r^{th} moment of $MIWD(\alpha,\beta,\gamma,t)$

Theorem 4.1: If T has the $MIWD(\alpha,\beta,\gamma,t)$, the r^{th} moment of T, say μ_r is given as follows

$$\mu_r=\begin{cases}\sum_{i=0}^{\infty}\frac{(-1)^i\gamma^i}{i!}\left[\alpha^{r-i\beta}\Gamma(i\beta-r+1)+\beta\gamma\alpha^{r-\beta(i+1)}\Gamma(\beta(i+1)-r)\right] & for\ \alpha,\beta,\gamma>0\\ \gamma^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right) & for\ \alpha=0,\beta,\gamma>0\\ \alpha^r\Gamma(1-r) & for\ \alpha>0,\beta,\gamma=0\end{cases} \tag{4.3}$$

The proof of this theorem is provided in Appendix.
Based on the above results given in theorem (4.1), the coefficient of variation, coefficient of skewness and coefficient of kurtosis of MIWD can be obtained according to the following relation

$$CV_{MIW} = \sqrt{\frac{\mu_2}{\mu_1} - 1} \tag{4.4}$$

$$CS_{MIW} = \frac{\mu_3 - 3\mu_2\mu_1 + 2\mu_1^{3}}{\left(\mu_2 - \mu_1^{2}\right)^{\frac{3}{2}}} \tag{4.5}$$

$$CK_{MIW} = \frac{\mu_4 - 4\mu_3\mu_1 + 6\mu_2\mu_1^{2} - 3\mu_1^{4}}{\left(\mu_2 - \mu_1^{2}\right)^{2}} \tag{4.6}$$

The coefficient of variation is the quantity used to measure the consistency of life time data. The coefficient of skewness is the quantity used to measure the skewness of life time data analysis. The coefficient of kurtosis is the quantity used to measure the kurtosis or peaked ness of the of the life time distribution. So the above models are helpful for accessing these characteristics.

4.4Moment Generating Function

The following theorem gives the moment generating function (mgf) of $MIWD(\alpha,\beta,\gamma,t)$.
Theorem 4.2: If T has the $MIWD(\alpha,\beta,\gamma,t)$, the moment generating function (mgf) of T, say $M(t)$ is given as follows

$$M(t) = \begin{cases} \sum_{i=0}^{\infty} \frac{(-1)^i \gamma^i}{i!}\left[\frac{\alpha\,\Gamma(i\beta+1)}{(\alpha-t)^{i\beta+1}} + \frac{\beta\gamma\,\Gamma\left(\beta(i+1)\right)}{(\alpha-t)^{\beta(i+1)}}\right] & for\ \alpha,\beta,\gamma>0 \\ \sum_{i=0}^{\infty} \frac{t^i \gamma^{\frac{i}{\beta}}}{i!}\Gamma\left(1-\frac{i}{\beta}\right) & for\ \alpha=0,\beta,\gamma>0 \\ \frac{\alpha}{\alpha-t} & for\ \alpha>0,\beta,\gamma=0 \end{cases} \tag{4.7}$$

The proof of theorem (4.2) is provided in Appendix.
Based on the above results given in theorem (4.2), the measure of central tendency, measure of dispersion, coefficient of variation, coefficient of skewness and coefficient of kurtosis of MIWD can be obtained according to the above relation.

5 Least square estimation

Case A: Let $T_1, T_2, \ldots\ldots, T_n$ be a random sample of Modified Inverse Weibull distribution with cdf $F_{MIW}(t)$ and suppose that $T_{(i)}$, $i = 1,2,\ldots\ldots,n$ denote the ordered sample. For sample of size n, we have

$$E(F(T_{(i)})) = \frac{\mathrm{i}}{\mathrm{n}+1}$$

The least square estimators (LSE_S) are obtained by minimizing

$$Q(\alpha,\beta,\gamma) = \sum_{\mathrm{i}=1}^{\mathrm{n}}\left(F(T_{(\mathrm{i})}) - \frac{\mathrm{i}}{\mathrm{n}+1}\right)^2 \tag{5.1}$$

In case of $MIWD(\alpha,\beta,\gamma,\eta,t)$, Equation (5.1) becomes

$$Q(\alpha,\beta,\gamma)=\sum_{i=1}^{n}\left(Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)-\frac{i}{n+1}\right)^{2} \tag{5.2}$$

To minimize Equation (5.2) with respect to α,β and γ, we differentiate with respect to these parameters, which leads to the following equations:

$$\sum_{i=1}^{n}\frac{1}{t}\left(Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)-\frac{i}{n+1}\right)Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)=0 \tag{5.3}$$

$$\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{\gamma}\left(Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)-\frac{i}{n+1}\right)Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)=0 \tag{5.4}$$

$$\sum_{i=1}^{n}\beta\left(\frac{1}{t}\right)^{\gamma}\ln\left(\frac{1}{t}\right)\left(Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)-\frac{i}{n+1}\right)Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)=0 \tag{5.5}$$

Case B: Let $T_1,T_2,.....,T_n$ be a random sample of $MIWD(\alpha,\beta,\gamma,t)$ Modified Inverse Weibull distribution with cdf $F_{MIW}(t)$, for sample of size n, we have
The least square estimators (LSE_S) are obtained by minimizing $Q(\alpha,\beta,\gamma)$ of $MIWD(\alpha,\beta,\gamma,t)$, equation (5.6) becomes

$$Q(\alpha,\beta,\gamma)=\sum_{i=1}^{n}\left(y_i-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right)^{2} \tag{5.6}$$

To minimize equation (5.6) with respect to α,β and γ, we differentiate with respect to these parameters, which leads to the following equations:

$$\sum_{i=1}^{n}y\frac{1}{t}-\alpha\sum_{i=1}^{n}\frac{1}{t^2}-\gamma\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}=0 \tag{5.7}$$

$$\sum_{i=1}^{n}y\left(\frac{1}{t_i}\right)^{\beta}-\alpha\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}-\gamma\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2\beta}=0 \tag{5.8}$$

$$\sum_{i=1}^{n}y\left(\frac{1}{t_i}\right)^{\beta}\ln\left(\frac{1}{t_i}\right)-\alpha\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}\ln\left(\frac{1}{t_i}\right)-\gamma\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2\beta}\ln\left(\frac{1}{t_i}\right)=0 \tag{5.9}$$

From the first two equations (5.7) and (5.8) we get

$$\hat{\gamma}_R=\frac{\sum_{i=1}^{n}y\left(\frac{1}{t_i}\right)^{\beta}\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2}-\sum_{i=1}^{n}\left(\frac{1}{t}\right)y\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}}{\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2\beta}\sum_{i=1}^{n}\frac{1}{t^2}-\left(\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}\right)^{2}} \tag{5.10}$$

$$\hat{\alpha}_R=\frac{\sum_{i=1}^{n}y\left(\frac{1}{t_i}\right)^{\beta}\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}-\sum_{i=1}^{n}\left(\frac{1}{t}\right)y\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2\beta}}{\left(\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{1+\beta}\right)^{2}-\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{2\beta}\sum_{i=1}^{n}\frac{1}{t^2}} \tag{5.11}$$

Substituting (5.7) and (5.8) into (5.9) we get a non-linear equation in β. By solving the obtained non-linear equation with respect to β we get $\hat{\beta}_R$. As it seems, such non-linear equation has no closed form solution in β. So, we have to use a numerical technique, such as Newton Raphson method, to solve it.

Case C: Let $T_1, T_2, ..., T_n$ be a random sample of $MIWD(\alpha, \beta, \gamma, t)$ with cdf $F_{MIW}(t)$, and suppose that $T_{(i)}$, $i = 1, 2, ..., n$ denote the ordered sample. For sample of size n, we have

$$E(F(T_{(i)})) = \frac{\text{i - 0.3}}{\text{n} + 0.4}, \qquad T_1 < T_2 < ..... < T_n$$

The rank regression and correlation method of $MIWD(\alpha, \beta, \gamma, t)$ are obtained by using the cdf $F_{MIW}(t)$, here minimum life is zero and $\alpha = 0$

$$\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right) = \ln\gamma + \beta\ln\left(\frac{1}{t}\right) \tag{5.12}$$

$$\text{Let } y = \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right), a = \ln\gamma, b = \beta, x = \ln\left(\frac{1}{t}\right)$$

$$\hat{a} = \frac{\sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)^2 - \sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\left(\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\right)}{n\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)^2 - \left(\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\right)^2} \tag{5.13}$$

$$\hat{b} = \frac{n\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right) - \sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)}{n\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)^2 - \left(\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\right)^2} \tag{5.14}$$

The correlation coefficient of $MIWD(\alpha, \beta, \gamma, t)$ by taking above assumptions

$$cc = \frac{n\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right) - \sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)}{\sqrt{\left(n\sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)^2 - \left(\sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\right)^2\right)\left(n\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)^2 - \left(\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\right)^2\right)}} \tag{5.15}$$

The standard error of estimate of $MIWD(\alpha, \beta, \gamma, t)$ by taking above assumptions

$$S_{y.x} = \sqrt{\frac{\sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)^2 - \ln\gamma\sum_{i=1}^{n}\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right) - \beta\sum_{i=1}^{n}\ln\left(\frac{1}{t}\right)\ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)}{n-k}} \tag{5.16}$$

The coefficient of determination of $MIWD(\alpha, \beta, \gamma, t)$ by taking above assumptions

$$R^2{}_{y.x} = 1 - \frac{\sum_{i=1}^{n} \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)^2 - \ln\gamma \sum_{i=1}^{n} \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right) - \beta \sum_{i=1}^{n} \ln\left(\frac{1}{t}\right) \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)}{\sum_{i=1}^{n} \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{n} \ln\left(\ln\left(\frac{1}{F_{MIW}(t)}\right)\right)\right)^2} \quad (5.17)$$

6 Order Statistics

Let $T_{1:n} \le T_{2:n} \le .... \le T_{n:n}$ be the order statistics, then the pdf of $T_{r:n}$ $(1 \le r \le n)$ is given by

$$f_{r:n}(t) = C_{r:n}(F(t))^{r-1}(1-F(t))^{n-r} f(t), \qquad t > 0 \quad (6.1)$$

The joint pdf of $T_{r:n}$ and $T_{s:n}$ $(1 \le r \le s \le n)$ is given by

$$f_{r,s:n}(t,u) = C_{r,s:n}(F(t))^{r-1}(F(u)-F(t))^{s-r-1}(1-F(t))^{n-s} f(t) f(u), \ (0 \le t \le u < \infty) \quad (6.2)$$

Where $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$ and $C_{r,s:n} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$

6.1Distribution of Minimum and Maximum

Let $t_1, t_2, ...t_n$ be n given random variables. Here we define $T_1 = Min(t_1, t_2, ...t_n)$ and $T_n = Max(t_1, t_2, ...t_n)$. We find the distributions of the modified Inverse Weibull distribution for the minimum and maximum observations Y_1 and Y_n

Theorem 6.1: Let $t_1, t_2, ...t_n$ are independently identically distributed random variables from modified Inverse Weibull distribution with fourparameters having probability density function (pdf) and cumulative distribution function is,

$$f_{1:n}(t) = n(1-F_{MIW}(t))^{n-1} f_{MIW}(t), \quad f_{n:n}(t) = n(F_{MIW}(t))^{n-1} f_{MIW}(t)$$

Proof: For the minimum and maximum order statistic of the four parameters modified Inverse Weibulldistribution $MIWD(\alpha, \beta, \gamma, \eta, t)$ pdf is given by

Case A: Minimum Order Statistics

$$f_{1:n}(t) = n\left[1-\exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right)\right]^{n-1}\left(\alpha + \beta\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t_1-\eta}\right)^2 \exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right) \quad (6.3)$$

1. The minimum order statistic of the $MIRD(\alpha, \gamma, \eta, t)$ by substituting $\beta = 2$

$$f_{1:n}(t) = n\left[1-\exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)^2\right)\right]^{n-1}\left(\alpha + 2\gamma\left(\frac{1}{t_1-\eta}\right)\right)\left(\frac{1}{t_1-\eta}\right)^2 \exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)^2\right) \quad (6.4)$$

2. The minimum order statistic of the $MIED(\alpha, \gamma, \eta, t)$ by substituting $\beta = 1$

$$f_{1:n}(t) = n\left[1-\exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)\right)\right]^{n-1}(\alpha + \gamma)\left(\frac{1}{t_1-\eta}\right)^2 \exp\left(-\frac{\alpha}{t_1-\eta} - \gamma\left(\frac{1}{t_1-\eta}\right)\right) \quad (6.5)$$

3. The minimum order statistic of the $IWD(\beta, \gamma, \eta, t)$ by substituting $\alpha = 0$

$$f_{1:n}(t) = n\left[1-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right)\right]^{n-1} \beta\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta+1} \exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right) \quad (6.6)$$

4. The minimum order statistic of the $IRD(\gamma, \eta, t)$ by substituting $\alpha = 0$, $\beta = 2$

$$f_{1:n}(t)=n\left[1-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^2\right)\right]^{n-1}2\gamma\left(\frac{1}{t_1-\eta}\right)^3\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^2\right) \quad (6.7)$$

5. The minimum order statistic of the $IED(\gamma,\eta,t)$ by substituting $\alpha=0,\ \beta=1$

$$f_{1:n}(t)=n\left[1-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)\right)\right]^{n-1}\gamma\left(\frac{1}{t_1-\eta}\right)^2\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)\right) \quad (6.8)$$

Case B: Maximum Order Statistics

$$f_{n:n}(t)=n\left[\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^\beta\right)\right]^n\left(\alpha+\beta\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t_n-\eta}\right)^2\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^\beta\right) \quad (6.9)$$

1. The maximum order statistic of the $MIRD(\alpha,\gamma,\eta,t)$ by substituting $\beta=2$

$$f_{n:n}(t)=n\left[\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^2\right)\right]^n\left(\alpha+2\gamma\left(\frac{1}{t_n-\eta}\right)\right)\left(\frac{1}{t_n-\eta}\right)^2\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^2\right) \quad (6.10)$$

2. The maximum order statistic of the $MIED(\alpha,\gamma,\eta,t)$ by substituting $\beta=1$

$$f_{n:n}(t)=n\left[\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)\right)\right]^n(\alpha+\gamma)\left(\frac{1}{t_n-\eta}\right)^2\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)\right) \quad (6.11)$$

3. The maximum order statistic of the $IWD(\beta,\gamma,\eta,t)$ by substituting $\alpha=0$

$$f_{n:n}(t)=n\left[\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^\beta\right)\right]^n\beta\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta+1}\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^\beta\right) \quad (6.12)$$

4. The maximum order statistic of the $IRD(\gamma,\eta,t)$ by substituting $\alpha=0,\ \beta=2$

$$f_{n:n}(t)=n\left[\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^2\right)\right]^n2\gamma\left(\frac{1}{t_n-\eta}\right)^3\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^2\right) \quad (6.13)$$

5. The maximum order statistic of the $IED(\gamma,\eta,t)$ by substituting $\alpha=0,\ \beta=1$

$$f_{n:n}(t)=n\left[\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)\right)\right]^n\gamma\left(\frac{1}{t_n-\eta}\right)^2\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)\right) \quad (6.14)$$

Theorem 6.2: The four parameters $MIWD(\alpha,\beta,\gamma,\eta,\tilde{t})$ modified Inverse Weibulldistribution of the median $\tilde{t}$ of is T_{m+1} is given by

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(F(\tilde{t})\right)^m(1-F(\tilde{t}))^m f(\tilde{t}) \qquad -\infty<\tilde{t}<\infty$$

Proof: For the median order statistic of the four parameters modified Inverse Weibulldistribution $MIWD(\alpha,\beta,\gamma,\eta,\tilde{t})$ pdf is given by

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^\beta\right)\right)^m\left(1-\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^\beta\right)\right)^m$$
$$\left(\left(\alpha+\beta\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{\beta-1}\right)\left(\frac{1}{\tilde{t}-\eta}\right)^2\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^\beta\right)\right) \quad (6.15)$$

We have different cases by substituting the parametric values in the above median order statistic of equation (6.15). So the derived special cases are

1. The median order statistic of the $MIRD(\alpha,\gamma,\eta,\tilde{t})$ by substituting $\beta=2$

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right)^{m}\left(1-\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right)^{m}\left(\left(\alpha+2\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right) \quad (6.16)$$

2. The median order statistic of the $MIED(\alpha,\gamma,\eta,\tilde{t})$ by substituting $\beta=1$

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right)^{m}\left(1-\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right)^{m}\left((\alpha+\gamma)\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\exp\left(-\frac{\alpha}{\tilde{t}-\eta}-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right) \quad (6.17)$$

3. The median order statistic of the $IWD(\beta,\gamma,\eta,\tilde{t})$ by substituting $\alpha=0$

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{\beta}\right)\right)^{m}\left(1-\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{\beta}\right)\right)^{m}\left(\beta\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{\beta+1}\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{\beta}\right)\right) \quad (6.18)$$

4. The median order statistic of the $IRD(\gamma,\eta,\tilde{t})$ by substituting $\alpha=0$, $\beta=2$

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right)^{m}\left(1-\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right)^{m}\left(2\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{3}\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\right)\right) \quad (6.19)$$

5. The median order statistic of the $IED(\gamma,\eta,\tilde{t})$ by substituting $\alpha=0$, $\beta=1$

$$g(\tilde{t})=\frac{(2m+1)!}{m!m!}\left(\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right)^{m}\left(1-\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right)^{m}\left(\gamma\left(\frac{1}{\tilde{t}-\eta}\right)^{2}\exp\left(-\gamma\left(\frac{1}{\tilde{t}-\eta}\right)\right)\right) \quad (6.20)$$

6.2 Joint Distribution of the rth order Statistic T_r and the sth order statistic T_S

The joint pdf of T_r and T_S with $T_r=t$ and $T_s=u$ $1\le r\le s\le n$ is given by

$$g(t,u)=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}(F(t))^{r-1}(F(u)-F(t))^{s-r-1}(1-F(t))^{n-s}f(t_1)f(t_n) \quad (6.21)$$

By taking $r=1$ and $s=n$ in (6.21), the min and max joint density can be written as

$$g(t_1,t_n)=n(n-1)(F(t_n)-F(t_1))^{n-2}f(t_1)f(t_n),\ t_1<t_n \quad (6.22)$$

Theorem 6.3: By using (6.22), the joint density function $g(t_1,t_n)$ of modified Inverse Weibulldistribution $MIWD(\alpha,\beta,\gamma,\eta,t)$ pdf is given by

Proof: The joint pdf of T_r and T_S with $T_r=t$ and $T_s=u$ $1\le r\le s\le n$ is given by

$$g(t_1,t_n)=n(n-1)\left(Exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta}\right)-Exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right)\right)^{n-2}$$
$$\left(\alpha+\beta\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t_1-\eta}\right)^{2}Exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right) \tag{6.23}$$
$$\left(\alpha+\beta\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t_n-\eta}\right)^{2}Exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta}\right),\qquad t_1<t_n$$

1. The minimum and maximum order statistic of the joint density function $g(t_1,t_n)$ of the $MIRD(\alpha,\gamma,\eta,t)$ by substituting $\beta=2$

$$g(t_1,t_n)=n(n-1)\left(\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^{2}\right)-\exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)^{2}\right)\right)^{n-2}$$
$$\left(\alpha+2\gamma\left(\frac{1}{t_1-\eta}\right)\right)\left(\frac{1}{t_1-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)^{2}\right) \tag{6.24}$$
$$\left(\alpha+2\gamma\left(\frac{1}{t_n-\eta}\right)\right)\left(\frac{1}{t_n-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)^{2}\right),\qquad t_1<t_n$$

2. The minimum and maximum order statistic of the joint density function $g(t_1,t_n)$ of the $MIED(\alpha,\gamma,\eta,t)$ by substituting $\beta=1$

$$g(t_1,t_n)=n(n-1)\left(\exp\left(-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)\right)-\exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)\right)\right)^{n-2}$$
$$(\alpha+\gamma)^2\left(\frac{1}{t_1-\eta}\right)^{2}\left(\frac{1}{t_n-\eta}\right)^{2}\exp\left(-\frac{\alpha}{t_1-\eta}-\gamma\left(\frac{1}{t_1-\eta}\right)-\frac{\alpha}{t_n-\eta}-\gamma\left(\frac{1}{t_n-\eta}\right)\right),t_1<t_n \tag{6.25}$$

3. The minimum and maximum order statistic of the joint density function $g(t_1,t_n)$ of the $IWD(\beta,\gamma,\eta,t)$ by substituting $\alpha=0$

$$g(t_1,t_n)=n(n-1)\beta^2\gamma^2\left(\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta}\right)-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}\right)\right)^{n-2}\left(\frac{1}{t_1-\eta}\right)^{\beta+1}\left(\frac{1}{t_n-\eta}\right)^{\beta+1}$$
$$\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^{\beta}-\gamma\left(\frac{1}{t_n-\eta}\right)^{\beta}\right),\qquad t_1<t_n \tag{6.26}$$

4. The minimum and maximum order statistic of the joint density function $g(t_1,t_n)$ of the $IRD(\gamma,\eta,t)$ by substituting $\alpha=0$, $\beta=2$

$$g(t_1,t_n)=n(n-1)4\gamma^2\left(\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)^{2}\right)-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)^{2}\right)\right)^{n-2}\left(\frac{1}{t_1-\eta}\right)^{3}\left(\frac{1}{t_n-\eta}\right)^{3}$$
$$\exp\left(-\gamma\left(\left(\frac{1}{t_1-\eta}\right)^{2}+\left(\frac{1}{t_n-\eta}\right)^{2}\right)\right),\qquad t_1<t_n \tag{6.27}$$

5. The minimum and maximum order statistic of the joint density function $g(t_1,t_n)$ of the $IED(\gamma,\eta,t)$ by substituting $\alpha=0$, $\beta=1$

$$g(t_1,t_n)=n(n-1)\gamma^2\left(\exp\left(-\gamma\left(\frac{1}{t_n-\eta}\right)\right)-\exp\left(-\gamma\left(\frac{1}{t_1-\eta}\right)\right)\right)^{n-2}\left(\frac{1}{t_1-\eta}\right)^2\left(\frac{1}{t_n-\eta}\right)^2 \exp\left(-\gamma\left(\left(\frac{1}{t_1-\eta}\right)+\left(\frac{1}{t_n-\eta}\right)\right)\right), \quad t_1<t_n \tag{6.28}$$

7 Maximum Likelihood Estimation of the MIWD

Consider the random samples $t_1,t_2,......t_n$ consisting of n observations when equation (2.1) of three parameter of modified Inverse Weibulldistribution $MIWD(\alpha,\beta,\gamma,t)$ pdf is taken as probability density function. The likelihood function of equation (2.1) taking $\eta=0$ is defined as

$$L(t_1,t_2,......t_n;\alpha,\beta,\gamma,\eta)=\prod_{i=1}^{n}\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)\left(\frac{1}{t}\right)^2 Exp\left(-\frac{\alpha}{t}-\gamma\left(\frac{1}{t}\right)^{\beta}\right) \tag{7.1}$$

By taking logarithm of equation (7.1), differentiating with respect to α,β,γ and equating it to zero, we obtain the estimating equations are

$$\ln L(t_1,t_2,......t_n;\alpha,\beta,\gamma,\eta)=\sum_{i=1}^{n}\ln\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)+\sum_{i=1}^{n}\left(\frac{1}{t}\right)^2-\sum_{i=1}^{n}\left(\frac{\alpha}{t}\right)-\gamma\sum_{i=1}^{n}\left(\frac{1}{t}\right)^{\beta} \tag{7.2}$$

$$\frac{\partial \ln L}{\partial \alpha}=\sum_{i=1}^{n}\frac{1}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)}-\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)=0 \tag{7.3}$$

$$\frac{\partial \ln L}{\partial \beta}=\sum_{i=}^{n}\frac{\left(\frac{1}{t}\right)^{\beta-1}\left(1+\beta\ln\left(\frac{1}{t}\right)\right)}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)}-\gamma\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}\ln\left(\frac{1}{t_i}\right)=0 \tag{7.4}$$

$$\frac{\partial \ln L}{\partial \gamma}=\sum_{i=}^{n}\frac{\beta\left(\frac{1}{t}\right)^{\beta-1}}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)}-\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}=0 \tag{7.5}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=\sum_{i=1}^{n}\frac{-1}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2}=\sum_{i=}^{n}\frac{\left(\frac{1}{t}\right)^{\beta-1}\left(1+\beta\ln\left(\frac{1}{t}\right)\right)\left(\alpha+\beta\ln\left(\frac{1}{t}\right)^{\beta-1}(\gamma-1)\right)}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)^2}-\gamma\sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}\ln^2\left(\frac{1}{t_i}\right)$$

$$\frac{\partial^2 \ln L}{\partial \gamma^2}=-\sum_{i=}^{n}\frac{\beta^2\left(\frac{1}{t}\right)^{2(\beta-1)}}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial \alpha\partial \beta}=\sum_{i=1}^{n}\frac{-\gamma\left(\frac{1}{t_i}\right)^{\beta-1}\left(\beta\ln\left(\frac{1}{t}\right)+1\right)}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\partial\gamma} = \sum_{i=1}^{n} \frac{-\beta\left(\frac{1}{t_i}\right)^{\beta-1}}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)^2}$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\gamma} = \sum_{i=1}^{n} \frac{\left(\frac{1}{t_i}\right)^{\beta-1}\left(\beta\ln\left(\frac{1}{t}\right)+1\right)(1-\beta\gamma)}{\left(\alpha+\beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)} - \sum_{i=1}^{n}\left(\frac{1}{t_i}\right)^{\beta}\ln\left(\frac{1}{t}\right)$$

By solving equations (7.3), (7.4) and (7.5) these solutions will yield the ML estimators $\hat{\alpha},\hat{\beta}$ and $\hat{\gamma}$.

7.1 Fisher Information matrix of theMIWD

Suppose T is a random variable with probability density function $f_\Theta(.)$, where $\Theta=(\theta_1,......,\theta_k)$. Then the information matrix $I(\Theta)$ the $k\times k$ symmetric matrix with elements

$$I_{IJ}(\Theta) = E_\Theta\left[\frac{\partial \log f_\Theta(t)}{\partial\theta_i}\frac{\partial \log f_\Theta(t)}{\partial\theta_j}\right] \quad (7.6)$$

If the density $f_\Theta(.)$ has second derivatives $\partial^2 \log f_\Theta(t)/\partial\theta_i\partial\theta_j$ for all i and j, Then the general expression is

$$I_{IJ}(\Theta) = -E_\Theta\left[\frac{\partial^2 \log f_\Theta(t)}{\partial\theta_i\partial\theta_j}\right]$$

For the three parameter Modified Inverse Weibulldistribution $MIWD(\alpha,\beta,\gamma,t)$ pdf all the second order derivatives are exist. Thus we have $L(t_1,t_2,......t_n;\alpha,\beta,\gamma)$, the inverse dispersion matrix is

$$V^{-1} = \left(V_{rs}^{-1}\right) = -E\begin{bmatrix} \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\alpha\partial\gamma} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L}{\partial\beta^2} & \frac{\partial^2 \ln L}{\partial\beta\partial\gamma} \\ \frac{\partial^2 \ln L}{\partial\alpha\partial\gamma} & \frac{\partial^2 \ln L}{\partial\beta\partial\gamma} & \frac{\partial^2 \ln L}{\partial\gamma^2} \end{bmatrix} \quad (7.7)$$

By solving this inverse dispersion matrix, these solutions will yield the asymptotic variance and co-variances of these ML estimators for $\hat{\alpha},\hat{\beta}$ and $\hat{\gamma}$. For the two parameter Modified Inverse Rayleigh distribution is the special type of Modified Inverse Weibull distribution when $\beta=2$ and for the two parameter Modified Inverse Exponential distribution is the special type of Modified Inverse Weibull distribution when $\beta=1$, all the second order derivatives are exist. By using (7.7), approximately $100(1-\alpha)\%$ confidence intervals for α,β,γ can be determined as

$$\hat{\alpha}\pm Z_{\alpha/2}\sqrt{\hat{V}_{11}},\quad \hat{\beta}\pm Z_{\alpha/2}\sqrt{\hat{V}_{22}} \quad \text{and } \hat{\gamma}\pm Z_{\alpha/2}\sqrt{\hat{V}_{33}} \quad (7.8)$$

Where $Z_{\alpha/2}$ is the upper αth percentile of the standard normal distribution.

8 Conclusions

In this paper we introduce the four parameter Modified Inverse Weibulldistribution and presented its theoretical properties. This distribution is very flexible reliability model that approaches to different life time distributions when its parameters are changes. From the instantaneous failure rate analysis it is observed that it has increasing and decreasing failure rate pattern for life time data.

Appendix A

The Proof of Theorem 4.1

$$\mu_r = \int_\eta^\infty t^r f(\alpha,\beta,\gamma,\eta,t)dt$$

By substituting from equation (2.1) into the above relation we have

$$\mu_r = \int_\eta^\infty t^r \left(\alpha + \beta\gamma\left(\frac{1}{t-\eta}\right)^{\beta-1}\right)\left(\frac{1}{t-\eta}\right)^2 Exp\left(-\frac{\alpha}{t-\eta} - \gamma\left(\frac{1}{t-\eta}\right)^\beta\right)dt \qquad \text{(A1)}$$

Case A: In this case $\alpha > 0, \beta > 0, \gamma > 0$ and $\eta = 0$. The exponent quantity $Exp\left(-\gamma\left(\frac{1}{t}\right)^\beta\right)$

$$Exp\left(-\gamma\left(\frac{1}{t}\right)^\beta\right) = \sum_{i=0}^\infty \frac{(-1)^i \gamma^i \left(\frac{1}{t}\right)^{i\beta}}{i!} \qquad \text{(A2)}$$

Here equation (A1) takes the following form

$$\mu_r = \sum_{i=0}^\infty \frac{(-1)^i \gamma^i}{i!} \int_0^\infty t^r \left(\alpha + \beta\gamma\left(\frac{1}{t}\right)^{\beta-1}\right)\left(\frac{1}{t}\right)^{i\beta+2} Exp\left(-\frac{\alpha}{t}\right)dt$$

$$\mu_r = \sum_{i=0}^\infty \frac{(-1)^i \gamma^i}{i!} \left(\int_0^\infty \alpha\left(\frac{1}{t}\right)^{-t+i\beta+2} Exp\left(-\frac{\alpha}{t}\right)dt + \int_0^\infty \beta\gamma\left(\frac{1}{t}\right)^{-t+i\beta+\beta+1} Exp\left(-\frac{\alpha}{t}\right)dt \right)$$

$$\mu_r = \sum_{i=0}^\infty \frac{(-1)^i \gamma^i}{i!} \left[\alpha^{r-i\beta}\Gamma(i\beta - r + 1) + \beta\gamma\alpha^{r-\beta(i+1)}\Gamma(\beta(i+1) - r)\right] \qquad \text{(A3)}$$

Case B: In the second case we assume that $\alpha = 0, \beta > 0, \gamma > 0$ and $\eta = 0$

$$\mu_r = \int_0^\infty \beta\gamma\left(\frac{1}{t}\right)^{-r+\beta+1} Exp\left(-\gamma\left(\frac{1}{t}\right)^\beta\right)dt$$

By substituting $w = \gamma\left(\frac{1}{t}\right)^\beta$ then we get

$$\mu_r = \gamma^{\frac{r}{\beta}}\Gamma\left(1 - \frac{r}{\beta}\right) = \gamma^{\frac{r}{\beta}}\xi_r, \quad \xi_r = \Gamma\left(1 - \frac{r}{\beta}\right), \quad r = 1,2,3,4 \qquad \text{(A4)}$$

Case C: In the third case we assume that $\alpha > 0, \beta = 0$ and $\eta = 0$

$$\mu_r = \int_0^\infty \alpha t^{r-2} Exp\left(-\frac{\alpha}{t}\right)dt$$

By substituting $w = \left(\frac{\alpha}{t}\right)$ then we get

$$\mu_r = \alpha^r \Gamma(1-r) = \alpha^r \omega_r \quad \omega_r = \Gamma(1-r), \qquad r = 1,2,3,4 \tag{A5}$$

The Proof of Theorem 4.2

$$M(t) = \int_{\eta}^{\infty} e^{tx} f(\alpha,\beta,\gamma,\eta,x)dx$$

By substituting from equation (2.1) into the above relation we have

$$M_{X^{-1}}(t) = \int_{\eta}^{\infty} \left(\alpha + \beta\gamma \left(\frac{1}{x-\eta}\right)^{\beta-1} \right) \left(\frac{1}{x-\eta}\right)^2 Exp\left(\frac{t}{x} - \frac{\alpha}{x-\eta} - \gamma\left(\frac{1}{x-\eta}\right)^{\beta} \right) dx \tag{A6}$$

By taking assumption that the minimum life is zero

$$M_{X^{-1}}(t) = \int_{0}^{\infty} \left(\alpha + \beta\gamma \left(\frac{1}{x}\right)^{\beta-1} \right) \left(\frac{1}{x}\right)^2 Exp\left(\frac{t}{x} - \frac{\alpha}{x} - \gamma\left(\frac{1}{x}\right)^{\beta} \right) dx$$

Case A: In this case $\alpha > 0, \beta > 0, \gamma > 0$ and $\eta = 0$. By using equation (A2), equation (A6) takes the following form

$$M_{X^{-1}}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i \gamma^i}{i!} \int_{0}^{\infty} \left(\alpha + \beta\gamma \left(\frac{1}{x}\right)^{\beta-1} \right) \left(\frac{1}{x}\right)^{i\beta+2} Exp\left(\frac{t}{x} - \frac{\alpha}{x} \right) dx$$

$$M_{X^{-1}}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i \gamma^i}{i!} \left[\frac{\alpha\Gamma(i\beta+1)}{(\alpha-t)^{i\beta+1}} + \frac{\beta\gamma\Gamma(\beta(i+1))}{(\alpha-t)^{\beta(i+1)}} \right] \tag{A7}$$

Case B: In this case $\alpha = 0, \beta > 0, \gamma > 0$ and $\eta = 0$. By using equation (A2), equation (A6) takes the following form

$$M_X(t) = \int_{0}^{\infty} \beta\gamma \left(\frac{1}{x}\right)^{\beta+1} Exp\left(tx - \gamma\left(\frac{1}{x}\right)^{\beta} \right) dx$$

$$M_X(t) = \sum_{i=0}^{\infty} \frac{t^i \gamma^{\frac{i}{\beta}}}{i!} \Gamma\left(1 - \frac{i}{\beta}\right) \tag{A8}$$

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