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Treatment of Operator Algebras through Cohomology Theory

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Abstract: In this paper, we study invariant properties of the dihedral cohomology of operator algebras. We focus on the Banach algebras with an involution as an example of operator algebra. It is shown that, important observations can be obtained from our results on discussing the relations between the dihedral and cyclic cohomology of operator algebras.

Keywords: Banach algebra, Cyclic cohomology, C*- algebras, dihedral cohomology.

1 Introduction

(Co)Homology theory is a main tool in algebraic topology. It`s plays an important role in topology, algebra and many branches of mathematics. The (co)homology theory are related with many subjects as discrete algebra (pure algebra), and indiscrete algebra (operator algebra). There are many styles of homology and cohomology groups as: Hochschild, cyclic, reflexive, dihedral, symmetry, bisymmetry and Weil (octahedral), following [9] and [8]. Recently the entire relative cyclic cohomology of Banach algebra have been designed by tell in [2].

Our central interest in dihedral (co)homology group, which can be obtained, by acting on the simplicial complex by the dihedral group of order $2(n+1)$ (see [9]). The dihedral homologies of polynomial algebra and Banach algebra have been studied in ([4],[5]). The nontrivial dihedral cohomology groups of Banach algebra have been planned in [6].

Firstly, we remember some cohomology groups, simplicial, cyclic and reflexive that related to dihedral cohomology of algebra. Using the actuality that the cyclic cohomology group is a subgroup of dihedral cohomology group. We get the long exact sequences of Connes-Tsygan for cyclic cohomology of the Banach algebra with unity [7],[9] as two exact sequences. We obtained the long exact sequences of Connes-Tsygan of the dihedral cohomology with Helemckii conditions. At last, we compute the dihedral cohomology of C*-algebras.

2 Simplicial Complexes in the Category of Banach Spaces

By the Banach complex we mean the family $X = \{X_n\}_{n\in\mathbb{Z}}$ of the Banach spaces X_n and the continuous operators; $d = \{d_n\}$, $d_n: X_n \to X_{n-1}$, called differentials and satisfying the relations $d_n \circ d_n = 0$. Consider Banach complexes $X = \{X_n\}$ for which $X_p = 0$ for $n < 0$ or $X_n = 0$ for $n > 0$.

A Banach complex X will be admissible if there is a family continuous operators $s = \{s_n\}$, $s_n: X_{n-1} \to X_n$, and satisfy the relations $d_n o\ s_n\ o\ d_n = d_n$.

From works in [1] and [4], the spaces $Z_n(X) = Ker\{d_n: X_n \to X_{n-1}\}$ are called spaces of n-dimensional cycles. The spaces $B_n(X) = Im\{d_{n+1}: X_{n+1} \to X_n\}$ are called spaces of n-dimensional boundaries. It is clear that $B_n(X) \subset Z_n(X)$.

The quotient spaces $H_n(X) = Z_n(X)/B_n(X)$ is the n-dimensional homology of X. The family $H^*(X) = \{H_n(X)\}$ is the homology of a Banach complex X. If X is an admissible Banach complex, then the map $d_{n+1}\ o\ s_n: Z_n(X) \to B_n(X)$ will be the inverse of the embedding $in: B_n(X) \to Z_n(X)$. Therefore, the homology $H^*(X)$ in this case will consist of Banach spaces $H_n(X)$. Short exact sequences

$$0 \to B_n(X) \overset{i_n}{\to} Z_n(X) \overset{p_n}{\to} H_n(X) \to 0 \qquad (1)$$

split,

i.e. there exist continuous operators $q_n: H_n(X) \to Z_n(X)$, for which the relations

$$p_n \circ q_n = Id: H_n(X) \to H_n(X),$$
$$Id - q_n \circ p_n = d_{n+1} \circ s_n: Z_n(X) \to B_n(X)$$

For the Banach complex X, the notation $\overline{X}$ is the dual complex, $\overline{X} = \{\overline{X}_n\}$ for which $\overline{X}_n$ is a Banach space conjugate to X_n. The differentials $d_n: X_n \to X_{n-1}$ induce the differentials $\overline{d}_n: \overline{X}_{-n+1} \to \overline{X}_{-n}$. Clear, that if X was an admissible Banach complex, then $\overline{X}$ would also be an

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admissible Banach complex. The homology of the dual complex $\overline{X}$ is the cohomology of X and given by:

$$H^{*}(X) = \{H^{n}(X)\}, \; H^{n}(X) = H_{-n}(\overline{X}). \qquad (2)$$

By a mapping of Banach complexes $f: X \to Y$ of dimension m we mean the family $f = \{f_n\}$ of continuous operators $f_n: X_n \to Y_{n+m}$. Such mappings form a linear space denoted by $Hom_m(X, Y)$. We define the linear operators $d: Hom_m(X,Y) \to Hom_{m-1}(X,Y)$, called the differentials, getting

$$d(f)_n = d_{n+m} \circ f_n + (-1)^m f_{n-1} \circ d_n : X_n \to Y_{n+m-1}$$

Such that $d \circ d = 0$, and thus the family $Hom\,(X,Y) = \{Hom_m\,(X,Y)\}$ forms a complex of the category of linear spaces.

The chain map of Banach complexes is called the map $f: X \to Y$ of dimension zero, satisfying the relation $d(f) = 0$, or same thing, $d_n \circ f_n = f_{n-1} \circ d_n$.

The chain map of Banach complexes $f: X \to Y$ induces a map homology

$$H_*(f) = \{H_n(f)\}: H_*(Y) \to H_*(X),$$
$$H_n(f): H_n(Y) \to H_n(X)$$

And the cohomology

$$H^*(f) = \{H^n(f)\}: H^*(X) \to H^*(Y),$$
$$H^n(f): H^n(X) \to H^n(Y) \qquad (3)$$

Two chain mappings $f, g: X \to Y$ Banach complexes are called homotopic ($f \cong g$) if there is a mapping $h: X \to Y$ of dimension 1 such that $d(h) = g - f$, or something same thing $d_{n+1} \circ h_n + h_{n-1} \circ d_n = g_n - f_n$. The mapping h is called the homotopy between the mappings $f \circ g$ and the homotopy ratio is an equivalence relation.

Banach complexes X and Y are said to be homotopy equivalent ($X \cong Y$) if there exist chain mappings $f: X \to Y$, $g: Y \to X$ such that $g \circ f \cong Id_X$, $f \circ g \cong Id_Y$. If the Banach complex is homotopy equivalent, then it is called contractible.

It is easy to see that if $f \cong g: X \to Y$, to $\overline{f} \cong \overline{g}: \overline{X} \to \overline{Y}$ and if $X \cong Y$, then $\overline{X} \cong \overline{Y}$. In addition, homotopic mappings induce the same homomorphisms of homology and cohomology, while homotopy equivalent Banach complexes have isomorphic homology and cohomology.

2.1 Proposition

Consider the admissible Banach complex X, then its homology $H_*(X)$, regarded as the Banach complex with zero differential, will be homotopy equivalent to $X: H_*(X) \cong X$

Indeed, we consider the map $Id - s \circ d: X \to Z\,(X)$. We denote its composition with the projection $p: Z\,(X) \to H_*(X)$, $\eta: X \to H_*(X)$. Reviewed above, the map $q: H_*(X) \to Z\,(X)$ gives a map $\xi: H_*(X) \to X$. It is easy to see that $\eta \circ \xi = Id: H_*\,(X) \to H_*(X)$. We show that the maps $s_n: X_n \to X_{n+1}$ give the required homotopy between the mappings Id and $\eta \circ \xi$ we have

$$Id - \xi_n \circ \eta_n =$$
$$Id - q_n \circ p_n(Id - s_n \circ d_n) = d_{n+1} \circ s_n + s_n \circ d_n.$$

From this proposal we obtain the fact;

2.2 Corollary [3]

For an admissible Banach complex X, the isomorphism $H^*(X) \cong \overline{H_*(X)}$.

Indeed, the homotopy equivalence $X \cong H_*(X)$ implies the homotopy equivalence $\overline{X} \cong \overline{H_*(X)}$, and therefore we have the indicated isomorphism.

Let X be a Banach complex. We define a Banach complex SX called superstructure over X, setting $(SX)_{n+1} = X_n$. We denote the elements in SX by $[x]$, where $x \in X$. If $x \in X$ has dimension n, then $[x]$ has dimension $n+1$. We define the differential on SX by the formula $[x] = -[d\,(x)]$.

It is clear that if X is an admissible Banach complex, then the superstructure SX is an admissible Banach complex, the homology of the superstructure is isomorphic to the superstructure over homology, i.e. $H_*(SX) = SH_*(H)$. Similarly, for a Banach complex X, we can define a structure $S^{-1}X$, setting $(S^{-1}\,X)_{n-1} = X_n$. Items in the denast when it will not cause confusion, we will denote $[x]$, where $x \in X$. If $x \in X$ has dimension n, then $[x]$ will have dimension $n-1$. The differential on SX define by $d[x] = -[d(x)]$.

The tensor product of the Banach complexes $X^{`}$ and $X^{``}$ is called the Banach the complex $X = X^{`} \otimes X^{``}$ for which

$$(X^{`} \otimes X^{``})_n = \sum_{p+q=n} X^{`}_p \otimes X^{``}_q,$$

the differential is defined as;

$$d_n\big(x^{`}_p \otimes x^{``}_q\big) = d^{`}_p\big(x^{`}_p\big) \otimes x^{``}_q + (-1)^p x^{`}_p \otimes d^{``}_q\big(x^{``}_q\big),$$
$$p + q = n$$

For the maps $f^{`}: X^{`} \to Y^{`}$, $f^{``}: X^{``} \to Y^{``}$ of dimension $m^{`}$ and $m^{``}$, respectively, we define the map $f = f^{`} \otimes f^{``}: X^{`} \otimes X^{``} \to Y^{`} \otimes Y^{``}$ of dimension $m = m^{`} + m^{``}$, putting on generators $x^{`}_p \otimes x^{``}_q \in X^{`}_p \otimes X^{``}_q$ **:**

$$(f^{`} \otimes f^{``})\big(x^{`}_p \otimes x^{``}_q\big) = (-1)^{m^{``}p} f^{`}_p\big(x^{`}_p\big) \otimes f^{``}_q\big(x^{``}_q\big)$$

The formula for the differential in the tensor product of Banach complexes can be rewritten as;

$$d = d^{`} \otimes 1 + 1 \otimes d^{``}.$$

2.3 Corollary

The tensor product of an admissible Banach complexes is an admissible Banach complex.

Indeed, let $(X^{`}, d^{`}, s^{`})$, $(X^{``}, d^{``}, s^{``})$ be admissible Banach complexes. We define $s: X^{`} \otimes X^{``} \to X^{`} \otimes X^{``}$ by setting

$$s = s^{`} \otimes 1 + 1 \otimes s^{``} - (d^{`} \circ s^{`} + s^{`} \circ d^{`}) \otimes s^{``}.$$

Direct calculations show that the required relation $d \circ s \circ d = d$. If the maps $f^{`}: X^{`} \to Y^{`}$, $f^{``}: X^{``} \to Y^{``}$ are respectively homotopic to the maps $g^{`}: X^{`} \to Y^{`}$, $g^{``}: X^{``} \to Y^{``}$ then $f^{`} \otimes f^{``}: X^{`} \otimes X^{``} \to Y^{`} \otimes Y^{``}$ will be homotopic to the mapping $g^{`} \otimes g^{``}: X^{`} \otimes X^{``} \to Y^{`} \otimes Y^{``}$. Required homotopy $h: X^{`} \otimes X^{``} \to Y^{`} \otimes Y^{``}$ is defined as $h = h^{`} \otimes g^{``} + f^{`} \otimes h^{``}$, where $h^{`}, h^{``}$ corresponding homotopies between $f^{`}, g^{`}$ and $f^{`}, f^{``}$.

From this it follows that homotopically equivalent chain complexes $X^{`} \cong Y^{`}$, $X^{``} \cong Y^{``}$ give homotopically equivalent tensor products $X^{`} \otimes X^{``} \cong Y^{`} \otimes Y^{``}$.

From Proposition *1* we get the following fact.

2.4 Corollary

For admissible Banach complexes X, Y, we have an isomorphism $H_*(X \otimes Y) = H_*(X) \otimes H_*(Y)$

3 (Co) Homology of Banach Algebras

In this section we review a few definitions and actualities required in continuation. Let X be an involutive Banach algebra with unity and $C^n(X)$ $(n \geq 0)$ is the Banach space of *(n+1)*-linear maps f on X which are called cochains. From ([1], [3]) and [9] we generate the following assertions. Consider the complex $C(X) = (C^n(X), b^n)$such that:

$$0 \to C^0(X) \xrightarrow{b^0} C^1(X) \to \ldots \to C^n(X) \xrightarrow{b^n} C^{n+1}(X) \to \ldots, \quad (4)$$

where an operator b^n is defined by the form,

$$b^n f(x_0, \ldots, x_n) = \sum_{i=1}^{n=1} (-1)^i f(x_0, \ldots, x_i x_{i+1}, \ldots, x_n) + (-1)^n f\ (x_n x_0, x_1, \ldots, x_{n-1}).$$

With the condition $b^{n+1} b^n = 0$*(*$Ker b^n \supset Im\, b^{n+1}$*)*. The group $H^n(X) = Ker b^n / Im\, b^{n+1}$ is called the simplicial cohomology of algebra *X*.

Note that;$(Ker b^n)$ is always closed but $(Im\, b^{n+1})$ is not closed. Let the action of the operators $t_n, r_n: C^n(X) \to C^n(X)$ on $C(X)$, since:

$$t_n f(x_0, \ldots, x_{n-1}, x_n) = (-1)^n f(x_n, x_0, \ldots, x_{n-1}), \quad (5)$$

$$r_n f(x_0, \ldots, x_{n-1}, x_n) = \varepsilon(-1)^{\frac{n(n+1)}{2}} f(x_0^*, x_n^*, \ldots, x_1^*), \quad (6)$$

Where x_n^* is the image of x_n, $t_n^{n+1} = 1$, $r_n^2 = 1$, $t_n r_n = r_n t_n^{n-1}$, $\varepsilon = \pm 1$.

Note that, the operators t_n, r_n generate $D_{2(n+1)}$ which is the dihedral group and if we dropped the family of generators r_n, the operators t_n generates the cyclic group $\frac{\mathbb{Z}}{n+1}$ of order *n+1*. The cochains that satisfy the relation (5) are called cyclic and the cochains that satisfy the relation (6) are called reflexive, if they satisfy them together, they called dihedral cochains. The cyclic, reflexive and dihedral cochains by the operator b^n are denoted by $CC^n(A)$, ${}^{\varepsilon}CR^n(A)$ and ${}^{\varepsilon}CD^n(A)$, respectively [7], [6] and the subcomplexes are:

$$\left.\begin{array}{l} CC(X): 0 \to CC^o(X) \xrightarrow{bC^o} CC^1(X), \ldots, CC^n(X) \xrightarrow{bC^n} CC^{n+1}(X) \to, \ldots, \\ CR(X): 0 \to {}^{\varepsilon}CR^o(X) \xrightarrow{bR^o} {}^{\varepsilon}CR^1(X), \ldots, {}^{\varepsilon}CR^n(X) \xrightarrow{bR^n} {}^{\varepsilon}CR^{n+1}(X) \to, \ldots, \\ CD(X): 0 \to {}^{\varepsilon}CD^o(X) \xrightarrow{bD^o} {}^{\varepsilon}CD^1(X), \ldots, {}^{\varepsilon}CD^n(X) \xrightarrow{bD^n} {}^{\varepsilon}CD^{n+1}(X) \to, \ldots, \end{array}\right\} \quad (7)$$

where

$CC^n(X) = \{f \in C^n(X); f = (-1)^n t_n f\}$, ,

${}^{\varepsilon}CR^n(X) = \left\{f \in C^n(X); f = \alpha(-1)^{\frac{n(n+1)}{2}} r_n f\right\}$,

${}^{\varepsilon}CD^n(X) = \{f \in C^n(X); f = (-1)^n t_n f,$

$f = \varepsilon(-1)^{\frac{n(n+1)}{2}} r_n f\}, \varepsilon = \pm 1.$

Let $CC^n(X)$, ${}^{\varepsilon}CR^n(X)$ and ${}^{\varepsilon}CD^n(X)$ be the cyclic, reflexive and dihedral complexes and the cyclic, reflexive and dihedral cohomology groups are denoted by: $HC^n(X)$, ${}^{\varepsilon}HR^n(X)$ and ${}^{\varepsilon}CD^n(X)$, respectively. the following assertion gives the relation between the cyclic and dihedral cohomologies of Banach algebra.

3.1 Theorem

The isomorphism, $HC^n(X) \approx {}^{-}HD^n(X) \oplus {}^{+}HD^n(X)$ is hold.

4 Connes-Tsygan Exact Sequence

For the dihedral cohomology of involutive Banach algebra, we get the exact Helemskii sequence:

$$0 \to CC(X) \xrightarrow{I} C(X) \xrightarrow{N} \tilde{C}(X) \xrightarrow{M} CC(X) \to 0, \quad (8)$$

to get the Connes-Tsygan long exact sequence. To show that, consider Helemskii sequence:

$$0 \to CC(X) \xrightarrow{I} C(X) \xrightarrow{N} \tilde{C}(X) \xrightarrow{M} CC(X) \to 0,$$

Where $C(X), CC(X)$ is defined above and $\tilde{C}(X) = (C^n(X), b'^n)$,

$$b^n f(x_o, x_1, \ldots, x_{n-1}) = b'^n f(x_o, x_1, \ldots, x_{n-1}) + (-1)^n f(x_n x_o, x_1, \ldots, x_{n-1}),$$

$N = 1 - t_n$, $M = 1 + t_n + t_n^2 + \ldots + t_n^n$, and I is obvious inclusion.

4.1 Lemma

The sequence

$$0 \to CC(X) \xrightarrow{I} C(X) \xrightarrow{N} \tilde{C}(X) \xrightarrow{M} CC(X) \to 0$$

is exact.

Proof.

We shall only show that the sequence (8) is exact on *C(X)*, that is $Ker M \supset Im\, N$, Since

$$MN = M(1 - t_n) = (1 + t_n + t_n^2 + \ldots + t_n^n)(1 - t_n) = 0.$$

It remains to show that if $x \in Ker M$ then there exist $y \in C(X)$, such that $My = x$. Suppose $y_n = (1 + t_n + t_n^2 + \ldots + t_n^n)x$, $t_n y_n = (t_n + t_n^2 + \ldots + t_n^n)x$ then $y_n - t_n y_n = (1 - t_n^n)x$. Let $y = y_1 + y_2 + \ldots + y_n$, then $y + t_n\, y = (n+1)x = (1 - t_n)y$.

Hence,

$y = \frac{1}{n+1}\{n + (n-1)t_n + (n-2)t_n^2 + \cdots + 2t_n^{n-2} + t_n^{n-1}\} \in C(A)$.

So $Ker M \subset Im\, N$. The lemma is proved.

4.2 Proposition

The sequence (8) instigates the accompanying commutative diagram:

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
CC(X) & \xrightarrow{\alpha} & {}^{+}CD(X) \oplus {}^{-}CD(X) \\
\downarrow & \downarrow & \downarrow \\
C(X) & \xrightarrow{\alpha} & {}^{+}CR(X) \oplus {}^{-}CR(X) \\
\downarrow & \downarrow & \downarrow \\
\tilde{C}(X) & \xrightarrow{\alpha'} & {}^{-}C\tilde{R}(X) \oplus {}^{+}C\tilde{R}(X) \\
\downarrow & \downarrow & \downarrow \\
CC(A) & \xrightarrow{\alpha''} & {}^{-}CD(A) \oplus {}^{+}CD(A) \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array} \qquad (9)
$$

Where

$$\alpha(f) = \frac{1}{2}\big((1+r)(f), (1-r)(f)\big), f \in CC(X),$$
$$\alpha'(f) = \frac{1}{2}\big((1-rt)(f), (1+rt)(f)\big), f \in \tilde{C}(X),$$
$$\alpha''(f) = \frac{1}{2}\big((1-r)(f), (1+r)(f)\big), f \in CC(X),$$

4.3 Proposition

For the exact sequences:

$$\left.\begin{array}{l}
0 \to {}^{+}CD\ (X) \xrightarrow{I} {}^{-}CR(X) \xrightarrow{N} {}^{-}C\tilde{R}(X) \xrightarrow{M} {}^{-}CD(X) \to 0 \\
0 \to {}^{-}CD\ (X) \xrightarrow{I} {}^{-}CR(X) \xrightarrow{N} {}^{+}C\tilde{R}(X) \xrightarrow{M} {}^{+}CD(X) \to 0
\end{array}\right\} \quad (10)$$

Such that, in (10), for all $x \in KerM$ which invariant with respect to rt, since $rt(x) = \varepsilon x$, then $\exists\, y \in {}^{\varepsilon}CR(X)$ $s.h.$ $ry = y$.

By sequence (10); consider the short exact sequence:

$$0 \to {}^{\varepsilon}CD\ (X) \xrightarrow{I} {}^{\varepsilon}CR(X) \xrightarrow{N} {}^{-\varepsilon}CSR(X) \to 0$$
$$0 \to {}^{-\varepsilon}CSR(X) \xrightarrow{N} {}^{-\varepsilon}C\tilde{R}(X) \xrightarrow{M} {}^{\varepsilon}CD(X) \to 0$$

Where $\varepsilon = \pm 1$, ${}^{\varepsilon}CSR(X) = Ker\ M$. Then we get two long exact sequences of group cohomologies as:

$$\left.\begin{array}{l}
{}^{\varepsilon}HD^{o}(X) \to {}^{\varepsilon}HR^{o}(X) \to {}^{-\varepsilon}HSR^{o}(X) \xrightarrow{\xi_{\varepsilon}^{o}} {}^{\alpha\varepsilon}HD^{1}(X) \to {}^{-\varepsilon}HR^{1}(X) \to \ldots \to \\
\to {}^{\varepsilon}HD^{n}(X) \to {}^{\varepsilon}HR^{n}(X) \to {}^{-\varepsilon}HSR^{n}(X) \xrightarrow{\xi_{\varepsilon}^{n}} {}^{\varepsilon}HD^{n+1}(X) \to {}^{\varepsilon}HR^{n+1}(X) \to \ldots
\end{array}\right\} (I)$$

$$\left.\begin{array}{l}
{}^{\varepsilon}HSR^{o}(X) \to {}^{\varepsilon}H\tilde{R}^{o}(X) \to {}^{\varepsilon}HD^{o}(X) \xrightarrow{\eta_{\varepsilon}^{o}} {}^{\varepsilon}HSR^{1}(X) \to \ldots \to \\
\to {}^{\varepsilon}H\tilde{R}^{n-1}(X) \to {}^{\varepsilon}HD^{n-1}(X) \xrightarrow{\eta_{\varepsilon}^{n-1}} {}^{\varepsilon}HSR^{n}(X) \to {}^{\varepsilon}H\tilde{R}^{n}(X) \to \ldots
\end{array}\right\} (II)$$

Suppose that, the connection maps η_{ε}^{n} $(n \geq 0)$ in (II) are the topological vector space isomorphism. Then from (I) we have:

$$\ldots \to {}^{\varepsilon}HD^{n}(X) \to {}^{\varepsilon}HR^{n}(X) \xrightarrow{\phi_{\varepsilon}^{n}} {}^{\varepsilon}HD^{n-1}(X) \xrightarrow{\beta_{\varepsilon}^{n}}$$
$$\xrightarrow{\beta_{\varepsilon}^{n}} {}^{\varepsilon}HD^{n+1}(X) \to {}^{-\varepsilon}HR^{n+1}(X) \xrightarrow{\phi_{\varepsilon}^{n+1}} {}^{\varepsilon}HD^{n}(X) \to \ldots, \qquad (11)$$

Where $\phi_{\varepsilon}^{n} = (\xi_{\varepsilon}^{n-1})^{-1} \circ H^{n}(N)$, $\beta_{\varepsilon}^{n} = \xi_{\varepsilon}^{n} \circ \eta_{\varepsilon}^{n-1}$.

A sequence (11) is the dihedral cohomology of Connes–Tsygan long exact sequence.

4.4 Theorem

For the C*-algebra X, the following are equivalent:

1) The maps η_{ε}^{n}, $n \geq 0$, $\varepsilon = \pm 1$ are isomorphism in topological vector.

2) The complex ${}^{\varepsilon}CR(X)$,$\varepsilon = \pm$ is exact.

3) $\big(Ext_{X}^{n}(C, X^{*})\big) = 0, n \geq 0, \varepsilon = \pm,$ X^{*} is a dual of X (X^{*} is X-bimodule $\mathbb{C}$ is a complex number set).

4.5 Theorem

The exact sequence of Connes–Tsygan of the dihedral cohomology is exist if:

1) For all unital C^{*}–algebra with right or left bounded approximate unit.

2) If $X = X^{2}$, X is a flat or right X-module and C is a flat left X-module, since C is the set of complex number.

Consider the exact sequence of Connes–Tsygan and the relation that relates the dihedral and cyclic cohomologies of operator algebras. Then we get the commutative diagram:

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{\varepsilon}HD^{n}(X) & \to & {}^{\varepsilon}HR^{n}(X) & \to & {}^{-\varepsilon}HD^{n-1}(X) & \to & {}^{\varepsilon}HD^{n+1}(X) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & HC^{n}(X) & \to & {}^{\varepsilon}H^{n}(X) & \to & HC^{n-1}(X) & \to & HC^{n+1}(X) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{-\varepsilon}HD^{n}(X) & \to & {}^{-\varepsilon}HR^{n}(X) & \to & {}^{\varepsilon}HD^{n-1}(X) & \to & {}^{-\varepsilon}HD^{n+1}(X) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array} \qquad (12)
$$

4.6 Theorem

Consider the Hilbert space h and the set of all bounded operators algebra $X = L(\mathrm{H})$ on H. Then we get

$${}^{\varepsilon}HD^{n}(X) = 0. \quad \varepsilon = \pm 1, \ n > 0.$$

Proof:

Consider the C^{*}-*algebra* X and it has no bounded traces. Then $H^{0}(X) = HC^{n}(X) = 0$. As Following [9]; $HC^{n}(X) = 0, \ \forall n \geq 0$. Let these facts in (11) we have:

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{\varepsilon}HD^{0}(X) & \to & {}^{\varepsilon}HR^{0}(X) & \to \ldots \to & {}^{-\varepsilon}HD^{n-1}(X) & \to & {}^{\varepsilon}HD^{n+1}(X) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & 0 & \to & 0 & \to \ldots \to & 0 & \to & 0 & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\ldots \to & {}^{-\varepsilon}HD^{0}(X) & \to & {}^{-\varepsilon}HR^{0}(X) & \to \ldots \to & {}^{\varepsilon}HD^{n-1}(X) & \to & {}^{-\varepsilon}HD^{n+1}(X) & \to \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array} \qquad (13)
$$

then ${}^{\varepsilon}HD^{n}(X) = 0. \quad \varepsilon = \pm 1, \ n > 0$.

We get the dihedral cohomology from the diagram (10) and from the well-known cyclic and Hochschild cohomology.

4.7 Theorem

Let X is arbitrary biflat algebra (nuclear C*-algebra). Then

1. ${}^{\varepsilon}HD^{n}(X) = 0. \quad \varepsilon = \pm 1, \ n > 0$, n is odd.

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${}^{\varepsilon}HD^n(X) = 0.$ ${}_{\varepsilon}X^{tr}$, $\varepsilon = \pm 1$, $n > 0$, n is even, since X^{tr} is the continuous trace set on X and ${}_{\varepsilon}X^{tr} = \varepsilon X^{tr}$.

4.8 Theorem

If the C*-algebra X is stable, then

$${}^{\varepsilon}HD^n(X) = 0, \ \varepsilon = \pm 1, \quad n \geq 0.$$

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