ℓ- Volterra Quadratic Stochastic Operators: Lyapunov Functions, Trajectories

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Abstract: We consider ℓ-Volterra quadratic stochastic operators defined on $(m-1)$-dimensional simplex, where $\ell \in \{0, 1, ..., m\}$. Under some conditions on coefficients of such operators we describe Lyapunov functions and apply them to obtain upper estimates for the set of ω- limit points of trajectories. We describe a set of fixed points of ℓ-Volterra operators.

Keywords: Quadratic stochastic operator, fixed point, trajectory, Volterra and non-Volterra operators, simplex.

1. Introduction

A quadratic stochastic operator (QSO) has meaning of a population evolution operator (see [6], [7], [8]), which arises as follows. Consider a population consisting of m species. Let $x^0 = (x_1^0, ..., x_m^0)$ be the probability distribution of species in the initial generations, and $P_{ij,k}$ the probability that individuals in the ith and jth species interbreed to produce an individual k. Then the probability distribution $x' = (x_1', ..., x_m')$ (the state) of the species in the first generation can be found by the total probability i.e.

$$x_k' = \sum_{i,j=1}^{m} P_{ij,k} x_i^0 x_j^0, \quad k = 1, ..., m. \qquad (1)$$

This means that the association $x^0 \to x'$ defines a map V called the evolution operator. The population evolves by starting from an arbitrary state x^0, then passing to the state $x' = V(x^0)$ (in the next "generation"), then to the state $x'' = V(V(x^0))$, and so on. Thus states of the population described by the following dynamical system

$$x^0, \; x' = V(x^0), \; x'' = V^2(x^0), \; x''' = V^3(x^0), ...$$

Note that V defined by (1) is a non linear (quadratic) operator, and it is higher dimensional if $m \geq 3$. Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood ([1], [2], [10]).

In [12], [13] we considered a class of nonlinear (quadratic) operators which is called ℓ-Volterra operators and the difference of ℓ- Volterra quadratic operators from known quadratic operators are discussed. Some invariant (in particular some fixed points) sets for ℓ-Volterra operators are described. Also we described a family of ℓ-Volterra operators each element of which has cyclic orbits generated by several vertices of the simplex. It is shown that the set of all ℓ-Volterra operators is convex, compact and its extremal points are constructed. For 1-Volterra operators and 2-Volterra operators defined on a two dimensional simplex the limit behavior of all trajectories (orbits) are studied.

In this paper we continue the investigations of ℓ-Volterra quadratic operators. Under some conditions on coefficients of such operators we describe Lyapunov functions and apply them to obtain upper estimates for the set of ω- limit points of trajectories. We describe a set of fixed points of the ℓ-Volterra operators. This paper also contains many remarks with comparisons of ℓ-Volterra operators and Volterra ones.

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2. Definitions

The quadratic stochastic operator (QSO) is a mapping of the simplex.

$$S^{m-1} = \left\{ x \in \mathbf{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\} \qquad (2)$$

into itself, of the form

$$V : x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \;\; k = 1, ..., m, \qquad (3)$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k} \geq 0, \;\; P_{ij,k} = P_{ji,k}, \;\; \sum_{k=1}^{m} P_{ij,k} = 1, \qquad (4)$$

where $i, j, k = 1, ..., m$. Thus each quadratic stochastic operator V can be uniquely defined by a cubic matrix $\mathbf{P} = (P_{ij,k})_{i,j,k=1}^{m}$ with conditions (4). Note that each element $x \in S^{m-1}$ is a probability distribution on $E = \{1, ..., m\}$. The population evolves by starting from an arbitrary state (probability distribution on E) $x \in S^{m-1}$ then passing to the state $V(x)$ (in the next "generation"), then to the state $V(V(x)) = V^2(x)$, and so on.

For a given $x^{(0)} \in S^{m-1}$ the trajectory (orbit)

$$\{x^{(n)}\}, \;\; n = 0, 1, 2, ... \text{ of } x^{(0)}$$

under the action of QSO (3) is defined by

$$x^{(n+1)} = V(x^{(n)}), \;\; \text{where} \;\; n = 0, 1, 2, ...$$

One of the main problem in mathematical biology consists in the study of the asymptotical behavior of the trajectories. The difficulty of the problem depends on given matrix $\mathbf{P}$. For the brief history of (particularly) studied QSOs see [11], [13].

The Volterra operators. (see [3]-[5]) A Volterra QSO is defined by (3), (4) and the additional assumption

$$P_{ij,k} = 0, \;\; \text{if} \;\; k \notin \{i, j\}, \;\; \forall i, j, k \in E. \qquad (5)$$

The biological treatment of condition (5) is clear: The offspring repeats the genotype of one of its parents. In [3] the general form of Volterra QSO

$$V : x = (x_1, ..., x_m) \in S^{m-1} \rightarrow$$
$$V(x) = x' = (x'_1, ..., x'_m) \in S^{m-1}$$

is given

$$x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \qquad (6)$$

where

$$a_{ki} = 2P_{ik,k} - 1 \;\text{for}\; i \neq k \;\text{and}\; a_{ii} = 0, i \in E.$$

Moreover

$$a_{ki} = -a_{ik} \;\; \text{and} \;\; |a_{ki}| \leq 1.$$

In [3],[4] the theory of QSO (6) was developed by using theory of the Lyapunov function and tournaments. But non-Volterra QSOs (i.e. which do not satisfy the condition (5)) were not in completely studied. Because there is no any general theory which can be applied for investigation of non-Volterra operators.

In this paper we consider the following class of non-Volterra operators.

ℓ-Volterra QSO. Fix $\ell \in E$ and assume that elements $P_{ij,k}$ of the matrix $\mathbf{P}$ satisfy

$$P_{ij,k} = 0 \;\; \text{if} \;\; k \notin \{i, j\} \;\; \text{for any} \;\; k = 1, ..., \ell; \;\; i, j \in E; (7)$$

$$P_{ij,k} > 0 \;\; \text{for at least one pair} \;\; (i, j), \;\; i \neq k, \;\; j \neq k \qquad (8)$$

for any $k \in \{\ell + 1, ..., m\}$.

Definition 1. *For any fixed $\ell \in E$, the QSO defined by (3), (4), (7) and (8) is called ℓ-Volterra QSO.*

Denote by $\mathcal{V}_\ell$ the set of all ℓ-Volterra QSOs.

Remark. 1. The biological treatment of the condition (7) is that only first ℓ offsprings repeat the genotype of one of their parents.

2. The condition (8) guarantees that $\mathcal{V}_{\ell_1} \bigcap \mathcal{V}_{\ell_2} = \emptyset$ for any $\ell_1 \neq \ell_2$.

3. Note that ℓ-Volterra QSO is Volterra if and only if $\ell = m$.

4. The class of ℓ-Volterra QSO for a given ℓ does not coincide with a class of non-Volterra QSOs mentioned in [11], [13].

3. Lyapunov functions of ℓ-Volterra QSO

Let $k \in \{1, ..., \ell\}$ then $P_{kk,i} = 0$ for $i \leq \ell, \; i \neq k$ and

$$\sum_{i=1}^{m} P_{kk,i} = P_{kk,k} + \sum_{i=\ell+1}^{m} P_{kk,i} = 1.$$

Using $P_{ij,k} = P_{ji,k}$ we get for $k = 1, ..., \ell$

$$x'_k = x_k \left(P_{kk,k} x_k + 2 \sum_{\substack{i=1 \\ i \neq k}}^{m} P_{ik,k} x_i \right) =$$

$$x_k \left(1 + (P_{kk,k} - 1) x_k + \sum_{\substack{i=1 \\ i \neq k}}^{m} (2P_{ik,k} - 1) x_i \right).$$

Denote $a_{ki} = 2P_{ik,k} - 1, k \neq i$ and $a_{kk} = P_{kk,k} - 1$ then we obtain

$$V : \begin{cases} x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right), \;\; k = 1, ..., \ell \\ x'_k = x_k \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right) + \\ \sum_{\substack{i,j=1 \\ i \neq k \\ j \neq k}}^{m} P_{ij,k} x_i x_j, \;\; k = \ell + 1, ..., m. \end{cases} \qquad (9)$$

Note that

$$a_{kk} \in [-1, 0]; \ |a_{ki}| \leq 1;$$

$$a_{ki} + a_{ik} = 2(P_{ik,i} + P_{ik,k}) - 2 \leq 0, \ i, k \in E. \quad (10)$$

*Remark.*Note that for Volterra case ($\ell = m$) one has $a_{kk} = 0$ and $a_{ki} + a_{ik} = 0$ i.e fixed (see [3],[4]) but (10) shows that in our case ($\ell < m$) one has $a_{kk} \in [-1, 0]$ and $a_{ki} + a_{ik} \in [-2, 0]$ i.e vary on the uncountable sets. In spite of this generality, we show that methods of [3]-[5] work for ℓ-Volterra QSOs.

Lemma 1.*If $\exists k_0 \in \{1, ..., \ell\}$ and $\delta \leq 0$ such that $a_{k_0 i} < \delta$ for any $i \in E$ then*

$$P_\delta = \Big\{ p = (p_1, ..., p_\ell) \in S^{\ell-1} :$$

$$\sum_{k=1}^\ell a_{ki} p_k < \delta, \text{ for any } i \in E \Big\} \neq \emptyset.$$

*Proof.*It is easy to see that

$$e^{(k_0)} = (0, ..., 0, e_{k_0}^{(k_0)} = 1, 0, ..., 0) \in P_\delta.$$

Thus for sufficiently small ε we have

$$\Big\{ p \in S^{\ell-1} : \|p - e^{(k_0)}\| < \varepsilon \Big\} \subset P_\delta,$$

where $\|p - e^{(k_0)}\| = \max_i |p_i - e_i^{(k_0)}|$. Indeed ε can be chosen as follows. Take $p = (p_1, ..., p_\ell)$ with $p_{k_0} = 1 - \varepsilon$, $\sum_{\substack{i=1 \\ i \neq k_0}}^\ell p_i = \varepsilon$ then $\|p - e^{(k_0)}\| \leq \varepsilon$ and

$$\sum_{k=1}^\ell a_{ki} p_k \leq \Big\{ \max_{\substack{k \in \{1, ..., \ell\} \\ k \neq k_0}} a_{ki} \Big\} \sum_{\substack{k=1 \\ k \neq k_0}}^\ell p_k + a_{k_0 i}(1 - \varepsilon) \leq$$

$$\Big\{ \max_{\substack{k \in \{1, ..., \ell\} \\ k \neq k_0}} \{a_{ki}, 0\} - a_{k_0 i} \Big\} \varepsilon + a_{k_0 i} < \delta$$

for any $i \in E$ if

$$\varepsilon < \min_{i \in E} \frac{\delta - a_{k_0 i}}{\max_{\substack{k \in \{1, ..., \ell\} \\ k \neq k_0}} \{a_{ki}, 0\} - a_{k_0 i}}.$$

This completes the proof.

Let $\{x^{(n)}\}_{n=1}^\infty$ be the trajectory of the point $x^0 \in S^{m-1}$ under operator (9). Denote by $\omega(x^0)$ the set of limit points of the trajectory. Since $\{x^{(n)}\} \subset S^{m-1}$ and S^{m-1} is compact, it follows that $\omega(x^0) \neq \emptyset$. Obviously, if $\omega(x^0)$ consists of a single point, then the trajectory converges, and $\omega(x^0)$ is a fixed point of (9). However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems (9). Therefore, it is of particular interest to obtain an upper bound for $\omega(x^0)$, i.e., to determine a sufficiently "small" set containing $\omega(x^0)$.

Denote

$$\text{int} S^{m-1} = \{x \in S^{m-1} : \prod_{i=1}^m x_i > 0\}.$$

Definition 2.*A continuous function $\varphi : \text{int} S^{m-1} \to \mathbf{R}$ is called a Lyapunov function for the dynamical system (9) if the limit $\lim_{n \to \infty} \varphi(x^{(n)})$ exists for any initial point x^0.*

Obviously, if $\lim_{n \to \infty} \varphi(x^{(n)}) = c$, then $\omega(x^0) \subset \varphi^{-1}(c)$. Consequently, for an upper estimate of $\omega(x^0)$ we should construct Lyapunov functions that are as large as possible.

Theorem 1.*If $\exists k_0 \in \{1, ..., \ell\}$ such that $a_{k_0 i} < 0$ for any $i \in E$ and $p = (p_1, ..., p_\ell) \in P_0$ then $\varphi_p(x) = x_1^{p_1} ... x_\ell^{p_\ell}$ is a Lyapunov function for (9).*

*Proof.*Compute $\varphi_p(Vx)$:

$$\varphi_p(Vx) = \prod_{k=1}^\ell x_k^{p_k} \Big(1 + \sum_{i=1}^m a_{ki} x_i \Big)^{p_k} =$$

$$\varphi_p(x) \prod_{k=1}^\ell \Big(1 + \sum_{i=1}^m a_{ki} x_i \Big)^{p_k}.$$

Using Young's inequality

$$b_1^{p_1} ... b_\ell^{p_\ell} \leq p_1 b_1 + ... + p_\ell b_\ell,$$

where $b_i > 0$ and $p_i \geq 0$, $\sum_{i=1}^\ell p_i = 1$ we obtain

$$\varphi_p(Vx) \leq \varphi_p(x) \Big(1 + \sum_{i=1}^m \Big(\sum_{k=1}^\ell a_{ki} p_k \Big) x_i \Big).$$

Since $p \in P_0$ we have $\sum_{k=1}^\ell a_{ki} p_k < 0$ for any $i \in E$. Consequently $\varphi(Vx) < \varphi(x)$.

Theorem 2.*If $a_{ki} < 0$ for any $k = 1, ..., r$, $(r \leq \ell)$ and $i = r+1, ..., m$ then*

$$\varphi(x) = x_1 + ... + x_r, \ x = (x_1, ..., x_m) \in \text{int} S^{m-1}$$

is a Lyapunov function for (9). Moreover $\sum_{n=0}^\infty \varphi(x^{(n)}) < +\infty$ for any trajectory $\{x^{(n)}\}$.

*Proof.*Compute $\varphi(Vx)$ according (9):

$$\varphi(Vx) = \sum_{k=1}^r x_k' = \varphi(x) + \sum_{k=1}^r \Big(\sum_{i=1}^m a_{ki} x_i \Big) x_k. \quad (11)$$

Since $a_{kk} \in [-1, 0]$, $a_{ki} + a_{ik} \leq 0$ (see (10)), it follows that

$$\sum_{k=1}^r \sum_{i=1}^r a_{ki} x_k x_i = \sum_{k=1}^r a_{kk} x_k^2 +$$

$$\sum_{1 \leq k < i \leq r} (a_{ki} + a_{ik}) x_k x_i \leq 0.$$

Therefore, by (11) we have

$$\varphi(Vx) \leq \varphi(x) + \sum_{k=1}^r \Big(\sum_{i=r+1}^m a_{ki} x_i \Big) x_k. \quad (12)$$

Let

$$\alpha = \min_{\substack{k \in \{1,\dots,r\} \\ i \in \{r+1,\dots,m\}}} \{-a_{ki}\},$$

since $\alpha > 0$, (12) gives us

$$\varphi(Vx) \le \varphi(x) - \alpha \sum_{k=1}^{r} \left(\sum_{i=r+1}^{m} x_i \right) x_k =$$

$$\varphi(x)[1 - \alpha + \alpha\varphi(x)]. \tag{13}$$

For any $x^0 \in \text{int}S^{m-1}$ we have $\varphi(x^0) < 1$. Since $\alpha \le 1$, it follows $1 - \alpha + \alpha\varphi(x^0) < 1$. Therefore, it follows from (13) that the inequality

$$\varphi(x^{(n+1)}) \le \varphi(x^{(n)})[1 - \alpha + \alpha\varphi(x^{(n)})] \le$$

$$\varphi(x^0)[1 - \alpha + \alpha\varphi(x^0)]^n$$

holds along the trajectory $\{x^{(n)}\}$. Thus, $\varphi(x^{(n)}) \to 0$ and also $\sum_{n=0}^{\infty} \varphi(x^{(n)}) < +\infty$.

It is known (see [9]) that if a_n and b_n are two bounded sequences of nonnegative numbers and if $a_{n+1} \le a_n + b_n$, $n = 1, 2, \dots$ then it follows from $b_n \to 0$ that $\{a_n\}$ is dense in $[\underline{\lim} a_n, \overline{\lim} a_n]$. Moreover, if $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

Below we use this fact to construct new Lyapunov functions.

Theorem 3. *If conditions of Theorem 2 are satisfied then*

$$\psi_p(x) = x_1^{p_1} \dots x_r^{p_r},\ r \le \ell, x = (x_1,\dots,x_m) \in \text{int}S^{m-1}$$

is a Lyapunov function of (9) for any $p = (p_1,\dots,p_r) \in S^{r-1}$.

Proof. Using Young's inequality, we get

$$\psi_p(x') \le \psi_p(x) \sum_{k=1}^{r} \left(1 + \sum_{i=1}^{m} a_{ki} x_i \right) p_k =$$

$$\psi_p(x) \left(1 + \sum_{k=1}^{r} \left(\sum_{i=1}^{m} a_{ki} x_i \right) p_k \right). \tag{14}$$

By conditions we have

$$\sum_{k=1}^{r} \sum_{i=1}^{r} a_{ki} p_k x_i \le \sum_{i=1}^{r} x_i;\ \ \sum_{k=1}^{r} \sum_{i=r+1}^{m} a_{ki} p_k x_i \le 0.$$

Hence by (14) we get

$$\psi_p(x') \le \psi_p(x) \left(1 + \sum_{k=1}^{r} x_k \right).$$

Consequently, along any trajectory $\{x^{(n)}\}$ we have

$$\psi_p(x^{(n+1)}) \le \psi_p(x^{(n)}) \left(1 + \varphi(x^{(n)}) \right), \tag{15}$$

where $\varphi(x^{(n)}) = \sum_{k=1}^{r} x_k^{(n)}$. According to Theorem 2, the series $\sum_{n=0}^{\infty} \varphi(x^{(n)})$ converges, and so it follows from (15) that $\lim_{n\to\infty} \psi_p(x^{(n)})$ exists along any trajectory.

Remark. When the functions φ_p and ψ_p are extended from $\text{int}S^{m-1}$ to S^{m-1} the expression 0^0 can arise, and we set it equal to 1.

Now we shall describe Lyapunov functions of other forms.

Theorem 4. *If there exists $p \in \{1,\dots,\ell\}$ and $q \in E$ such that $a_{pi} - a_{qi} \le 0$ for any $i \in E$ then*

$$f_{pq}(x) = \frac{x_p}{x_q},\ x = (x_1,\dots,x_m) \in \text{int}S^{m-1}$$

is a Lyapunov functions of (9). Moreover $f_{pq}(x)$ is monotonically decreasing along the trajectory $\{x^{(n)}\}$, where $x^0 \in \text{int}S^{m-1}$ and $x^0 \ne V(x^0)$.

Proof. We have

$$f_{pq}(x') = \frac{x'_p}{x'_q} = f_{pq}(x) \times$$

$$\frac{1 + \sum_{i=1}^{m} a_{pi} x_i}{1 + \sum_{i=1}^{m} a_{qi} x_i + \mathbf{1}_{\{q>\ell\}} x_q^{-1} \sum_{\substack{i,j=1 \\ i\ne q, j\ne q}}^{m} P_{ij,q} x_i x_j}, \tag{16}$$

where $\mathbf{1}_{\{q>\ell\}} = 0$ (resp. =1) if $q \le \ell$ (resp. $q > \ell$). Clearly,

$$\mathbf{1}_{\{q>\ell\}} x_q^{-1} \sum_{\substack{i,j=1 \\ i\ne q, j\ne q}}^{m} P_{ij,q} x_i x_j \ge 0,\ \text{for any } x \in \text{int}S^{m-1}.$$

Consequently, from (16) by condition of theorem we get

$$f_{pq}(x') \le \alpha f_{pq}(x), \tag{17}$$

where

$$\alpha = \max_{x \in \text{int}S^{m-1}} \frac{1 + \sum_{i=1}^{m} a_{pi} x_i}{1 + \sum_{i=1}^{m} a_{qi} x_i} \le 1.$$

This implies $f_{pq}(x^{(n+1)}) < f_{pq}(x^{(n)}), n \ge 0$. Thus sequence $f_{pq}(x^{(n)})$ is a monotonically decreasing. Since it is bounded we conclude that f_{pg} is a Lyapunov function.

Remark. The Lyapunov functions mentioned in Theorems 1,2 and 4 are monotonically decreasing along any trajectory. Note that under conditions of Theorem 4 we can also construct the function $f_{qp}^{+}(x) = \frac{x_q}{x_p}$ which is monotonically increasing along any trajectory $\{x^{(n)}\}$ with $x^0 \in \text{int}S^{m-1}$. But the limit $\lim_{n\to\infty} f_{qp}^{+}(x^{(n)})$ can be equal to $+\infty$.

4. Upper estimations of $\omega(x^0)$

In this section we shall apply the Lyapunov functions described in the previous section to obtain an upper bound of $\omega(x^0)$. Denote by $\text{Fix}(V)$ the set of all fixed points of the operator (9) i.e.

$$\text{Fix}(V) = \{x \in S^{m-1} : V(x) = x\}.$$

Theorem 5.*If there exists $k_0 \in \{1, ..., \ell\}$ and $\delta > 0$ such that $a_{k_0 i} < -\delta$ for any $i \in E$ then for $x^0 \notin \text{Fix}(V)$,*

$$\omega(x^0) \subset \{x \in S^{m-1} : \prod_{i=1}^{\ell} x_i = 0\}.$$

*Proof.*Consider Lyapunov function $\varphi_p(x) = \prod_{i=1}^{\ell} x_i^{p_i}$ for $p \in P_\delta$. By proof of Theorem 1 we have

$$\varphi_p(Vx) < (1-\delta)\varphi_p(x), \; \delta > 0.$$

Iterating this inequality we obtain

$$\varphi_p(x^{(n)}) < (1-\delta)^n \varphi_p(x^0).$$

Hence

$$\lim_{n\to\infty} \varphi_p(x^{(n)}) = \lim_{n\to\infty} \prod_{i=1}^{\ell} (x_i^{(n)})^{p_i} = 0.$$

This completes the proof.

As a corollary of Theorem 2 we have

Theorem 6.*Suppose conditions of Theorem 2 are satisfied. If $i \in \{1, ..., r\}$ then $x_i^{(n)} \to 0$, at the rate of a geometric progression as $n \to \infty$.*

This Theorem gives the estimation $\omega(x^0) \subset S^{m-r-1}$, where S^{m-r-1} is the face of S^{m-1} spanned by the vertices $e^{(r+1)} = (0, ..., 0, e_{r+1} = 1, 0, ..., 0),..., e^{(m)} = (0, ..., 0, 1)$.

If in Theorem 4 we consider more stronger condition i.e. $a_{pi} - a_{qi} < 0$ instead of $a_{pi} - a_{qi} \leq 0$, for any $i \in E$. Then we get (17) with $\alpha < 1$. In this case it follows that $f_{pq}(x^{(n)}) \to 0$. Using the fact that $0 < x_q^{(n)} < 1$, we get $x_p^{(n)} \to 0$. This enables us to get a more precise estimate for $\omega(x^0)$: it is a subset of the simplex S^{m-1} with $x_p = 0$ where $p \in \{1, ..., \ell\}$ such that there exists $q = q(p) \in E$ which satisfies conditions of Theorem 4.

By these results and results of [12] we make following remarks.

*Remark.*1. For Volterra operators the estimate $\omega(x^0) \subset S^{m-r-1}$ can be improved to the estimation

$$\omega(x^0) \subset \partial S^{m-r-1} = \{x \in S^{m-r-1} : \prod_{i=1}^{m-r-1} x_i = 0\}$$

(see [3]). In general, if $\ell < m$ then such an improvement is impossible.

2. If $\ell \leq m - 2$ then ℓ-Volterra operators can have cyclic trajectories this is quite different behavior from the behavior of Volterra operators, since Volterra operators have no cyclic trajectories.

3. One of the main goal by introducing the notion of ℓ-Volterra operators was to give an example of QSO which has more rich dynamics than Volterra QSO. It is well known

[3], [5] that for Volterra operators (see (6)) if $a_{ij} \neq 0$ ($i \neq j$) then for any non-fixed initial point λ^0 the set $\omega(\lambda^0)$ of all limit points of the trajectory $\{\lambda^{(n)}\}$ is subset of the boundary of simplex. But for ℓ-Volterra operators, in general, the limit set can be subset of the inside of simplex.

4. It is known [3] that Volterra operators are homomorphisms. Consequently, for any initial point $x^0 \in S^{m-1}$ the "negative" trajectory $\{V^{-n}(x^0)\}, n = 0, 1, 2, ...$ exists. Moreover the negative trajectories always converge. But such kind of result is not true for ℓ-Volterra operators.

5. The fixed points of the operator (9)

It is easy to see that a vertex $e^{(i)} = (0, ..., 0, 1_i, 0, ..., 0)$ of S^{m-1} is a fixed point of V iff $P_{ii,i} = 1$. We consider the question of the existence of other fixed points.

For $j \in E$ denote

$$X_j = \left\{ x \in S^{m-1} : x_k' = V(x)_k = x_k, \; k = 1, ..., j \right\}.$$

Note that $X_{m-1} = X_m = \text{Fix}(V)$ and $X_j \subset X_{j-1}$ for any $j = 1, ..., m - 1$.

It is easy to see that $x \in X_\ell$ if and only if

$$x_k \sum_{i=1}^{m} a_{ki} x_i = 0, \; k = 1, ..., \ell. \qquad (18)$$

Set $\text{supp}_\ell x = \{i \in \{1, ..., \ell\} : x_i \neq 0\}$ then equations (18) are equivalent to the following

$$\text{supp}_\ell x \cap \text{supp}_\ell Ax = \emptyset, \qquad (19)$$

where $A = (a_{ij})_{i,j=1}^m$ is $m \times m$ matrix with a_{ij} defined in (10).

Lemma 2.*If $x, y \in X_\ell$ and $\text{supp}_\ell x = \text{supp}_\ell y$ then $\lambda x + (1-\lambda)y \in X_\ell$ for any $\lambda \in [0, 1]$.*

*Proof.*By (19) and $\text{supp}_\ell x = \text{supp}_\ell y$ we have

$$\text{supp}_\ell x \cap (\text{supp}_\ell Ax \cup \text{supp}_\ell Ay) = \emptyset.$$

Since $\text{supp}_\ell(\lambda u + (1-\lambda)v) \subset \text{supp}_\ell u \cup \text{supp}_\ell v$ we have

$$\text{supp}_\ell(\lambda x + (1-\lambda)y) \cap \text{supp}_\ell(\lambda Ax + (1-\lambda)Ay) \subset$$

$$\text{supp}_\ell x \cap (\text{supp}_\ell Ax \cup \text{supp}_\ell Ay) = \emptyset.$$

Hence $\lambda x + (1-\lambda)y \in X_\ell$.

For $p, q, r \in E$ put $\Delta = (a_{pr} - a_{pp})(a_{qr} - a_{qq}) - (a_{pr} - a_{pq})(a_{qr} - a_{qp})$;

$$\Delta_1 = a_{qr}a_{pq} - a_{pr}a_{qq}; \; \Delta_2 = a_{pr}a_{qp} - a_{qr}a_{pp}.$$

Theorem 7.*If*
(a) $p, q, r \in E$ such that $\{p, q, r\} \cap \{\ell+1, ..., m\}$ contains at most one element, say r;
(b) $P_{ij,k} = 0$ for any $i, j \in \{p, q, r\}, k \in \{\ell+1, ..., m\} \setminus \{p, q, r\}$;

(c) $\Delta \neq 0$, $\text{sign}(\Delta) = \text{sign}(\Delta_1) = \text{sign}(\Delta_2) = \text{sign}(\Delta - \Delta_1 - \Delta_2)$.

Then the interior of the two-dimensional face S^2_{pqr} spanned by the vertices $e^{(p)}, e^{(q)}$ and $e^{(r)}$ of S^{m-1} contains exactly one fixed point of V.

Proof. Under conditions (a) and (b) the restriction of V (see (9)) to the face S^2_{pqr} has the form

$$
\begin{aligned}
x'_p &= x_p \left(1 + a_{pp}x_p + a_{pq}x_q + a_{pr}x_r\right), \\
x'_q &= x_q \left(1 + a_{qp}x_p + a_{qq}x_q + a_{qr}x_r\right), \\
x'_r &= x_r \left(1 + a_{rp}x_p + a_{rq}x_q + a_{rr}x_r\right) + \\
&\quad P_{pp,r}x_p^2 + 2P_{pq,r}x_px_q + P_{qq,r}x_q^2.
\end{aligned}
\tag{20}
$$

Using $x_p + x_q + x_r = 1$, $x_px_qx_r > 0$ and (20) the equation $V(x) = x$ on S^2_{pqr} can be written as

$$
\begin{aligned}
(a_{pr} - a_{pp})x_p + (a_{pr} - a_{pq})x_q &= a_{pr}, \\
(a_{qr} - a_{qp})x_p + (a_{qr} - a_{qq})x_q &= a_{qr}.
\end{aligned}
\tag{21}
$$

Elementary computations show that $x^* = (x^*_1, ..., x^*_m)$, where

$$
x^*_p = \frac{\Delta_1}{\Delta}, \quad x^*_q = \frac{\Delta_2}{\Delta}, \quad x^*_r = \frac{\Delta - \Delta_1 - \Delta_2}{\Delta},
$$

and all the rest of the coordinates are zero, is a fixed point of V. It follows from condition (c) that $x^*_p, x^*_q, x^*_r > 0$. Therefore, x^* is a fixed point satisfying the condition of the theorem. Uniqueness is verified by a simple computation.

Example. Consider $m = 5$, $\ell = 2$ and the following 2-Volterra operator

$$
\begin{aligned}
x'_1 &= x_1 \left(1 + \sum_{i=1}^5 a_{1i}x_i\right), \\
x'_2 &= x_2 \left(1 + \sum_{i=1}^5 a_{2i}x_i\right), \\
x'_3 &= x_3 \left(1 + \sum_{i=1}^5 a_{3i}x_i\right) + P_{55,3}x_5^2, \\
x'_4 &= x_4 \left(1 + \sum_{i=1}^5 a_{4i}x_i\right) + \\
&\quad P_{11,4}x_1^2 + 2P_{12,4}x_1x_2 + P_{22,4}x_2^2, \\
x'_5 &= x_5 \left(1 + \sum_{i=1}^5 a_{5i}x_i\right) + \\
&\quad P_{33,5}x_3^2 + 2P_{13,5}x_1x_3 + 2P_{23,5}x_2x_3.
\end{aligned}
\tag{22}
$$

It is easy to see that the operator (22) satisfies conditions (a) and (b) of Theorem 7 with $p = 1, q = 2, r = 4$. Condition (c) of Theorem 7 is necessary and sufficient for the existence and uniqueness of solution to system (21). One can check that the operator (22) satisfies the condition (c), for example, if $P_{11,1} = P_{22,2} = a$, $P_{12,1} = P_{12,2} = b$, $P_{14,1} = P_{24,2} = c$, with $a \neq 2b$ and $c > 1/2$. In this case the unique fixed point x^* in the interior of S^2_{124} is

$$
x^* = \left(\frac{1 - 2c}{a + 2b - 4c}, \frac{1 - 2c}{a + 2b - 4c}, 0, \frac{a + 2b - 2}{a + 2b - 4c}, 0 \right).
$$

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