Information Sciences Letters

Volume 9
Issue 2 *May 2020*

Article 3

2020

Some Fixed Point Theorems under E.A. and (CLR) Properties on C∗-Algebra Valued Metric Spaces

Manoj Kumar
Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat, 131039, India, mhimdad@gmail.com

Mohammad Imdad
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, mhimdad@gmail.com

Mohammad Asim
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India, mhimdad@gmail.com

Inf. Sci. Lett. **9**, No. 2, 75-82 (2020) 75

Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/090203

Some Fixed Point Theorems under E.A. and (CLR) Properties on C^*-Algebra Valued Metric Spaces

Manoj Kumar[1], *Mohammad Imdad*[2,*] *and Mohammad Asim*[2]

[1]Department of Mathematics, Deenbandhu Chhotu Ram University of Science and Technology, Murthal, Sonepat, 131039, India
[2]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Received: 22 Aug. 2019, Revised: 12 Apr. 2020, Accepted: 20 Apr. 2020
Published online: 1 May 2020

Abstract: In this paper, we prove some common fixed point theorems for four weakly compatible self maps satisfying E.A. property and (CLR) property in C^*-algebra valued metric spaces. An example is also adopted to substantiate the significance of our newly proved results. In addition, we use one of our main results to establish the existence and uniqueness of a solution for the system of integral equations.

Keywords: C^*-algebra valued metric space, common fixed point, weakly compatible maps, E.A. property, (CLR) property.

1 Introduction

The Banach contraction principle [1] is a very useful, simple and effective tool in modern analysis. It has extensive applications within several domains, such as Mathematical Sciences, Physical Sciences and Social Sciences. Several authors have investigated the existence of fixed points for several types of mappings which are spread throughout the existing literature. For a relevant work of this kind, one can consult [2,3,4,5,6,7,8,9,10,11]. In particular, it is an important tool for solving existing problems within several domains, such as integral equations, differential equations, fractional differential equations and partial differential equations [12,13,14,15,16,17] etc.

Recently, Ma et al. [18] has extended Banach contraction principle to C^*-algebra valued metric spaces using the set of all positive members of a unital C^*-algebra instead of the set of real numbers. Later on, some researchers have done some work on this (see [19,20,21,22,23]).

In this paper, we prove some common fixed point theorems for four weakly compatible self maps along with E.A. property and (CLR) property of C^*-algebra valued metric spaces.

The following definitions and notations will be used in a sequel to prove our theorems. For the details, we refer to [22,23].

A $*$-algebra $\mathbb{A}$ is a complex algebra with linear involution $*$ such that for all $p,q \in \mathbb{A}$ $(pq)^* = q^*p^*$ and $p^{**} = p$. The pair $(\mathbb{A},*)$ is called a unital $*$-algebra if it contains the unity element $1_{\mathbb{A}}$. A unital $*$-algebra $(\mathbb{A},*)$ is called a Banach $*$-algebra if it satisfies $\|p^*\| = \|p\|$ along with a complete sub-multiplicative norm. A Banach C^*-algebra satisfying $\|p^*p\| = \|p\|^2$, for all $p \in \mathbb{A}$ is called C^*-algebra. If $p = p^*$ and $\sigma(p) = \{\beta \in \mathbb{R} : \beta 1_{\mathbb{A}} - p \text{ is non-invertible }\}$, p is called the positive element of $\mathbb{A}$. If $p \in \mathbb{A}$ is positive, we write it as $p \succcurlyeq 0_{\mathbb{A}}$. The partial ordering on $\mathbb{A}$ can be defined as follows: $p \succcurlyeq q$ if and only if $p - q \succcurlyeq 0_{\mathbb{A}}$.

Throughout the paper, by $\mathbb{A}$, we denote a unital C^*-algebra with the unity element $1_{\mathbb{A}}$.

In 2014, Ma et al. [18] introduced the notion of C^*-algebra valued metric spaces in the following way:

Definition 1.1. Suppose X is a non-empty set. The mapping $d : X \times X \to \mathbb{A}$ is called C^*-algebra valued metric on X if it satisfies the following for all $p,q,r \in \mathbb{A}$:

(i) $d(p,q) \succcurlyeq 0_{\mathbb{A}}$ and $d(p,q) = 0_{\mathbb{A}}$ if $p = q$;
(ii) $d(p,q) = d(q,p)$;
(iii) $d(p,r) \preccurlyeq d(p,q) + d(q,r)$.

The triplet $(X,\mathbb{A},d)$ is called a C^*-algebra valued metric space.

Definition 1.2. A sequence $\{x_n\}$ in $(X,\mathbb{A},d)$ is said to be

* Corresponding author e-mail: mhimdad@gmail.com

(i) convergent with respect to $\mathbb{A}$, if for given $\varepsilon > 0$, there exists a positive integer k such that $\|d(x_n, x)\| < \varepsilon$, for all $n > k$,
(ii) Cauchy sequence with respect to $\mathbb{A}$ if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \varepsilon$, for all $n, m > k$.

The triplet $(X, \mathbb{A}, d)$ is called complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

The following definition is used in our subsequent discussions.

Definition 1.3.[20] The max function on $\mathbb{A}$ (C^*-algebra) with the partial order relation '$\preccurlyeq$' is defined by (for all $x, y \in \mathbb{A}$):

$$\max\{x, y\} = y \Leftrightarrow x \preccurlyeq y \text{ and } \|x\| \leq \|y\|.$$

Jungck [24] and Vetro [25] introduced the concept of weakly compatible maps as follows:

Definition 1.4. Let f and g be two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to be weakly compatible if they commute at coincidence points.

Example 1.1. Define $f, g : \mathbb{R} \to \mathbb{R}$ by $fx = \frac{x}{3}$, for all $x \in \mathbb{R}$ and $gx = x^2$, for all $x \in \mathbb{R}$. Here, 0 and $\frac{1}{3}$ are two coincidence points for the maps f and g. Note that f and g commute at 0, i.e. $fg(0) = gf(0) = 0$, but $fg(\frac{1}{3}) = f(\frac{1}{9}) = \frac{1}{27}$ and $gf(\frac{1}{3}) = g(\frac{1}{9}) = \frac{1}{81}$ so f and g are not weakly compatible on $\mathbb{R}$.

In 2002, Aamri and Moutawakil [26] introduced the notion of E.A. property, as follows:

Definition 1.5. Let f and g are two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Example 1.2. Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ and $fx = \frac{1}{5}x$ and $gx = \frac{3}{5}x$ for each $x \in X$. Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$, where $0 \in X$. Hence, the pair (f, g) satisfies the E.A. property.

In 2011, Sintunavarat and Kumam [27] introduced the notion of (CLR) property as follows.

Definition 1.6. Let f and g are two self-mappings of a metric space (X, d). Then, the pair (f, g) is said to satisfy (CLR_f) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fx$ for some $x \in X$.

Example 1.3. Let $X = [0, 1]$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ and $fx = x$ and $gx = x^2$ for each $x \in X$. Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$ so that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = f(0)$, where $0 \in X$. Hence, the pair (f, g) satisfies the (CLR_f) property.

2 Main Results

In this section, we will prove some common fixed point theorems for four weakly compatible self maps along with E.A. property and (CLR) property.

Theorem 2.1. Let f, g, S and T be four self maps of C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ satisfying the followings:

$$SX \subseteq gX, TX \subseteq fX, \tag{1}$$

$$d(Sx, Ty) \preccurlyeq a^* m(x, y) a,$$

for any $x, y \in X$, where $a \in \mathbb{A}$ with $\|a\| < 1$ and

$$m(x, y) = \max\{d(fx, gy), d(Sx, fx), d(Tx, gy), d(Sx, gy), d(Ty, fx)\}. \tag{2}$$

If one of fX, gX, SX and TX is a complete subspace of X, the pairs (f, S) and (g, T) have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, f, g, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (1), we can construct a sequence $\{y_n\}$ in X, as follows (for all $n = 0, 1, 2, \ldots$):

$$y_{2n+1} = Sx_{2n} = gx_{2n+1}, y_{2n+2} = Tx_{2n+1} = fx_{2n+2}, \tag{3}$$

Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ for some n. Then, $y_{2n} = y_{2n+1}$, i.e. $Tx_{2n-1} = fx_{2n} = Sx_{2n} = gx_{2n+1}$, f and S have a coincidence point. Hence, we are done. Now, we suppose that $d_{2n} > 0$ for all $n \in \mathbb{N}$. Then, from (2), we have

$$d(Sx_{2n}, Tx_{2n+1}) \preccurlyeq a^* m(x_{2n}, x_{2n+1}) a, \tag{4}$$

where,

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\{d(fx_{2n}, gx_{2n+1}), d(Sx_{2n}, fx_{2n}), \\ &\quad d(Tx_{2n+1}, gx_{2n+1}), d(Sx_{2n}, gx_{2n+1}), \\ &\quad d(Tx_{2n+1}, fx_{2n})\} \\ &= \max\{d_{2n}, d_{2n+1}\}. \end{aligned} \tag{5}$$

Thus, from (4), we have

$$d(Sx_{2n}, Tx_{2n+1}) \preccurlyeq a^* \max\{d_{2n}, d_{2n+1}\} a. \tag{6}$$

If possible, assume $d_{2n+1} \geq d_{2n}$, for all $n \in \mathbb{N}$, so that (6) reduces to

$$d_{2n+1} \preccurlyeq a^* d_{2n+1} a,$$

that is,

$$d(y_{2n+1}, y_{2n+2}) \preccurlyeq a^* d(y_{2n+1}, y_{2n+2}) a,$$

which together with $\|a\| < 1$ implies that

$$\|d(y_{2n+1}, y_{2n+2})\| \leq \|a\|^2 \|d(y_{2n}, y_{2n+1})\| < \|d(y_{2n}, y_{2n+1})\|,$$

a contradiction. Thus, $d_{2n} > d_{2n+1}$ for all $n \in \mathbb{N}$ from (6). We have

$$d_{2n+1} \preccurlyeq a^* d_{2n} a.$$

Similarly, $d_{2n} \preccurlyeq a^*d_{2n-1}a$, and $d_{2n-1} \preccurlyeq a^*d_{2n-2}a$. In general, we have (for all $n \in \mathbb{N}$)

$$d_n \preccurlyeq a^*d_{n-1}a,$$

i.e.

$$\begin{aligned} d(y_n,y_{n+1}) &\preccurlyeq a^*d(y_{n-1},y_n)a \\ &\preccurlyeq (a^*)^2 d(y_{n-2},y_{n-1})a^2 \\ &\dots \\ &\preccurlyeq (a^*)^n d(y_0,y_1)a^n. \end{aligned}$$

Thus, for any $k \in \mathbb{N}$ and using the triangle inequality, we have

$$\begin{aligned} d(y_{n+k},y_n) \preccurlyeq\ & d(y_{n+k},y_{n+k-1}) + d(y_{n+k-1},y_{n+k-2}) + \\ & \dots + d(y_{n+1},y_n) \\ \preccurlyeq\ & \sum_{m=n}^{n+k-1} (a^*)^m d(y_1,y_0)a^m \\ \preccurlyeq\ & \sum_{m=n}^{n+k-1} (ba^m)^* ba^m \\ \preccurlyeq\ & \sum_{m=n}^{n+k-1} |ba^m|^2 \\ \preccurlyeq\ & \sum_{m=n}^{n+k-1} \| |ba^m|^2 \| 1_{\mathbb{A}} \\ \preccurlyeq\ & \|b\|^2 1_{\mathbb{A}} \sum_{m=n}^{n+k-1} (a^m)^2 \to 0_{\mathbb{A}} (n \to \infty), \end{aligned}$$

where $1_{\mathbb{A}}$ is the unit element in $\mathbb{A}$ and $d(y_1,y_0) = |b|^2$ for some $b \in \mathbb{A}_+$. This can be done since $d(y_1,y_0) \in \mathbb{A}_+$. Therefore, $\{y_n\}$ is a Cauchy sequence since fX is complete. Note that $\{y_n\}$ is contained in fX and has a limit in fX, say u, that is, $\lim_{n\to\infty} y_n = u$. Let $v \in f^{-1}u$, then $fv = u$. Now, we shall prove that $Sv = u$. Let, if possible, $Sv \neq u$, that is, $d(Sv,u) = p \succ 0_{\mathbb{A}}$. Putting $x = v$ and $y = x_{n-1}$ in (2), we get

$$d(Sv,Tx_{n-1}) \preccurlyeq a^*m(v,x_{n-1})a.$$

Putting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} d(Sv,Tx_{n-1}) \preccurlyeq a^* \lim_{n\to\infty} m(v,x_{n-1})a, \quad (7)$$

where

$$\begin{aligned} \lim_{n\to\infty} m(v,x_{n-1}) &= \lim_{n\to\infty} \max\{d(u,y_{n-1}),d(Sv,u),d(y_n,y_{n-1}), \\ & \quad d(Sv,y_{n-1}),d(y_n,u)\} \\ &= \max\{d(u,u),d(Sv,u),d(u,u), \\ & \quad d(Sv,u),d(u,u)\} \\ &= d(Sv,u) = p. \end{aligned}$$

Thus, by (7), we have

$$d(Sv,u) \preccurlyeq a^*d(Sv,u)a,$$

which together with $\|a\| < 1$, implies that

$$\|d(Sv,u)\| \leq \|a\|^2 \|d(Sv,u)\| < \|d(Sv,u)\|,$$

a contradiction. Thus, we get $Sv = u = fv$ where v is the coincidence point of the pair (f,S). Since $SX \subseteq gX$, $Sv = u$ which implies that $u \in gX$. Let $w \in g^{-1}u$. Then $gw = u$. Using the same arguments as above, one can easily verify that $Tw = u = gw$, i.e. u is the coincidence point of the pair (g,T).

The same result holds if we assume that gX is complete instead of fX. Now, if TX is complete, then by (1), $u \in TX \subseteq fX$. Similarly, if SX is complete, $u \in SX \subseteq gX$. Now, since the pair (f,S) and (g,T) are weakly compatible,

$$u = Sv = fv = Tw = gw,$$

then

$$\begin{aligned} fu &= fSv = Sfv = Su, \\ gu &= gTw = Tgw = Tu. \end{aligned}$$

Now, we claim that $Tu = u$. Let, if possible, $Tu \neq u$. By (2), we have

$$d(u,Tu) = d(Sv,Tu) \preccurlyeq a^*m(v,u)a,$$

where

$$\begin{aligned} m(v,u) &= \max\{d(fv,gu),d(Su,fu),d(Tv,gv), \\ & \quad d(Su,gv),d(Tv,fu)\} \\ &= \max\{d(u,Tu),d(u,u),0_{\mathbb{A}},d(u,Tu),d(Tu,u)\} \\ &= d(u,Tu). \end{aligned}$$

Thus, we have

$$d(u,Tu) \preccurlyeq a^*d(u,Tu)a,$$

which together with $\|a\| < 1$ implies

$$\|d(u,Tu)\| \leq \|a\|^2 \|d(u,Tu)\| < \|d(u,Tu)\|,$$

a contradiction. Thus, we have $Tu = u$. Similarly, one can prove that, $Su = u$ and we get $fu = Su = gu = Tu = u$. Hence, u is the common fixed point of f,g,S and T.

To prove the uniqueness, let z be another common fixed point of f,g,S and T with $u \neq z$. From (2), we have

$$d(u,z) = d(Su,Tz) \preccurlyeq a^*m(u,z)a,$$

where

$$\begin{aligned} m(u,z) &= \max\{d(fu,gz),d(Su,fu),d(Tz,gz), \\ & \quad d(Su,gz),d(Tz,fu)\} \\ &= \max\{d(u,z),0_{\mathbb{A}},0_{\mathbb{A}},d(u,z),d(z,u)\} \\ &= d(u,z). \end{aligned}$$

Thus, we have $d(u,z) \preccurlyeq a^*d(u,z)a$, which together with $\|a\| < 1$, provides

$$\|d(u,z)\| \leq \|a\|^2 \|d(u,z)\| < \|d(u,z)\|,$$

a contradiction. Thus, $u = z$ and the uniqueness follows.

Theorem 2.1 of [28] can be deduced from our Theorem 2.1 in the following form.

Corollary 2.1. Let S and T be two self maps of complete C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ satisfying

$$\|d(Sx, Ty)\| \leq \|a\| \|d(x, y)\|,$$

for any $x, y \in X$, where $a \in \mathbb{A}$ with $\|a\| < 1$. Then, S and T have a unique common fixed point in X.

Proof. Putting $f = g = I_X$ and taking $m(x, y) = d(x, y)$, one can easily verify the result.

Remark 2.1. Putting $S = T$ in the aforementioned corollary, we can generalize the Theorem 2.1 of [2].

Lemma 2.1. Let f, g, S and T be four self maps of C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ satisfying the followings:

(Li) $\{f, S\}$ (or $\{g, T\}$) satisfies (E.A.) property,
(Lii) $S(X) \subset g(X)$ (or $T(X) \subset f(X)$),
(Liii) f, g, S and T satisfy condition (2).

Then, $\{f, S\}$ and $\{g, T\}$ satisfy (E.A.) common property.

Proof. If the pair $\{f, S\}$ satisfies the (E.A.) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = z, \text{ for some } z \in X.$$

Since $S(X) \subset g(X)$, for each $\{x_n\}$, there exists $\{y_n\}$ in X such that $Sx_n = gy_n$. Therefore, $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$. Thus, we have that $gy_n \to z$, $Sx_n \to z$ and $fx_n \to z$. Now , we assert that $Ty_n \to z$. On the contrary, let $Ty_n \to t \neq z$. Then from (2), we have

$$d(Sx_n, Ty_n) \preccurlyeq a^* m(x_n, y_n) a.$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} \lim_{n\to\infty} d(Sx_n, Ty_n) = \lim_{n\to\infty} d(z, t) &\preccurlyeq \lim_{n\to\infty} a^* m(x_n, y_n) a \\ &= a^* \lim_{n\to\infty} d(t, z) a \qquad (8) \end{aligned}$$

where,

$$\begin{aligned} \lim_{n\to\infty} m(x_n, y_n) &= \lim_{n\to\infty} \max \{d(fx_n, gy_n), d(Sx_n, fx_n), \\ &\quad d(Ty_n, gy_n), d(Sx_n, gy_n), d(Ty_n, fx_n)\} \\ &= \max \{0_{\mathbb{A}}, 0_{\mathbb{A}}, d(t, z), 0_{\mathbb{A}}, d(t, z)\} \\ &= \lim_{n\to\infty} d(t, z). \end{aligned}$$

Now, condition (8) gives rise $d(z, t) \preccurlyeq a^* d(z, t) a$, which together with $\|a\| < 1$, implies

$$\|d(z, t)\| \leq \|a\|^2 \|d(z, t)\| < \|d(z, t)\|,$$

a contradiction. Therefore, $\lim_{n\to\infty} Ty_n = z$ which shows that the pairs $\{f, S\}$ and $\{g, T\}$ satisfy (E.A.) common property.

Theorem 2.2. Let f, g, S and T be four self mappings of C^*-algebra-valued metric space $(X, \mathbb{A}, d)$ satisfying (2) and the followings:

(i) pairs (f, S) and (g, T) are weakly compatible,
(ii) pairs pairs (f, S) and (g, T) satisfy the E.A. common property,
(iii) fX and gX are closed subsets of X.

Then, f, g, S and T have a unique common fixed point.

Proof. Since the pairs (f, S) and (g, T) satisfy the E.A. common property. Then, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = z, \text{ for some } z \in X.$$

If fX is a closed subset of X, $\lim_{n\to\infty} fx_n = z \in gX$. Therefore, there exists a point $v \in X$ such that $z = fv$. Now, we shall show that $fv = Sv$. Let, if possible, $fv \neq Sv$. Using (2.2), we have

$$d(Sv, Ty_n) \preccurlyeq a^* m(v, y_n) a. \qquad (9)$$

Defining limit as $n \to \infty$, we have $\lim_{n\to\infty} d(Sv, Ty_n) \preccurlyeq a^* \lim_{n\to\infty} m(v, y_n) a$, where

$$\begin{aligned} \lim_{n\to\infty} m(v, y_n) &= \lim_{n\to\infty} \max \{d(fv, gy_n), d(Sv, fv), d(Ty_n, gy_n), \\ &\quad d(Sv, gy_n), d(Ty_n, fv)\} \\ &= \max\{d(z, z), d(Sv, z), d(z, z), \\ &\quad d(Sv, z), d(z, z)\} \\ &= d(Sv, z). \qquad (10) \end{aligned}$$

Making limit as $n \to \infty$ in (11) and using (12), we get

$$d(Sv, z) \preccurlyeq a^* d(Sv, z) a,$$

which together with $\|a\| < 1$, implies

$$\|d(Sv, z)\| \leq \|a\|^2 \|d(Sv, z)\| < \|d(Sv, z)\|,$$

a contradiction. Hence, $Sv = z = fv$. Therefore, v is a coincidence point of the pair (f, S). If gX is a closed subset of X, $\lim_{n\to\infty} gy_n = z \in gX$. Therefore, there exists a point $u \in X$ such that $z = gu$. Now, we shall show that $gu = Tu$. Let, if possible, $gu \neq Tu$. Using (2.2), we have

$$d(Sx_n, Tu) \preccurlyeq a^* m(x_n, u) a. \qquad (11)$$

Defining limit as $n \to \infty$, we have $\lim_{n\to\infty} d(Sx_n, Tu) \preccurlyeq a^* \lim_{n\to\infty} m(x_n, u) a$, where

$$\begin{aligned} \lim_{n\to\infty} m(x_n, u) &= \lim_{n\to\infty} \max \{d(fx_n, gu), d(Sx_n, fx_n), \\ &\quad d(Tu, gu), d(Sx_n, gu), d(Tu, fx_n)\} \\ &= \max\{d(z, z), d(z, z), d(Tu, z), \\ &\quad d(z, z), d(Tu, z)\} \\ &= \max \{0_{\mathbb{A}}, 0_{\mathbb{A}}, d(Tu, z), d(Tu, z)\} \\ &= d(Tu, z). \qquad (12) \end{aligned}$$

Making limit as $n \to \infty$ in (13) and using (14), we get

$$d(z, Tu) \preccurlyeq a^* d(z, Tu) a,$$

which together with $\|a\| < 1$, implies that

$$\|d(z,Tu)\| \leq \|a\|^2\|d(z,Tu)\| < \|d(z,Tu)\|,$$

a contradiction. Hence, $Tu = z = gu$. Therefore, u is a coincidence point of the pair (g,T). Since the pair (g,T) is weakly compatible, $gTu = Tgu$ which implies that $TTu = Tgu = gTu = ggu$. Since $TX \subseteq fX$, there exists $v \in X$ such that $Tu = fv$. Now, we claim that $fv = Sv$. Let, if possible, $fv \neq Sv$. From (2), we have

$$d(Sv,Tu) \preccurlyeq a^*m(v,u)a \tag{13}$$

where

$$\begin{aligned} m(v,u) &= \max\{d(fv,gu),d(Sv,fv),d(Tu,gu),\\ &\quad d(Sv,gu),d(Tu,fv)\}\\ &= d(Sv,fv) = d(Sv,Tu). \end{aligned}$$

Thus, from (15), we have

$$d(Sv,Tu) \preccurlyeq a^*d(Sv,Tu)a,$$

which together with $\|a\| < 1$ implies that

$$\|d(Sv,Tu)\| \leq \|a\|^2\|d(Sv,Tu)\| \leq \|d(ASv,Tu)\|,$$

a contradiction. Therefore, $Sv = Tu = fv$. Thus, we have $Tu = gu = Sv = fv$. The weak compatibility of the pair (f,S) implies that $fSv = Sfv = SSv = ffv$. Now, we claim that Tu is the common fixed point of f,g,S and T. Suppose that $TTu \neq Tu$. From (2), we have

$$\begin{aligned} d(Tu,TTu) &= d(Sv,TTu)\\ &\preccurlyeq a^*m(v,tu)a, \end{aligned} \tag{14}$$

where

$$\begin{aligned} m(v,u) &= \max\{d(fv,gTu),d(Sv,fv),d(gTu,TTu),\\ &\quad d(Sv,gTu),d(TTu,fv)\}\\ &= \max\{d(Tu,TTu),0_{\mathbb{A}},0_{\mathbb{A}},d(Tu,TTu)\}\\ &= d(Tu,TTu). \end{aligned}$$

Using this value in (16) and $\|a\| < 1$, we get

$$\|d(Tu,TTu)\| \leq \|a\|^2\|d(Tu,TTu)\| < \|d(Tu,TTu)\|,$$

a contradiction. Therefore, $Tu = TTu = gTu$. Hence, Tu is the common fixed point of g and T. Similarly, we can prove that Sv is the common fixed point of f and S. Since $Tu = Sv$, Tu is the common fixed point of f,g,S and T.

For the uniqueness of common fixed point, suppose that p and q are two common fixed points of f,g,S and T such that $p \neq q$. From (2), we have

$$d(Sp,Tq) \preccurlyeq a^*m(p,q)a, \tag{15}$$

where

$$\begin{aligned} m(p,q) &= \max\{d(fp,gq),d(Sp,fp),d(Tq,gq),\\ &\quad d(Sp,gq),d(Tq,fp)\}\\ &= \max\{d(p,q),0_{\mathbb{A}},0_{\mathbb{A}},d(p,q)\}\\ &= d(p,q). \end{aligned}$$

Using this value in (17) together with $\|a\| < 1$, we get

$$\|d(p,q)\| \leq \|a\|^2\|d(p,q)\| < \|d(p,q)\|,$$

a contradiction. Therefore, $p = q$, and the uniqueness follows.

Theorem 2.3. The conclusions of Theorem 2.2 remain true if the condition (iii) is replaced by the following

(iii)' $\overline{SX} \subset gX$ and $\overline{TX} \subset fX$.

As a corollary of Theorem 2.3, we have the following result which is also a variant of Theorem 2.1. **Corollary 2.2.** The conclusions of Theorems 2.2 and 2.3 remain true if the conditions (iii) and (iii)' are replaced by the following

(iii)" SX and TX are closed subsets of X provided $SX \subset gX$ and $TX \subset fX$.

Theorem 2.4. Let f,g,S and T be four self mappings of C^*-algebra-valued metric space $(X,\mathbb{A},d)$ satisfying (2), (9) and the following:

(i) $SX \subseteq gX$ and the pair (f,S) satisfies (CLR_f) property, or

(ii) $TX \subseteq fX$ and the pair (g,T) satisfies (CLR_f) property.

Then f,g,S and T have a unique common fixed point.

Proof. Without loss of generality, assume that $SX \subseteq gX$ and the pair (f,S) satisfies (CLR_g) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = fx,$$

for some x in X. Since $SX \subseteq gX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = gy_n$. Hence, $\lim_{n\to\infty} gy_n = fx$. We shall show that $\lim_{n\to\infty} Ty_n = fx$. Let, if possible, $\lim_{n\to\infty} Ty_n = z \neq fx$. Using (2), we have $d(Sx_n,Ty_n) \preccurlyeq a^*m(x_n,y_n)a$, that is

$$\lim_{n\to\infty} d(Sx_n,Ty_n) \preccurlyeq a^* \lim_{n\to\infty} m(x_n,y_n)a$$

and

$$d(fx,z) \preccurlyeq a^* \lim_{n\to\infty} m(x_n,y_n), \tag{16}$$

where

$$\begin{aligned} \lim_{n\to\infty} m(x_n,y_n) &= \lim_{n\to\infty} \max\{d(fx_n,gy_n),d(Sx_n,fx_n),\\ &\quad d(Ty_n,gy_n),d(Sx_n,gy_n),d(Ty_n,fx_n)\}\\ &= \max\{0_{\mathbb{A}},0_{\mathbb{A}},d(z,fx),0_{\mathbb{A}},d(z,fx)\}\\ &= d(z,fx). \end{aligned}$$

Using this value in (18) together with $\|a\| < 1$, we get

$$\|d(fx,z)\| \leq \|a\|^2\|d(fx,z)\| < \|d(fx,z)\|,$$

a contradiction. Thus, $fx = z$, that is, $\lim_{n\to\infty} Ty_n = fx$. Then, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Ty_n = fx = z.$$

Now, we shall show that $Sx = z$. Let, if possible, $Sx \neq z$. From (2), we have $d(Sx,Ty_n) \preccurlyeq a^*m(x,y_n)a$, i.e. $\lim_{n\to\infty} d(Sx,Ty_n) \preccurlyeq a^* \lim_{n\to\infty} m(x,y_n)a$ which implies that

$$d(Sx,z) \preccurlyeq a^* \lim_{n\to\infty} m(x,y_n)a, \tag{17}$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x,y_n) &= \lim_{n\to\infty} \max\{d(fx,gy_n), d(Sx,fx),\\ &\quad d(Ty_n,gy_n), d(Sx,gy_n), d(Ty_n,fx)\}\\ &= \max\{d(z,z), d(Sx,z), d(z,z),\\ &\quad d(Sx,z), d(z,z)\}\\ &= d(Sx,z).\end{aligned}$$

Using this value in (19) together with $\|a\| < 1$, we get

$$\|d(Sx,z)\| \leq \|a\|^2 \|d(Sx,z)\| < \|d(Sx,z)\|,$$

a contradiction. Therefore, we get $Sx = z = Ax$. Since, the pair (f,S) is weakly compatible, it follows that $fx = Sz$. Moreover, since $Sx \subseteq gX$, there exists some y in X such that $Sx = gy$, i.e. $gy = z$. Now, we show that $Ty = z$. Let, if possible, $Ty \neq z$. Using (2), we get

$$\lim_{n\to\infty} d(Sx_n,Ty) \preccurlyeq a^* \lim_{n\to\infty} m(x_n,y)a, \tag{18}$$

this implies that

$$d(z,Ty) \preccurlyeq a^* \lim_{n\to\infty} m(x_n,y)a,$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x_n,y) &= \max\{d(fx_n,gy), d(Sx_n,fx_n), d(Ty,gy),\\ &\quad d(Sx_n,gy), d(Ty,fx_n)\}\\ &= \max\{d(z,z), d(z,z), d(z,Ty),\\ &\quad d(z,z), d(Ty,z)\}\\ &= d(z,Ty).\end{aligned}$$

Making use of this value in (20) alongwith $\|a\| < 1$, we get

$$\|d(z,Ty)\| \leq \|a\|^2 \|d(z,Ty)\| < \|d(z,Ty)\|,$$

a contradiction. Thus, $z = Ty = gy$. Since the pair (g,T) is weakly compatible, it follows that $Tz = gz$. Now, we claim that $Sz = Tz$. Let, if possible, $Sz \neq Tz$. From (2), we have

$$d(Sz,Tz) \preccurlyeq a^*m(z,z)a, \tag{19}$$

where

$$\begin{aligned}m(z,z) &= \max\{d(fz,gz), d(Sz,fz), d(gz,Tz),\\ &\quad d(Sz,gz), d(Tz,fz)\}\\ &= d(Sz,Tz).\end{aligned}$$

Using this value in (21) along with $\|a\| < 1$, we get

$$\|d(Sz,Tz)\| \leq \|a\|^2 \|d(Sz,Tz)\| < \|d(Sz,Tz)\|,$$

a contradiction. Therefore, $Sz = Tz$, i.e. $fz = Sz = Tz = gz$. Now, we shall show that $z = Tz$. Let, if possible, $z \neq Tz$. Putting $y = z$ in (2), we get

$$d(z,Tz) = d(Sx,Tz) \preccurlyeq a^*m(x,z)a, \tag{20}$$

where

$$\begin{aligned}m(x,z) &= \max\{d(fx,Sz), d(Sx,fx), d(gz,Tz),\\ &\quad d(Sx,gz), d(Tz,fx)\}\\ &= d(Sx,Tz)\\ &= d(z,Tz).\end{aligned}$$

Putting this value in (22) and using $\|a\| < 1$, we get

$$\|d(z,Tz)\| \leq \|a\|^2 \|d(z,Tz)\| < \|d(z,Tz)\|,$$

a contradiction. Therefore, $z = Tz = gz = fz = Sz$. Hence z is the common fixed point of f,g,S and T. Uniqueness follows easily as in the previous theorem.

Now, we provide the following example, which illustrates Theorem 2.1.

Example 2.1. $X = [0,1]$, and $\mathbb{A} = M_2(\mathbb{C})$, the set of bounded linear operators on a Hilbert space $\mathbb{C}^2$. Define $d : X \times X \to \mathbb{A}$ by

$$d(x,y) = \begin{bmatrix} |x-y| & 0 \\ 0 & k|x-y| \end{bmatrix},$$

where $k \geq 0$ is a constant. Then, $(X,\mathbb{A},d)$ is a C^*-algebra-valued metric space.

Define the four self maps f,g,S and T on X by $fx = x$, $gx = \frac{x}{4}$, $Sx = \frac{x}{8}$, $Tx = \frac{x}{2}$ for all $x \in X$. Clearly,

$$SX = [0,\frac{1}{8}] \subseteq [0,\frac{1}{4}] = gX,\ TX = [0,\frac{1}{2}] \subseteq [0,1] = fX.$$

Also, fX is complete subspace of X and the pairs (f,S) and (g,T) are weakly compatible.

$$d(Sx,Ty) = \begin{bmatrix} |\frac{x}{8}-\frac{y}{2}| & 0 \\ 0 & k|\frac{x}{8}-\frac{y}{2}| \end{bmatrix},\ d(fx,gy) = \begin{bmatrix} |x-\frac{y}{4}| & 0 \\ 0 & k|x-\frac{y}{4}| \end{bmatrix},$$

$$d(Sx,fx) = \begin{bmatrix} |\frac{x}{8}-x| & 0 \\ 0 & k|\frac{x}{8}-x| \end{bmatrix},\ d(Ty,gy) = \begin{bmatrix} |\frac{y}{2}-\frac{y}{4}| & 0 \\ 0 & k|\frac{y}{2}-\frac{y}{4}| \end{bmatrix},$$

$$d(Ty,fx) = \begin{bmatrix} |\frac{y}{2}-x| & 0 \\ 0 & k|\frac{y}{2}-x|\} \end{bmatrix},\ d(Sx,gy) = \begin{bmatrix} |\frac{x}{8}-\frac{y}{4}| & 0 \\ 0 & k|\frac{x}{8}-\frac{y}{4}| \end{bmatrix}$$

Observe that $d(Sx,Ty) \preccurlyeq a^*m(x,y)a$ (for all $x,y \in X$) where $m(x,y)$ is defined in (2) with

$$a = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \in A \text{ and } \|a\| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} < 1.$$

Here, 0 is the unique common fixed point of f,g,S and T.

3 Application

As an application, we solve the following integral equation:
Consider the integral equation

$$x(t) = \int_E (K_1(t,s,x(s)+K_2(t,s,x(s))))ds + g(t), t \in E,$$

where E is a measurable set. Let $X = L^\infty(E)$ and $H = L^2(E)$. Define $d : X \times X \to L(H)$ by (for all $f,g \in X$ and $\| a \| = k < 1$):

$$d(f,g) = \pi_{|f-g|},$$

where $\pi_h : H \to H$ is the multiplicative operator defined by: $\pi_h(\psi) = h.\psi$.

Now, we are equipped to state and prove our result, as follows:

Theorem 3.1. Assume that, for all $u,v \in L^\infty(E)$,

(1) $K_1 : E \times E \times \mathbb{R} \to [0,+\infty), K_2 : E \times E \times \mathbb{R} \to (-\infty,0]$ and $g \in L^\infty(E)$;
(2) there exists a continuous function $\psi : E \times E \to \mathbb{R}$ and $k \in (0,1)$ such that (for $t,s \in E$ and $u,v \in \mathbb{R}$)

$$| K_1(t,s,u) - K_2(t,s,v) | \leq k | \psi(t,s) || u - v |,$$

(3) $sup_{t\in E} \int_E | \psi(t,s) | ds \leq 1.$

Then, the integral equation has a solution in $L^\infty(E)$.

Proof. Let $S,T : L^\infty(E) \to L^\infty(E)$ be given by:

$$Sx(t) = \int_E K_1(t,s,x(s))ds + g(t)$$

and

$$Tx(t) = \int_E K_2(t,s,x(s))ds + g(t).$$

Set $a = kI$, then $a \in L(H)$. For any $h \in H$, we have

$$\begin{aligned}
\| d(Sx,Ty) \| &= sup_{\|h\|=1}(\pi_{|Sx-Ty|}h,h)\\
&= sup_{\|h\|=1} \int_E [| \int_E K_1(t,s,x(s)) -\\
&\quad K_2(t,s,x(s))ds |]h(t)\bar{h(t)}dt\\
&\leq sup_{\|h\|=1} \int_E [\int_E | K_1(t,s,x(s)) -\\
&\quad K_2(t,s,x(s)) | ds] | h(t) |^2 dt\\
&\leq sup_{\|h\|=1} \int_E [\int_E | k\psi(t,s)(x(s) -\\
&\quad y(s)) | ds] | h(t) |^2 dt\\
&\leq ksup_{\|h\|=1} \int_E [\int_E | \psi(t,s) | ds] | h(t) |^2 dt\\
&\quad \times \| x - y \|_\infty\\
&= k \| x - y \|_\infty\\
&= \| a \| \| d(x,y) \| .
\end{aligned}$$

Since, $\| a \| < 1$, so using Corollary 2.1, we can say that S and T have a unique common fixed point.

Acknowledgement

The authors are grateful to the anonymous referee for the constructive and beneficial comments that improved the present paper.

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals, *Fund. Math.*, **3**, 133-181, 1922.
[2] Y.J. Cho and N. Petrot, Existence theorems for fixed fuzzy points with closed α-cut sets in complete metric spaces, *Commun. Korean Math. Soc.*, **26(1)**, 115-124, 2011.
[3] D. Qiu and L. Shu, Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings, *Information Sciences*, **178**, 3595-3604, 2008.
[4] R. A. Rashwan and M. A. Ahmad, Common fixed point theorems for fuzzy mappings, *Arch. Math. (Brno)*, **38(3)**, 219-226, 2002.
[5] Z. Shi-sheng, Fixed point theorems for fuzzy mappings (II), *Appl. Math. Mech.*, **7(2)**, 147-152, 1986.
[6] W. Liguang, L. Bo and B. Ran, Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach, *Fixed Point Theory and Applications*, **2010**, Article ID 283827, 2010.
[7] X. Du and Z. Zhao, On Fixed Point Theorems of Mixed Monotone Operators, *Fixed Point Theory and Applications*, **2011**, ArticleID 563136, 1-8 ,doi:10.1155/ /563136, 2011.
[8] X. Li, Z. Zhao, On a fixed point theorem of mixed monotone operators and applications, *Electronic Journal of Qualitative Theory of Differential Equations*, **(94)**, 1-7, 2012.
[9] X. Zhang, L. Liu, Y. Wu, Fixed point theorems for the sum of three classes of mixed monotone operators and applications, *Fixed Point Theory and Applications*, **2016:49**, DOI 10.1186/s13663-016-0533-4, 2016.
[10] Z. Zhao, Existence and uniqueness of fixed points for some mixed monotone operators, *Nonlinear Analysis*, **73(6)**, 1481-1490, 2010.
[11] Z. Zhao, Fixed points of $\tau - \varphi -$convex operators and applications, *Applied Mathematics Letters*, **23(5)**, 561-566, 2010.
[12] J. Mao, Z. Zhao, N. Xu, The existence and uniqueness of positive solutions for integral boundary value problems, *Electronic Journal of Qualitative Theory of Differential Equations*, **16**, 1-8, 2010.
[13] T. Wang, R. Xu, Bounds for Some New Integral Inequalities With Delay on Time Scales, *Journal of Mathematical Inequalities*, **6(1)**, 1-12, 2012.
[14] X. Hao, L. Liu, Y. Wu, Positive solutions for second order differential systems with nonlocal conditions, *Fixed Point Theory*, **13(2)**, 507-516, 2012.
[15] G. Wang, H.T.,Che, H.B., Chen, Feasibility-solvability theorems for generalized vector equilibrium problem in reflexive Banach spaces, *Fixed Point Theory and Applications*, **2012**, 1-13, 2012.
[16] L. Liu, X. Zhang, J. Jiang, Y. Wu, The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems, *J. Nonlinear Sci. Appl.*, **9(5)**, 2943-2958, 2016.

[17] B. Singh, V. Gupta and S. Kumar, Common Fixed Point Theorems Using the E.A. and CLR Properties in 2-Menger Spaces, *International Journal of Analysis*, **2013**, Article ID 934738, 7 pages, 2013.
[18] Z. Ma, L. Jiang, H. Sun, C^*-algebra valued metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, **2014**, 11 pages, 2014.
[19] M. Asim and M. Imdad, C^*-algebra valued extended *b*-metric spaces and fixed point results with an application, *U.P.B. Sci. Bull., Series A*, **82(1)**, 207-218, 2020.
[20] M. Asim and M. Imdad, C^*-algebra valued symmetric spaces and fixed point results with an application, *Korean J. Math.*, **28(1)**, 17-30, 2020.
[21] S. Radenovic, P. Vetro, A. Nastasi, LT. Quan, Coupled Fixed Point Theorems in C^*-Algebra-Valued b-Metric Spaces, *S.P.S.U. of Novi pazar, Ser. A: appl. math. informarm. andmech.*, **9(1)**, 81-90, 2017.
[22] K.R. Davidson, C^*-algebras by example, *Fields institute monographs 6, American Mathematical Society, Providence, RI*, 1996.
[23] G.J. Murphy, C^*-algebras and operator theory, *Academic Press, Boston*, 1990.
[24] G. Jungck, Common fixed points for non-continuous non-self mappings on non-metric spaces, *Far East J. Math. Sci.*, **4**, 199-212, 1996.
[25] C. Vetro, On Branciaris theorem for weakly compatible mappings, *Appl. Math. Lett.*, **23**, 700-705, doi: 10.1016/j.aml.2010.02.011, 2010.
[26] M. Aamri, D.El. Moutawakil: Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, **270**, 181-188, doi: 10.1016/S0022-247X(02)00059-8, 2002.
[27] W. Sintunavarat, P. Kumam: Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, *J. Appl. Math.*, **2011**, Article ID 637958, 14 pages, doi: 10.1155/2011/637958, 2011.
[28] Q. Xin, L. Jiang, Z. Ma: Common fixed point theorems in C^*-algebra valued metric spaces , *Journal of Nonlinear Sciences and Applications*, **9**, 4617-4627, 2016.