

# Homomorphisms of $C^*$ -Algebras

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**Abstract:** In this note we give a straightforward proof of the fact that every continuous homomorphism from a  $C^*$ -algebra into a weakly sequential complete Banach algebra is a finite rank operator. We also study Dieudonne type homomorphisms of the unital  $C^*$ -algebras.

**Keywords:**  $C^*$ -algebra, homomorphism,  $V$ -algebra, Dieudonne operator.

## 1. Introduction

Recently, many authors have been interested in the structure of compact and weakly compact homomorphisms of Banach algebras [4,5,7]. In particular, homomorphisms of  $C^*$ -algebras have been studied extensively in the literature. In [5], Ghahramani proved that every compact homomorphism from a  $C^*$ -algebra is a finite rank operator. Extending this result, Galé-Ransford-White [4] proved that every weakly compact homomorphism from a  $C^*$ -algebra is a finite rank operator. Mathieu [7] give more elementary proof of the Galé-Ransford-White result.

Let  $K$  be a compact Hausdorff space and let  $C(K)$  be the space of all continuous functions on  $K$ . It is well known [6] that an arbitrary bounded linear operator from  $C(K)$  into a weakly sequentially complete Banach space is weakly compact. Generalizing this result, Akemann-Dodds-Gamlen [1] proved that an arbitrary bounded linear operator from a  $C^*$ -algebra into a weakly sequentially complete Banach space is weakly compact. Combining the Akemann-Dodds-Gamlen result with the Galé-Ransford-White result, we can assert that every continuous homomorphism from a  $C^*$ -algebra into a weakly sequential complete Banach algebra is a finite rank operator. In this note, we give more elementary proof of the last result without using of Akemann-Dodds-Gamlen Theorem. We also study Dieudonne type homomorphisms of the unital  $C^*$ -algebras.

## 2. $C^*$ -Algebras

Let  $X$  be a complex Banach space and let  $X^*$  be its dual. A sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  such that  $(\varphi(x_n))_{n \in \mathbb{N}}$  is a Cauchy sequence of scalars for each  $\varphi \in X^*$  is called a *weak Cauchy sequence*. Recall that the space  $X$  is said to be *weakly sequentially complete* if every weak Cauchy sequence has a weak limit. In this section, we prove the following

**Theorem 1.** *Every continuous homomorphism from a  $C^*$ -algebra into a weakly sequentially complete Banach algebra is of finite rank.*

For the proof we need some preliminary results.

Let  $A$  be an arbitrary complex unital Banach algebra with the unit element  $1_A$ . We will denote by  $S(A)$  the set of all normalized states on  $A$ , namely,

$$S(A) = \{\Phi \in A^* : \|\Phi\| = \Phi(1_A) = 1\}.$$

An element  $h \in A$  is said to be *Hermitian* if  $\Phi(h) \in \mathbb{R}$  for all  $\Phi \in S(A)$ . It is well known [2, Corollary 10.13] that  $h \in A$  is Hermitian if and only if  $\|\exp(ith)\| = 1$  for all  $t \in \mathbb{R}$ . For example, if  $A$  is a unital  $C^*$ -algebra, then  $h \in A$  is Hermitian if and only if  $h$  is self-adjoint. Furthermore, each  $a \in A$  can be written as  $a = h + ik$ , where  $h$  and  $k$  are self-adjoint elements of  $A$ .

By  $Her(A)$  we will denote the set of all Hermitian elements of  $A$ . It can be seen that  $Her(A)$  is a closed real subspace of  $A$ . The algebra  $A$  is said to be a  *$V$ -algebra* if each  $a \in A$  is of the form  $a = h + ik$ , where  $h, k \in Her(A)$ . The Vidav-Palmer Theorem [2, Theorem

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38.14] states that a  $V$ -algebra with involution defined by  $(h + ik)^* = h - ik$  is a  $C^*$ -algebra. Recall also that for an arbitrary  $h \in \text{Her}(A)$ ,

$$\|h\| = \sup \{ |\Phi(h)| : \Phi \in S(A) \} \quad (1)$$

(see [2, Theorem 10.17 and Lemma 38.3]).

Let  $A$  be an arbitrary complex Banach algebra. It is well known [3] that the second dual  $A^{**}$  of  $A$  can be equipped with two Banach algebra multiplications  $\circ$  and  $*$  (the first and the second Arens multiplication) which extend the original multiplication in  $A$  (canonically embedded into  $A^{**}$ ). Namely, for  $a \in A$ ,  $\varphi \in A^*$ , and  $F, G \in A^{**}$  we set  $\langle F \circ G, \varphi \rangle = \langle F, G \cdot \varphi \rangle$  and  $\langle F * G, \varphi \rangle = \langle G, \varphi \cdot F \rangle$ , where  $G \cdot \varphi$  and  $\varphi \cdot F$  are functionals on  $A$  defined by  $\langle G \cdot \varphi, a \rangle = \langle G, \varphi \cdot a \rangle$  and  $\langle \varphi \cdot F, a \rangle = \langle F, a \cdot \varphi \rangle$ . If  $F \circ G = F * G$  for every  $F, G \in A^{**}$ , then  $A$  is said to be *Arens regular*. For example,  $C^*$ -algebras are Arens regular [3].

**Lemma 1.** *Let  $A$  be a unital  $C^*$ -algebra and let  $B$  be an arbitrary complex Banach algebra. If there exists a contractive homomorphism  $\omega : A \mapsto B$  with dense range, then  $B$  is also a  $C^*$ -algebra and  $B$  is  $*$ -isomorphic to a quotient  $C^*$ -algebra of  $A$ .*

*Proof.* Since  $\ker(\omega)$  is a closed two-sided ideal of  $A$ ,  $\ker(\omega)$  is self-adjoint. Hence the quotient algebra  $A/\ker(\omega)$  is a unital  $C^*$ -algebra. We can see that the induced mapping  $\tilde{\omega} : A/\ker(\omega) \mapsto B$ , defined by  $\tilde{\omega}(a + \ker\omega) = \omega(a)$  is an contractive and injective homomorphism with dense range. Hence, we can suppose without loss of generality that  $\omega$  is injective. Next, we will prove that  $B$  is a  $C^*$ -algebra and  $B$  is  $*$ -isomorphic to the algebra  $A$ . We can easily see that  $\omega(1_A)$  is the unit element of  $B$  and  $\|\omega(1_A)\| = 1$ . Hence,  $B$  is unital. Now let  $h \in \text{Her}(A)$ . Since  $\omega(\exp(it h)) = \exp(it \omega(h))$ ,  $t \in \mathbb{R}$ , we have  $\|\exp(it \omega(h))\| \leq \|\omega\| \|\exp(it h)\| \leq 1$ , for all  $t \in \mathbb{R}$ . It follows that  $\omega(h) \in \text{Her}(B)$ . Let us show that  $B$  is a  $V$ -algebra. To see this, let  $b \in B$  be given. Then, there exists a sequence  $(a_n)_{n \in \mathbb{N}}$  in  $A$  such that  $\omega(a_n) \rightarrow b$ . Let  $a_n = h_n + ik_n$  ( $n = 1, 2, \dots$ ), where  $(h_n)_{n \in \mathbb{N}}$  and  $(k_n)_{n \in \mathbb{N}}$  are the sequences in  $\text{Her}(A)$ . Then, we have that  $\omega(h_n) + i\omega(k_n) \rightarrow b$ . Hence, for an arbitrary  $\varepsilon > 0$ , there exists an integer  $N$  such that  $\|\omega(h_n) - \omega(h_m) + i(\omega(k_n) - \omega(k_m))\| \leq \varepsilon$ , for all  $n, m > N$ . Since  $\{\omega(h_n) - \omega(h_m)\}$  and  $\{i(\omega(k_n) - \omega(k_m))\}$  are in  $\text{Her}(B)$ , it follows that for all  $\Phi \in S(B)$ ,  $|\Phi(\omega(h_n) - \omega(h_m))| \leq \varepsilon$ ,  $|\Phi(i(\omega(k_n) - \omega(k_m)))| \leq \varepsilon$ . Taking into account (2.1), we obtain  $\|\omega(h_n) - \omega(h_m)\| \leq \varepsilon$ ,  $\|\omega(k_n) - \omega(k_m)\| \leq \varepsilon$ . Since  $\text{Her}(B)$  is a real Banach space, there exist Hermitian elements  $l$  and  $m$  in  $B$  such that  $\omega(h_n) \rightarrow l$  and  $\omega(k_n) \rightarrow m$ . Consequently, we have  $b = l + im$ , where  $l, m \in \text{Her}(B)$ . Thus  $B$  is a  $V$ -algebra. By the Vidav-Palmer Theorem [2, Theorem 38.14],  $B$  is a  $C^*$ -algebra with the involution defined by  $b^* = l - im$ . Furthermore, for an arbitrary

$a = h + ik \in A$ , we have

$$\begin{aligned} \omega(a^*) &= \omega(h - ik) = \omega(h) - i\omega(k) \\ &= (\omega(h) + i\omega(k))^* = (\omega(h + ik))^* = \omega(a)^*. \end{aligned}$$

Therefore,  $\omega$  is a  $*$ -homomorphism. By [12, Corollary 1.2.6],  $\omega$  is an isometry. Since  $\omega$  has dense range,  $\omega$  is a surjective isometry. Hence  $\omega$  is a  $*$ -isomorphism. This completes the proof.

*Proof(Proof of Theorem 1).* Let  $A$  be  $C^*$ -algebra and let  $B$  be a weakly sequentially complete Banach algebra. Let  $\omega : A \mapsto B$  be a continuous homomorphism. Since the space  $\overline{\omega(A)}$  is weakly sequentially complete, we lose no generality if we assume that  $\overline{\omega(A)} = B$ . Furthermore, since  $A$  is Arens regular and  $B$  is a weakly sequentially complete, by [13, Theorem 4.1],  $B$  has the unit element  $1_B$ . Let  $(e_i)_{i \in I}$  be an approximate identity for  $A$  such that  $\sup_i \|e_i\| \leq 1$ . Then,  $(\omega(e_i))_{i \in I}$  is a bounded approximate identity for  $B$ . It follows that  $\omega(e_i) \rightarrow 1_B$ . Let  $A \oplus C$  be the  $C^*$ -unitization of  $A$  with the norm  $\|a + \lambda\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$ . Then the mapping  $\tilde{\omega} : A \oplus C \mapsto B$ , defined by  $\tilde{\omega}(a + \lambda) = \omega(a) + \lambda 1_B$  is a homomorphism with dense range. Moreover, since  $\tilde{\omega}(a + \lambda) = \lim_i \omega(ae_i + \lambda e_i)$ ,  $\|\tilde{\omega}(a + \lambda)\| \leq \|\omega\| \sup_i \|ae_i + \lambda e_i\| \leq \|\omega\| \|a + \lambda\|$ . Hence,  $\tilde{\omega}$  is bounded. This shows that  $\omega$  can be extended to  $A \oplus C$  as a continuous homomorphism. Therefore, we may assume that  $A$  has a unit element. Renorming  $B$  if necessary, we can assume that  $\omega$  is contractive. By the preceding lemma,  $B$  is a  $C^*$ -algebra. But we know that weakly sequentially complete  $C^*$ -algebras are finite-dimensional [11, Proposition 2]. Hence,  $\omega$  is a finite rank operator. The proof is complete.

### 3. Dieudonne Type Homomorphisms

Let  $X$  and  $Y$  be two Banach spaces and let  $T : X \mapsto Y$  be a bounded linear operator. The operator  $T$  is said to be a *Dieudonne operator* if  $T$  sends weakly Cauchy sequences in  $X$  into weakly convergent ones (see [6]). For example, if either  $X$  or  $Y$  is a weakly sequentially complete, then every bounded linear operator  $T : X \mapsto Y$  is a Dieudonne operator. Assume that the operator  $\bar{T} : X/\ker T \mapsto Y$  is defined by  $\bar{T}(x + \ker T) = Tx$ . One can easily see that if  $\bar{T}$  is a Dieudonne operator, then so is  $T$ . The following example shows that the converse is not true in general.

**Example** Let  $G$  be a non-discrete locally compact abelian group and let  $A(G)$  be the Fourier algebra of  $G$ . For a compact subset  $K$  of  $G$ , we denote by  $A(K)$ , the algebra of all functions on  $K$  which are the restrictions to  $K$  of the functions in  $A(G)$  with the norm

$$\|f\|_{A(K)} = \inf \left\{ \|h\|_{A(K)} : h|_K = f \right\}.$$

Clearly, the algebra  $A(K)$  can be identified with the quotient algebra  $A(G)I_K$ , where  $I_K$  is the largest

closed ideal in  $A(G)$  whose hull is  $K; I_K = \{f \in A(G) : f(K) = \{0\}\}$ . Recall that  $K$  is said to be a *Helson set* if every  $f \in C(K)$  is the restriction to  $K$  of a member of  $A(G)$ . It can be seen that if  $K$  is a Helson set, then  $A(K)$  is isomorphic to  $C(K)$ . As is known [10, Chapter 5], there exists a Helson set in any non-discrete locally compact abelian group. Since  $A(G)$  is a weakly sequentially complete, the canonical quotient map  $\pi : A(G) \mapsto A(K)$  is a Dieudonne operator. Now, assume on the contrary that  $\bar{\pi}$  is also a Dieudonne operator. Since  $\bar{\pi}$  is the identity operator on  $A(K)$ , it follows that  $A(K)$  is a weakly sequentially complete. But this is not possible if  $K$  is an infinite Helson set. We say that  $T : X \mapsto Y$  is a *Dieudonne type operator* if  $\bar{T}$  is a Dieudonne operator.

**Proposition 1.** *If  $T : X \mapsto Y$  is a weakly compact linear operator, then  $T$  is a Dieudonne type operator.*

*Proof.* Assume that  $T$  is weakly compact. Let  $\pi : X \mapsto X/\ker T$  be the canonical quotient map. Then,  $T = \bar{T} \circ \pi$ . Since  $\pi$  is open, every bounded subset of  $X/\ker T$  is the image of some bounded set in  $X$ . It follows that  $\bar{T}$  is also a weakly compact operator. Thus we can assume that  $T$  is injective. Let us show that  $T$  is a Dieudonne operator. Let  $(x_n)_{n \in \mathbb{N}}$  be a weakly Cauchy sequence in  $X$ . Since the sequence  $(x_n)_{n \in \mathbb{N}}$  is bounded, there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  such that  $(Tx_{n_k})_{k \in \mathbb{N}}$  is weakly convergent. Since  $(Tx_n)_{n \in \mathbb{N}}$  is a weakly Cauchy sequence, it follows that the sequence  $(Tx_n)_{n \in \mathbb{N}}$  converges weakly.

Let  $X$  be an infinite dimensional non-reflexive and weakly sequentially complete Banach space (for instance, such as  $L^1(\mu)$ ). Then the identity operator on  $X$  is a Dieudonne operator but not weakly compact. Now assume that  $T : X \mapsto Y$  is a Dieudonne operator. We remark that if  $X$  does not contain an isomorphic copy of  $\ell^1$ , then  $T$  is weakly compact. To see this, let  $(x_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $X$ . By Rosenthal's  $\ell^1$ -theorem [9], the sequence  $(x_n)_{n \in \mathbb{N}}$  has a weakly Cauchy subsequence  $(x_{n_k})_{k \in \mathbb{N}}$ . Since  $\bar{T}$  is a Dieudonne operator, the sequence  $(Tx_{n_k})_{k \in \mathbb{N}}$  is weakly convergent.

Let  $A$  and  $B$  be two Banach algebras and let  $\theta : A \mapsto B$  be a continuous homomorphism. We say that  $\theta$  is a *Dieudonne type homomorphism* if  $\bar{\theta}$  is a Dieudonne type operator.

**Corollary 1.** *Every Dieudonne type homomorphism from a unital  $C^*$ -algebra into a Banach algebra is of finite rank.*

*Proof.* Let  $\theta : A \mapsto B$  be such a homomorphism. We can suppose without loss of generality that  $\theta$  has dense range. Renorming  $B$  if necessary, we can assume that  $\theta$  is contractive. By Theorem 2.1,  $B$  is a  $C^*$ -algebra and  $\bar{\theta}$  is a  $*$ -isomorphism between  $A/\ker \theta$  and  $B$ . Let  $(b_n)_{n \in \mathbb{N}}$  be a weakly Cauchy sequence in  $B$ . Then,  $(\bar{\theta}^{-1}(b_n))_{n \in \mathbb{N}}$  is a weakly Cauchy sequence in  $A/\ker \theta$ . Since  $\bar{\theta}$  is a Dieudonne operator, it follows that the sequence  $(b_n)_{n \in \mathbb{N}}$

converges weakly. Hence,  $B$  is a weakly sequentially complete  $C^*$ -algebra. However, weakly sequentially complete  $C^*$ -algebras are finite-dimensional [11, Proposition 2].

## References

- [1] A. Akemann, P.G. Dodds, and J.L.B. Gamlen, Weak compactness in the dual space of a  $C^*$ - algebra, *J. Funct. Anal.* **10**(1972), 446-450.
- [2] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, 1973.
- [3] J. Duncan and S.A.R. Hosseniun, The second dual of a Banach algebra, *Proc. Roy. Soc. Edinburgh*, A.**84** (1979), 309-325.
- [4] J.E. Galé, T J. Ransford, and M.C. White, Weakly compact homomorphisms, *Trans. Amer. Math. Soc.* **331**(1992), 815-824.
- [5] F. Ghahramani, Compact homomorphisms of  $C^*$ - algebras, *Proc. Amer. Math. Soc.* **103**(1988), 458-462.
- [6] A. Grothendieck, Sur les applications lineaires faiblement compactes d'espaces du type  $C(K)$ , *Canad. J. Math.* **5**(1953), 129-173.
- [7] M. Mathieu, Weakly compact homomorphisms from  $C^*$ - algebras are of finite rank, *Proc. Amer. Math. Soc.* **107**(1989), 761-762.
- [8] H. Mustafayev and C. Temel, Compact homomorphisms of regular Banach algebras, *Math. Nachr.* **284**(2011), 518-525.
- [9] H. P. Rosenthal, A characterization of Banach space containing  $\ell^1$ , *Proc. Nat. Acad. Sci. U.S.A.* **71**(1974).
- [10] W. Rudin, *Fourier Analysis on Groups*, New York, Interscience, 1962.
- [11] S. Sakai, Weakly compact operators on operator algebras, *Pacific J. Math.* **14**(1964), 659-664.
- [12] S. Sakai,  *$C^*$ -Algebras and  $W^*$ -Algebras*, Springer-Verlag, 1971.
- [13] A. Ülger, Arens regularity of weakly sequentially complete Banach algebras, *Proc. Amer. Math. Soc.* **127**(1999), 3221-3227.



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