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Solitons and Periodic Solutions to Nonlinear Partial Differential Equations by the Sine-Cosine Method

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Abstract: In this paper, we construct new solitary solutions to nonlinear PDES by the Sine-Cosine method. Moreover, the periodic solutions and bell-shaped solitons solutions to the Benjamin-Bona-Mahony and the Gardner equations are obtained. New solution to the Cassama-Holm equation is also obtained. Finally, the solution of a two-component evolutionary system of a homogeneous Kdv equations of order 2 has been investigated by the proposed method.

Keywords: Wave variables, Sine-Cosine Method, Nonlinear PDEs.

1. Introduction

In this paper, we study the solutions of the nonlinear Benjamin-Bona-Mahony equation, Gardner equation, Cassama-Holm equation and the two-component evolutionary system of a homogeneous KDV equations given, respectively, by

$$u_t = u_{xxt} - u_x - uu_x, \tag{1}$$

$$u_t = u_{xxx} + 6uu_x, \tag{2}$$

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{3}$$

$$u_t = -3v_{xx} \quad , \quad v_t = u_{xx} + 4u^2 \tag{4}$$

Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. In the past decade, many significant methods have been proposed for obtaining solutions of nonlinear partial differential equations such as the Sinc-Galerkin method [12,3], the finite difference method [2], the Adomian decomposition method [11], the Differential transform method [13], the extended tanh-function method [7,9,10,8], the Exp-function method [1], the direct algebraic method, Hirota's method, inverse scattering method, Backlund transformation, the Wadati trace method, Hirota bilinear forms, pseudo spectral method, the tanh-sech method, the Riccati equation expansion method and so on.

The main aim of this paper is to apply the Sine-Cosine function method with the help of symbolic computation to obtain new soliton solutions of (1, 2, 3) and the nonlinear system (4). By using Sine-Cosine function method, many kinds of nonlinear partial differential equations arising in mathematical physics have been solved successfully [4–6].

2. The Sine-Cosine Method

Since we restrict our attention to traveling waves, we use the transformation $u(x,t) = u(\zeta)$, where the wave variable $\zeta = x - ct$, converts the the nonlinear PDE to an equivalent ODE. The sine-cosine algorithm admits the use of the ansatzes

$$u(x,t) = \lambda \cos^{\beta}(\mu\zeta), \quad |\zeta| \leq \frac{\pi}{2\mu}, \tag{5}$$

$$u(x,t) = \lambda \sin^{\beta}(\mu\zeta), \quad |\zeta| \leq \frac{\pi}{2\mu}, \tag{6}$$

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where λ, μ, c and β are parameters that will be determined. Substituting (5) or ((6) into the reduced ODE gives a polynomial equation of cosine or sine terms. Balancing the exponents of the trigonometric functions cosine or sine, collecting all terms with same power in $\cos^{\beta}(\mu\zeta)$ or $\sin^{\beta}(\mu\zeta)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns λ, μ and β. The problem is now completely reduced to an algebraic one. Having determined λ, μ, c and β by algebraic calculations or by using computerized symbolic calculations, the solutions proposed in (5) and in (6) follow immediately.

3. Applications

In this section we apply the proposed method to four physical models that admit solitary solutions.

3.1. Benjamin-Bona-Mahony (BBM) equation

Consider the BBM equation

$$u_t = u_{xxt} - u_x - uu_x. \tag{7}$$

Using the wave variable $\zeta = x - ct$ carries (7) into the ODE

$$(1-c)u + \frac{1}{2}u^2 + cu'', \tag{8}$$

obtained after integrating the ODE and setting the constant of integration to zero. Substituting (5) into (8) gives

$$\frac{1}{2}\lambda\cos^{\beta-2}(\mu\zeta)(2c(\beta-1)\beta\mu^2 - 2(c\beta^2\mu^2$$

$$+c-1)\cos^2(\mu\zeta) + \lambda\cos^{\beta+2}(\mu\zeta)) = 0$$

The equation is satisfied only if the following system of algebraic equations hold

$$\begin{aligned} 2+\beta &= 0 \\ \lambda + 2c(\beta-1)\beta\mu^2 &= 0 \\ -2(c\beta^2\mu^2 + c - 1) &= 0 \end{aligned} \tag{9}$$

which leads to

$$\lambda = -\frac{12\mu^2}{1+4\mu^2} \quad , \quad c = \frac{1}{1+4\mu^2}, \tag{10}$$

where μ is any arbitrary constant. Therefore, the solution of (7) is

$$u(x,t) = -\frac{12\mu^2}{1+4\mu^2}\sec^2(x - \frac{1}{1+4\mu^2}t) \tag{11}$$

Now, if we use the ansatze (6) instead of (5), then we get the same system (9) and therefore, the solution is

$$u(x,t) = -\frac{12\mu^2}{1+4\mu^2}\csc^2(x - \frac{1}{1+4\mu^2}t) \tag{12}$$

Figures 3.1 and 3.2 shows the periodicity of the two obtained solutions for the BBM equation.

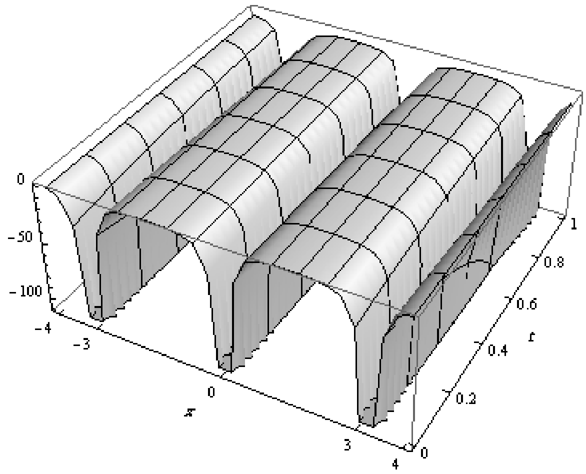

Figure 1 Plot of the first obtained solution for the BBM equation for $\mu = 1$, Example 3.1

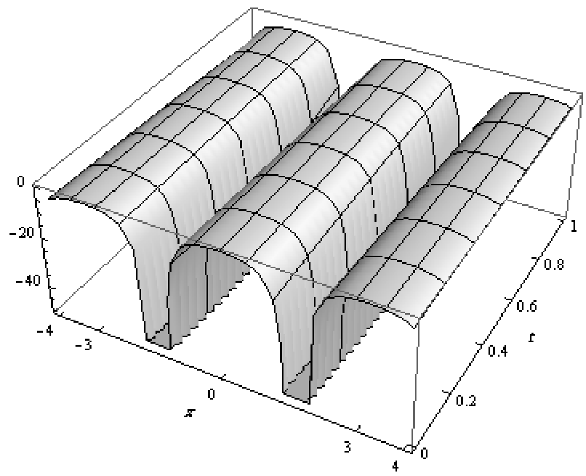

Figure 2 Plot of the Second obtained solution for the BBM equation for $\mu = 1$, Example 3.1

3.2. Gardner equation

Consider the Gardner equation

$$u_t = u_{xxx} + 6uu_x. \tag{13}$$

Using the wave variable $\zeta = x - ct$ carries (13) into the ODE

$$cu + 3u^2 + u'' = 0, \tag{14}$$

obtained after integrating the ODE and setting the constant of integration to zero. Substituting (5) into (14) gives

$$\lambda\cos^{\beta-2}(\mu\zeta)((\beta-1)\beta\mu^2 + (c-\beta^2\mu^2) = 0 \tag{15}$$

$$\cos^2(\mu\zeta) + 3\lambda\cos^{\beta+2}(\mu\zeta))$$

The equation is satisfied only if the following system of algebraic equations hold

$$\begin{aligned} 2+\beta &= 0 \\ 3\lambda + (\beta-1)\beta\mu^2 &= 0 \\ c-\beta^2\mu^2 &= 0 \end{aligned} \tag{16}$$

which leads to

$$\lambda = -2\mu^2 \quad , \quad c = 4\mu^2, \tag{17}$$

where μ is any arbitrary constant. Therefore, the solution of (13) is

$$u(x,t) = -2\mu^2 \sec^2\left(\mu(x-4\mu^2 t)\right). \tag{18}$$

Using (6) the solution is

$$u(x,t) = -2\mu^2 \csc^2\left(\mu(x-4\mu^2 t)\right). \tag{19}$$

3.3. Cassama-Holm equation

Consider the Cassama-Holm equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{20}$$

Using the wave variable $\zeta = x - ct$ carries (20) into the ODE

$$(2k-c)u + \frac{3}{2}u^2 + cu'' - \frac{1}{2}(u'^2 - uu'' = 0, \tag{21}$$

obtained after integrating the ODE and setting the constant of integration to zero. Substituting (5) into (14) gives

$$\begin{aligned} &2c(\beta-1)\beta\mu^2 - 2c(c-2k+c\beta^2\mu^2) \\ &\cos^2(\mu\zeta) - 2\beta(3\beta-2)\lambda\mu^2\cos^\beta(\mu\zeta) \\ &+3\lambda(1+\beta^2\mu^2)\cos^{\beta+2}(\mu\zeta) = 0. \end{aligned} \tag{22}$$

The equation is satisfied only if the following system of algebraic equations hold

$$\begin{aligned} \beta &= 2 \\ -2c(c-2k+c\beta^2\mu^2) - 2\beta(3\beta-2)\lambda\mu^2 &= 0 \\ 3\lambda(1+\beta^2\mu^2) &= 0 \\ 2c(\beta-1)\beta\mu^2 &= 0 \end{aligned} \tag{23}$$

which leads to

$$\lambda = -2k \quad , \quad c = 0,, \quad \mu = \frac{i}{2}, \tag{24}$$

where k is the constant given in the origin equation. Therefore, the solution of (20) is

$$u(x,t) = -2k\cosh^2\left(\frac{x}{2}\right) \tag{25}$$

Using (6) the solution is

$$u(x,t) = 2k\sinh^2\left(\frac{x}{2}\right). \tag{26}$$

respectively of the Cassama-Holm equation.

3.4. Two-component Kdv evolutionary system of order 2

Consider the two-component evolutionary system of a homogeneous Kdv equations of order 2

$$\begin{aligned} u_t &= -3v_{xx} \\ v_t &= u_{xx} + 4u^2 \end{aligned} \tag{27}$$

Using the wave variable $\zeta = x - ct$ carries (27) into the ODE

$$\begin{aligned} -cu' &= -3v'' \\ -cv' &= u''^2. \end{aligned} \tag{28}$$

From (28) we have

$$u = \frac{3}{c}v', \tag{29}$$

and therefore,

$$c^2 u + 12u^2 + 3u'' = 0 \tag{30}$$

Substituting (5) into (30) gives

$$\begin{aligned} &\lambda\cos^{\beta-2}(\mu\zeta)(3(\beta-1)\beta\mu^2 + (c^2 - 3\beta^2\mu^2) = 0 \\ &\cos^2(\mu\zeta) + 12\lambda\cos^{\beta+2}(\mu\zeta)) = 0 \end{aligned} \tag{31}$$

The equation is satisfied only if the following system of algebraic equations hold

$$\begin{aligned} \beta+2 &= 0 \\ 3(\beta-1)\beta\mu^2 + 12\lambda &= 0 \\ c^2 - 3\beta^2\mu^2 &= 0 \end{aligned} \tag{32}$$

which leads to

$$\lambda = -\frac{3\mu^2}{2} \quad , \quad c = 2\sqrt{3}\mu, \tag{33}$$

where μ is any arbitrary constant. Therefore, the solution of (27) is

$$\begin{aligned} u(x,t) &= -\frac{3}{2}\mu^2\sec^2(\mu(x-2\sqrt{3}\mu t) \\ v(x,t) &= -\sqrt{3}\mu^2\tan(\mu(x-2\sqrt{3}\mu t) \end{aligned} \tag{34}$$

Using (6) the solution is

$$\begin{aligned} u(x,t) &= -\frac{3}{2}\mu^2\csc^2(\mu(x-2\sqrt{3}\mu t) \\ v(x,t) &= \sqrt{3}\mu^2\cot(\mu(x-2\sqrt{3}\mu t) \end{aligned} \tag{35}$$

Figures 3.3 and 3.4 shows the periodicity of the solution of the Kdv system.

4. Conclusion

The Sine-Cosine method has been successfully implemented to establish new solitary wave solutions for various type of nonlinear PDEs. Also, we apply the method to solve a two-component evolutionary system of Kdv equations of order 2. The systems of algebraic equations in this paper have been solved by using Mathematica 7.

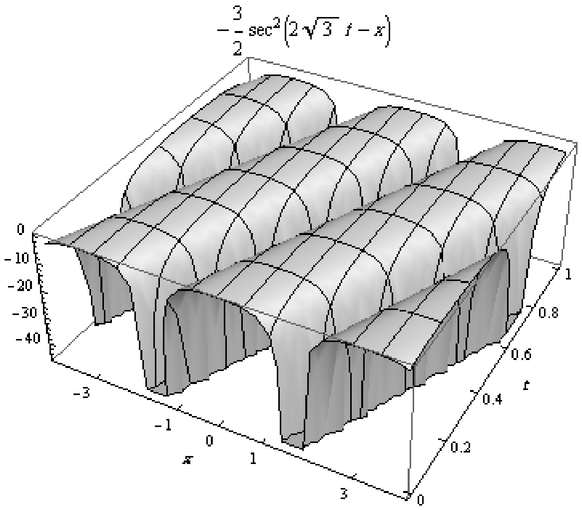

Figure 3 Plot of $u(x,t)$ for the Kdv system for $\mu = 1$, Example 3.4

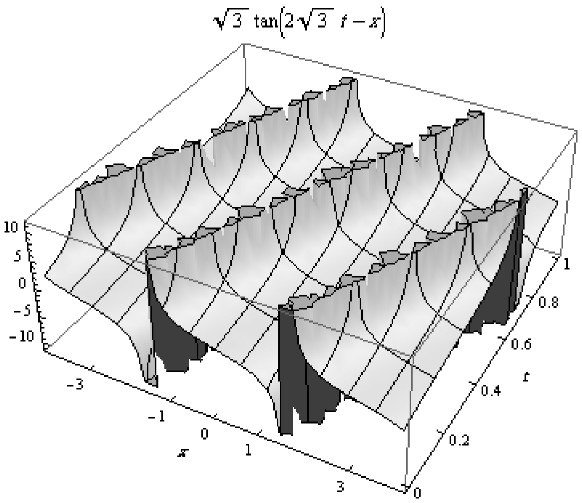

Figure 4 Plot of $v(x,t)$ for the Kdv system for $\mu = 1$, Example 3.4

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