Information Sciences Letters

Volume 9
Issue 3 *Sep. 2020*

Article 4

2020

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Inf. Sci. Lett. **9**, No. 3, 181-188 (2020) 181

Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/090304

Interpolants for linear approximation over convex Polyhedron

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Received: 20 April 2020, Revised: 13 July 2020, Accepted: 18 July 2020
Published online: 1 Sep. 2020

Abstract: Finite element simulation of a 3-D information, despite having enormous significance and applications, could not get its due. Because of the complexity in the formulation of the basis functions, this topic is not much researched. In this paper, we propose a simple formulation of the Wachspress coordinates over a 3-D domain, where each node is considered to be of order 3(i.e. 3 planes do intersect at each node). The concept of barycentric coordinates for a polyhedron was proposed by Wachspress(1975), but being dependent on the Exterior Triple Points (ETPs), the computation of denominator function (adjoint) was quite intricate. Inspired by the simple recursive relation proposed by Dasgupta (2003), and with the help of a property which is being explored in this paper that,"for a polyhedron, wedge functions corresponding to the consecutive nodes which are linear on the common face, attain the same value at the mid point of the edge joining them", a simple recursive relation has been derived in this paper. The entire analysis has been experimented over the convex hexahedron.

Keywords: GADJ, Polyhedronal discretization, Wedge functions, Adjoint, Approximation, Wachspress' coordinates.

1 Introduction

The usual process employed in the computer aided patchwork approximation is to obtain a patch locally on one member of the discretized domain and since C^0-continuity holds globally by construction, these patches are combined together to yield a global approximation over the entire domain. This method of subdividing the domain is widely known as the Finite Element Method. The use of finite element method algorithm in the field of computer graphics and geometric modeling got enhanced with the use of barycentric coordinates [1,2,3,4]. Barycentric coordinates initiated by Möbius [5], were the coordinates capable of representing any point within an m simplex in an n dimensional Euclidean space($m \leq n+1$) as a convex combination of its vertices [6,7,8]. Barycentric coordinates have a wide range of applications, its simplicity in interpolation makes it a very useful tool for interpolation. The shading methods like Gauraud shading method, Phong's shading methods, generalized Phong shading method etc., are some good applications of barycentric coordinates. There are several other fields of engineering and mathematical physics where the barycentric coordinates have a special importance.

Inspired by the barycentric coordinates Kalman [9] proposed the generalized barycentric coordinates in 1961. The following theorem due to Kalman plays a key role in the field of finite element method:

Theorem 1. *Let s_0, s_1,...,s_m be points in R^n and let $S = H(s_0, s_1, ..., s_m)$. Then there exist non-negative real-valued continuous functions λ_0, λ_1,..., λ_m on S with λ_0 a convex function such that, for each $x \in S$, $\sum_{i=0}^{m} \lambda_i(x)s_i = x$ and $\sum_{i=0}^{m} \lambda_i(x) = 1$.
(Here $H(s_0, s_1, ..., s_m)$ is the smallest convex set containing $\{s_0, s_1, ..., s_m\}$, also known as convex hull of the set $\{s_0, s_1, ..., s_m\}$.) At that time it was just a concept of pure mathematics and its applications were limited.*

The above theorem steerage to identify the following conditions which are mandatory for the generalized barycentric coordinated $\{W_i\}_{i=1}^{n}$ to obey:

- $\sum_{i=1}^{n} W_i = 1$.
- W_i attains the value 1 at the i^{th} node 0 on every other node.

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- $\sum_{i=1}^{n} xW_i = \forall x$ belonging to the considered element.
- $W_i \in C^{\infty}$.
- W_i is linear on the sides adjacent to i.

Wachspress [10], initiated the concept of rational basis functions over the polygons. These basis functions(Wachspress coordinates) were the first [6] known generalized barycentric coordinates for the polygonal discretization of the domain. Later many generalized barycentric coordinates, came into existence [11,12,13] even then the Wachspress coordinates remain unbeatable [11,6,14,15] due to their efficient execution to obtain favorable results.

The use of 3D elements in the field of computer graphics, grew rapidly in the last two decades [1,16,17]. Due to its applications in computer animation for movies [1], in medical field for surgery simulation [17], crack recognition [18], predicting crack initiation [19,20] in civil and mechanical engineering. Wachspress [21] proposed the construction of generalized barycentric coordinates for 3D elements by using an interesting algebraic geometry property of the element to compute the adjoint for the construction of wedge functions. Some other coordinates, have also been proposed, but either the study was restricted to some simple elements like tetrahedron or hexahedron [1,16] or [17]they failed to achieve the versatility of the Wachspress coordinates.

Dasgupta [22], proposed a recursive relation for the computation of the Wachspress coordinates. In the technique proposed by Dasgupta, one could compute the adjoint for the Wachspress coordinates for a convex polygon by simply substituting the Cartesian coordinates of the nodes in a recursive formula, and obtain the adjoint(denominator) function.

In the present work, a method for the computation of Wachspress coordinates for 3D domain has been proposed. Inheriting by the technique of Dasgupta, a simple recursive relation has been derived with the help of a property, that “for a polyhedron, wedge functions corresponding to the consecutive nodes which are linear on the common face, attain the same value at the mid point of the edge joining them”, and this property has also been explored in this paper. This recursive relation contains two(k_i, k_{i+1}(say)) consecutive unknown normalizing constants along with the ratio of two polynomials evaluated at the mid point of the common edge. Hence, by setting the value 1 for k_1, the remaining $k_j's$ have been computed exactly which in turn is used to get the value of adjoint for the wedge function.

It is quite interesting to note that the technique explored in this paper covers all shapes of convex polyhedron and computes appropriate adjoint function for the wedge construction.

In support of the technique described in this paper an adjoint for linear approximation over the hexahedron has been studied and discussed in detail.

2 Setup and Formulation

Let $\Omega \subseteq \mathbb{R}^3$, be the polyhedral discretization of the domain. An element $P_m \in \Omega$, is a geometric shape with m faces, 2(m-2) vertices and 3(m-2) edges [21] where each face is an n sided polygon contained in a plane $F^j = 0$. Hence associated with each vertex of P_m there is a wedge function $W_i = \frac{N_i}{D}, (i = 1, ..., 2(m-2))$, where N_i is a tri-variate polynomial of degree (m-3) and D is also a tri-variate polynomial of degree (m-4), in order to achieve degree one approximation. These $W_i's$ are defined in accordance with the following properties:

a)There is one node at each vertex of the polyhedron(throughout this work, the i^{th} node is considered to have the Cartesian coordinates (x_i, y_i, z_i))(cf. [21]). For each node there is an associated wedge within each polyhedron containing the node.

b)Wedge $W_i(x,y,z)$ associated with node i is normalized to unity at i.

c)Wedge $W_i(x,y,z)$ is of degree one on faces adjacent to i.

d)Wedge $W_i(x,y,z)$ vanishes on all nodes j($\neq i$).

e)The wedges associated with P_m form a basis for degree one approximation over it. For the polyhedron P_m, there must be at least $2(m-2)$ nodes. For these to suffice, we must have:

$$\sum_{k=1}^{2(m-2)} x_k^i y_k^j z_k^{\nu} W_k = x^i y^j z^{\nu} \qquad 0 \leq i+j+\nu \leq 1 \quad (1)$$

f)Each wedge function and all its derivatives are continuous within the polyhedron for which the wedge is a basis function.

3 Background

In the Wachspress method [23], for a well set convex polyhedron of order m, the Wachspress coordinate for degree one approximation associated with the i^{th} node is

$$W_i(x,y,z) = \frac{K_i P_{m-3}^i(x,y,z)}{Q_{m-4}(x,y,z)} \quad (2)$$

where P_{m-3}^i is polynomial of degree (m-3), which is the product of all the linear forms of the faces of the polyhedra which do-not contain the node i, K_i's are the normalizing constants and Q_{m-4} is the unique polynomial representing the curve passing through the Exterior Triple Points(ETPs) [23]. It has been noted that the computation of ETPs is quite lengthy and consequently the computation of adjoint becomes tedious. In particular, if two opposite faces are parallel then some of the ETPs lie on the absolute line and the computation of adjoint is not possible by following the technique of Wachspress [23].

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In order to make the process of constructing wedge functions for linear approximation independent of geometric properties of the element viz. EIPs, Dasgupta [22], proposed an alternative approach to deal with the adjoint as follows:

$$\sum W_i = 1 \Rightarrow D = \sum N_i$$

Instead of computing D directly, the aim is to compute $\sum N_i$ [22].

4 Governing Result

Let $i = i^{j,k,l}$ be the nodes of the polyhedral element $P_m \in \Omega$ under consideration, where $(i = 1, 2, \cdots, 2(m-2))$, $j, k, l \in B_m$ where B_m is a block of integers $\{1, 2, ..., m\}$ and F^j is the j^{th} face of the considered polyhedral element. The notation $i = i^{jkl}$ is devised to denote that the i^{th} node is the intersection of the faces F^j, F^k and F^l, where each face F^ν is the linear form of the plane containing all the vertices of the corresponding polygonal face of the polyhedron. Let e_i be the edge joining the vertices i and $i+1$.

The wedge function, corresponding to the node i is defined as

$$W_i = \frac{N_i}{D} = \frac{\prod_{\nu \neq j,k,l} K_i F^\nu}{D} \tag{3}$$

Theorem 2. *Let e_i be an edge of P_m, joining the vertices i and $i+1$ and $m_{(i,i+1)}$ be the mid point of e_i. Let $W_i(x,y,z)$ and $W_{i+1}(x,y,z)$ be the wedges for linear approximation associated with the nodes i and $i+1$ respectively then*

$$W_i(x,y,z)|_{m_{(i,i+1)}} = W_{i+1}(x,y,z)|_{m_{(i,i+1)}}$$

Proof. The restriction of the wedges W_i and W_{i+1} along the adjacent faces is a bivariate polynomial of degree one and when restricted over e_i, consequently the restrictions of both W_i and W_{i+1} is a univariate polynomial of degree one. In view of, properties (b) and (d), $W_i(x,y,z)$ attains value 0 at the node $i+1$ and 1 at the node i. Similarly, $W_{i+1}(x,y,z)$ attains value 1 at the node $i+1$ and 0 at the node i. Thus, forms a rectangle R with vertices $i, i+1, P, Q$(see Fig.1) whose diagonals mutually bisect at $m'_{i,(i+1)}$.

Let L be the perpendicular, dropped from $m'_{(i,i+1)}$, on the edge e_i and it is obvious that the foot of the perpendicular is the point $m_{(i,i+1)}$ i.e. the mid point of e_i and hence, $W_i(x,y,z)$ and $W_{i+1}(x,y,z)$ attain the same value at $m_{(i,i+1)}$(cf. Fig1)

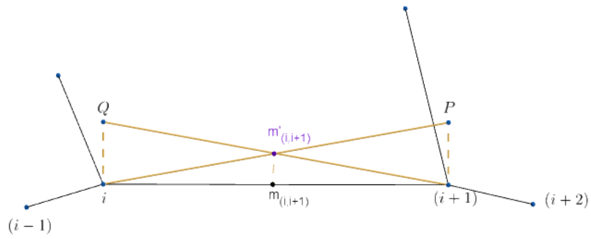

Fig. 1: Figure representing rectangle R.

Thus,

$$W_i(x,y,z)|_{m_{(i,i+1)}} = W_{i+1}(x,y,z)|_{m_{(i,i+1)}}$$

5 Recurrence relation

In this section, a recurrence relation has been established with the help of the Lemma 2, by which all the normalizing constant K_i's can be computed instantly if the Cartesian coordinates of the vertices are known and thus the adjoint can be found easily without indulging in the complex geometry of the element.

To establish the recurrence relation we refer property (e)(cf. section 2) with $i = j = \nu = 0$, which gives

$$\sum_{i=1}^{2(m-2)} W_i = 1 \tag{4}$$

$$\Rightarrow \sum_{i=1}^{2(m-2)} \frac{N_i}{D} = 1 \tag{5}$$

$$\Rightarrow D = \sum_{i=1}^{2(m-2)} N_i \tag{6}$$

where N_i is the numerator of the wedge function W_i for the linear approximation over P_m.

In view of Lemma 2, we know that

$$W_i(x,y,z)|_{m_{(i,i+1)}} = W_{i+1}(x,y,z)|_{m_{(i,i+1)}} \tag{7}$$

Clearly, every pair of consecutive nodes have two faces in common, let the common faces of i and $(i+1)$ be F^j and F^k. Let F^{l^i} be the face of the polyhedro containing i but not $i+1$ and similarly $F^{l^{i+1}}$ is the face containing $(i+1)$ but not i. Then,

$$W_i = \prod_{\nu \neq j,k,l^i} \frac{K_i F^\nu}{D} \tag{8}$$

$$W_{i+1} = \prod_{\nu \neq j,k,l^{i+1}} \frac{K_{i+1} F^\nu}{D} \tag{9}$$

Substituting the values of W_i and W_{i+1} in (7) the following recurrence relation is obtained

$$K_{i+1} = K_i \frac{F^{l^{i+1}}}{F^{l^i}}|_{m_{(i,i+1)}} \tag{10}$$

Which is the desired recurrence relation.

6 Numerical Example

It is quite interesting to observe that, following the recurrence relation(10) the adjoint function for hexahedral element(P_6 say) can be obtained conveniently without knowing ETPs. The elaborative study of this element has been presented in this section by considering a specific choice of P_6. Moreover, it has been verified that the restriction of the wedge function associated with four vertices lying in the same plane, along the plane turns out to be the same as the Wachspress coordinates for the corresponding quadrilateral element of $\mathbb{R}^2$.
Let the domain $\Omega \subseteq \mathbb{R}^3$ be discretized using polyhedron of order 6, $P_6 = (1, 2, \cdots, 8) \in \Omega$. The Cartesian coordinates of the vertices are $1 = (0, 0, 0)$, $2 = (1, 0, 0)$, $3 = (2, 2, 0)$, $4 = (0, 1, 0)$, $5 = (0, \frac{5}{26}, \frac{21}{26})$, $6 = (\frac{3}{2}, \frac{1}{4}, \frac{3}{2})$, $7 = (\frac{5}{4}, 0, 1)$ and $8 = (0, 0, \frac{1}{2})$(cf. Fig. 2).

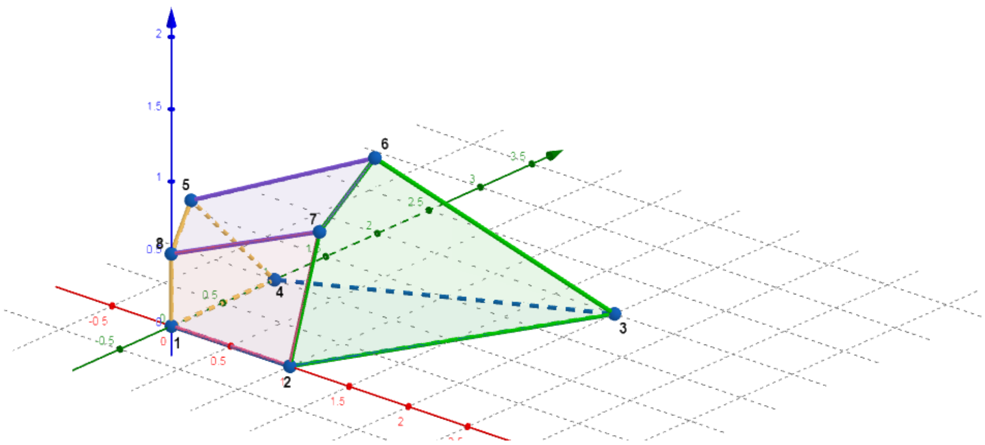

Fig. 2: Hexahedral element.
(i,j,k,l) denotes the face passing through the nodes i, j, k, l of P_6, and are computed as follows:

$$\begin{aligned}
F^1 &\cong (7, 8, 1, 2) = x \\
F^2 &\cong (1, 2, 3, 4) = z \\
F^3 &\cong (2, 3, 6, 7) = -12 + 12x - 6y - 3z \\
F^4 &\cong (5, 6, 7, 8) = -75 - 60x - 240y + 150z \\
F^5 &\cong (3, 4, 5, 6) = 42 + 21x - 42y - 42z \\
F^6 &\cong (1, 2, 7, 8) = y
\end{aligned}$$

Using relation(6), the numerator N_i of wedge function corresponding to the i^{th} node (i=1,2,3,4) of the considered hexahedron are defined as

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$$N_1 = K_1 \prod_{\nu \neq 1,2,6} F^{\nu}, N_2 = K_2 \prod_{\nu \neq 2,3,6} F^{\nu},$$
$$N_3 = K_3 \prod_{\nu \neq 2,3,5} F^{\nu}, N_4 = K_4 \prod_{\nu \neq 1,2,5} F^{\nu},$$
$$N_5 = K_5 \prod_{\nu \neq 1,4,5} F^{\nu}, N_6 = K_6 \prod_{\nu \neq 3,4,5} F^{\nu},$$
$$N_7 = K_7 \prod_{\nu \neq 3,4,6} F^{\nu}, N_8 = K_8 \prod_{\nu \neq 1,4,6} F^{\nu}$$

On substituting $K_1 = 1$ in relation(10), we obtain $K_2 = -6$ $K_3 = -\frac{756}{65}$, $K_4 = \frac{168}{65}$, $K_5 = \frac{63}{52}$, $K_6 = -\frac{8505}{1352}$, $K_7 = -\frac{405}{416}$and $K_8 = \frac{75}{416}$.
Thus, by (6) we obtain

$$D = \frac{945(40 + 52x + 16x^2 + 148y + 80xy + 64y^2 - 30z - 27xz + 12yz - 10z^2)}{43264} \tag{11}$$

Referring relation(11), we are now set to define wedge functions W_i(i=1,2,...,8):

$$W_i = \frac{N_i}{D}(i = 1, 2, ..., 8) \tag{12}$$

It has been verified that W_i(cf relation(12)) satisfies all the properties specified in section 2.

Validation of Wachspress' averments It has been asserted by Wachspress that the restriction of all the wedge functions along the six quadrilateral faces of hexahedral element gives precisely the corresponding wedge functions which have been already defined corresponding quadrilateral element(cf. Fig. 3).

–**Restriction along the face** F^1 It may be noted that, $W_i|_{F^1} = 0$ for $i \neq 1, 4, 5, 8$ and

$$W_1|_{F^1} = -\frac{(5 + 16y - 10z)(-1 + y + z)}{5 + 16y - 5z},$$
$$W_4|_{F^1} = \frac{y(5 + 16y - 10z)}{5 + 16y - 5z},$$
$$W_5|_{F^1} = \frac{26yz}{5 + 16y - 5z},$$
$$W_8|_{F^1} = -\frac{10z(-1 + y + z)}{5 + 16y - 5z}$$

One may easily verify that these four restrictions are precisely the the rational wedge functions corresponding to the quadrilateral element(1,2,7,8)(cf Fig. 3)

–Restriction along the face F^2

It is clear that $W_i|_{F^2} = 0$ $i \neq 1, 2, 3, 4$ and

$$W_1|_{F^1} = -\frac{(2 + x - 2y)(-2 + 2x - y)}{2(2 + x + y)},$$
$$W_2|_{F^1} = \frac{x(2 + x - 2y)}{2 + x + y},$$
$$W_3|_{F^1} = \frac{3xy}{2(2 + x + y)},$$
$$W_4|_{F^1} = -\frac{(-2 + 2x - y)y}{2 + x + y}$$

All the wedge properties for the quadrilateral element (1,2,3,4) (see Fig.4) follows similarly for the wedge functions $W_i|_{F^2}$ (i=1,2,3,4)

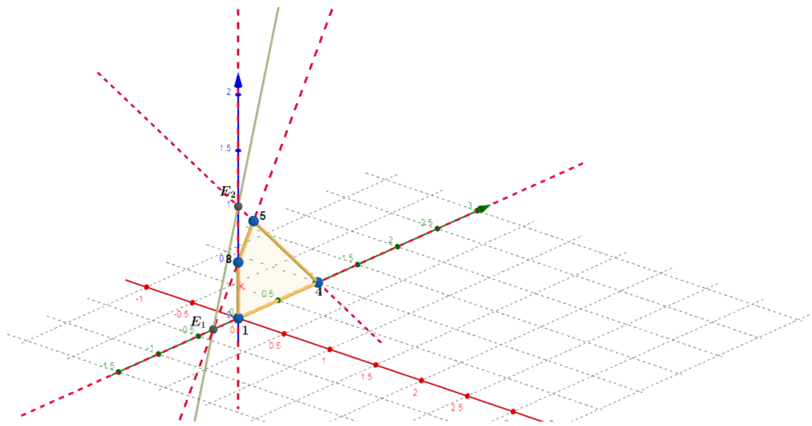

Fig. 3: Restriction along the face F^1.

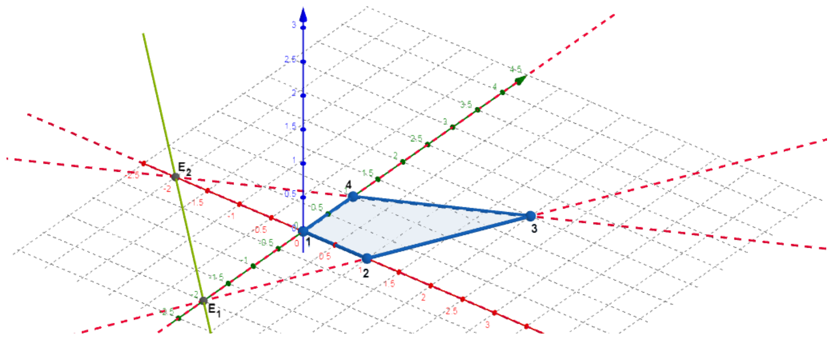

Fig. 4: Restriction along the face F^2.

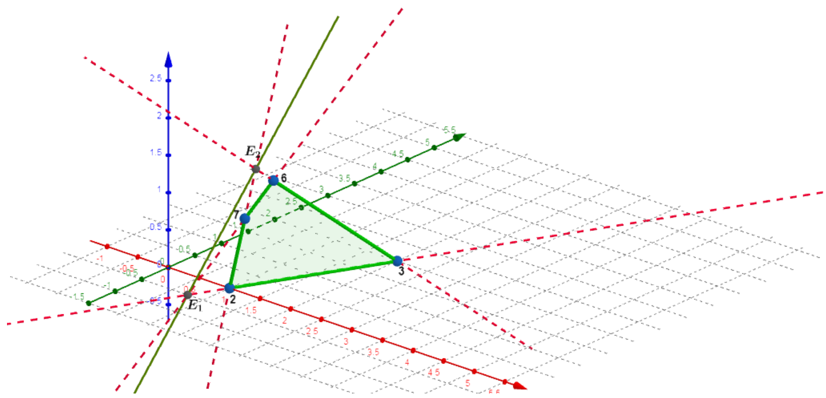

Fig. 5: Restriction along the face F^3.

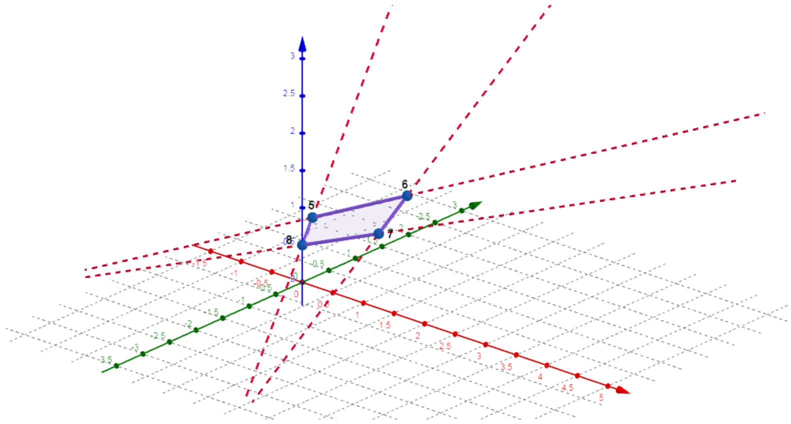

Fig. 6: Restriction along the face F^4.

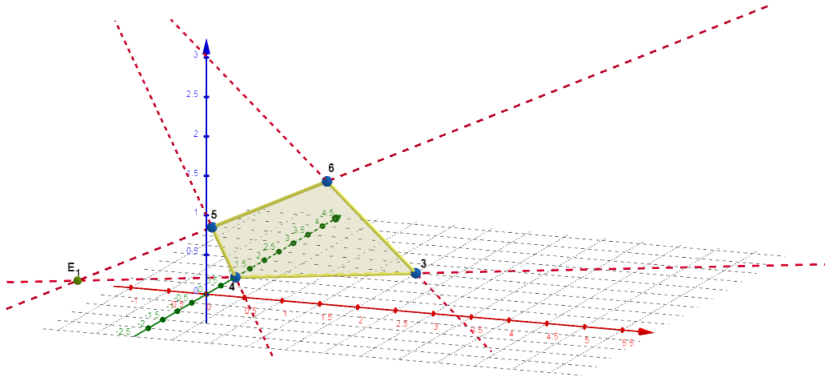

Fig. 7: Restriction along the face F^5.

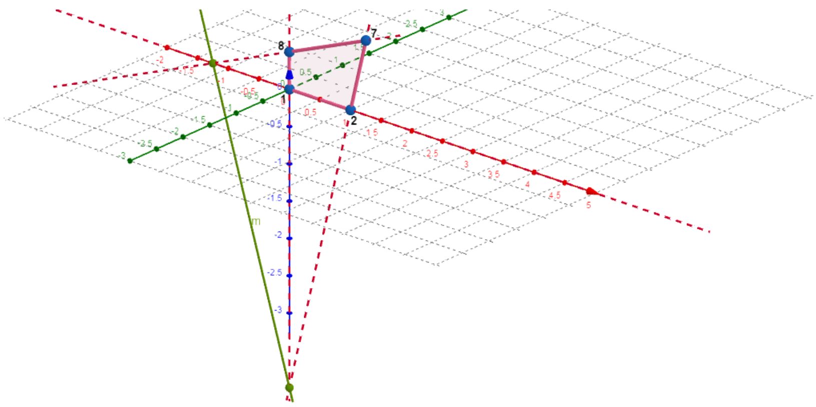

Fig. 8: Restriction along the face F^6.

Similarly, when restricted on the faces F^3, F^4, F^5 and F^6 the adjoint for the wedge functions are represented in the figures (5), (6), (7) and (8) respectively.

7 Conclusion

In this paper , we have presented a new as well as simple method for the computation of the interpolant for a 3D domain. Apart from this, a numerical implementation of this method is given which guarantees that the construction abides all the prescribed conditions. The claims of Wachspress [23], regarding the interpolants are also verified and found valid. This method provides a simple way to interpolate the data dependent on three variables and hence has a wide range of applications.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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