Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/030108

Estimation of Reinsurance Premium for Positive Strictly Stationary Sequence with Heavy-Tailed Marginals

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Received: 12 May 2013, Revised: 23 Jan 2014, Accepted: 9 Feb 2014
Published online: 1 Mar 2014

Abstract: Many authors have studied estimation of the reinsurance premium when sequences are i.i.d. for different distributions, particularly for heavy tailed ones. The goal of this paper is to extend this estimation for dependent sequences with heavy tailed marginals. Our work is limited to some mixing sequences using the distortion risk measure due to Wang. In this study it is shown that estimator of the reinsurance premium with high retention is asymptotically normal and without bias.

Keywords: Extremal index , Wang's premium principle, Extreme value statistics. **2000 Mathematics Subject Classification**: 60G52, 62G32, 91B30.

1 Introduction

Let χ denote the set of nonnegative random variables on the probability space $(\Omega,\mathscr{A},P)$. In this study, we use the following distortion risk measure :

$$\mathscr{R}_g : \chi \longrightarrow \mathbf{R}^+$$

$$X \longrightarrow \mathscr{R}_g(X) = \int_0^\infty g(S_X(x))dx,$$

where $S_X(x) = 1-F_X(x)$ is the survival function of X and g is a concave increasing function verifying $g(0) = 0, g(1) = 1$. One remarks that this distortion measure is sub-additive. It was introduced by Denneberg [9] and Wang [29]. It verifies also axioms of coherent risk measure proposed by Artzner et al [1]. In the field of insurance the risk measure $\mathscr{R}_g$ is called the risk premium. If g is the identity from $[0,1]$ to $[0,1]$, then $\mathscr{R}_g = E(X)$ wich is called net premium. Generally the safety loading is defined as $\mathscr{R}_g(X)-E(X)$. Building a function $h : [0,1] \to [0,1]$ verifying $h(x) = g(x)-x$, one obtains the safety loading which is written $\mathscr{R}_h(X) = \int_0^\infty h(S_X(x))dx$. This quantity isn't a distortion risk measure.

Reinsurance is a financial transaction by which risk is transferred from an insurance company to a reinsurance company (reinsurer) in exchange of a payment (reinsurance premium). Suppose that an insurance risk X is split into two parts as

$$X = [X-(X-R)_+] + (X-R)_+,$$

where $(X-R)_+ = \max(X-R,0)$, and R is a high retention level. Insurer retains the risk $X-(X-R)_+$, and transfers the risk $(X-R)_+$ to the reinsurer with the reinsurance premium

$$\mathscr{R}_{g,R}(X) = \int_R^\infty g(S_X(x)dx \le \mathscr{R}_g(X)$$

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If $g(x) = x^{1/\rho}, \rho \geq 1$ (called the distortion parameter), we obtain the proportional hazards transform (PHT) premium (Wang [29])

$$\Pi_{\rho,R}(X) = \int_R^\infty (S_X(x))^{1/\rho} dx \tag{1}$$

Now, consider i.i.d. random variables $X_1, \ldots, X_n$ with common distribution function F of an insured risk $X > 0$. We assume that $S_X(x)$ has regular variation function near infinity with index $-\alpha$, that is:

$$\lim_{v\to\infty} \frac{S(vx)}{S(v)} = x^{-\alpha}, \text{ for any } x > 0 \text{ and } 1 < \alpha < 2. \tag{2}$$

(see, e.g., de Haan and Ferreria, [12]). Such cdf's constitute a major subclass of the family of heavy-tailed distributions. It includes distributions such those Pareto's, Burr's, Student's, α-stable $(0 < \alpha < 2)$, and log-gamma, which are known to be appropriate models of fitting large insurance claims, large fluctuations of prices, log-returns, and so on (see Beirlant et al. [2]; Reiss and Thomas, [26] for more details). A high quantile x_p situated in the border or even beyond the range of the available data is denoted

$$x_{p_X} := Q_X(1-p),\ p = p_n \to 0,\ as\ n \to \infty,\ np_n \to \lambda \geq 0,$$

where $Q(s) = \inf\{x \in \mathbf{R} : F(x) \geq s\}, 0 < s < 1$ is the quantile function associated to the df F. Note that the condition (2) is equivalent to

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{1/\alpha},\ for\ any\ x > 0, \tag{3}$$

where $U(t) = Q(1-1/t), t \geq 1$.

To get asymptotic normality of estimators of parameters of extreme events, it is usual to assume the following extra second regular variation condition, that involves a second order parameter $\eta < 0$:

$$\lim_{t\to\infty} (A(t))^{-1} \left(\frac{U(tx)}{U(t)} - x^{1/\alpha} \right) = x^{1/\alpha} \frac{x^\eta - 1}{\eta}, for\ any\ \ x > 0, \tag{4}$$

where A is a suitably chosen function of constant sign near infinity. More restrictively, we consider that F belonged to the wide class of Hall [13], that is

$$1 - F_X(x) = cx^{-\alpha}(1 + dx^{\eta\alpha} + o(x^{\eta\alpha})),\ x \to \infty \tag{5}$$

for some constants $c > 0$ and $d \neq 0$, thus $A(t)$ is equivalent to $d\eta\alpha^{-1}c^\eta t^\eta$ as $t \to \infty$. The most popular estimator of α, is the Hill estimator [14], with the form

$$\widehat{\alpha} = \left(\frac{1}{k} \sum_{i=1}^{k} \log X_{n-i,n} - \log X_{n-k+1,n} \right)^{-1},$$

where $X_{i,n}$ denotes, the i-th ascending order statistics $1 \leq i \leq n$, associated to the random sample $(X_1, X_2, \ldots, X_n)$.

Hall and Welsh [13] proved that the asymptotic mean squared error of the Hill estimator is minimal for

$$k_{opt} = \left[-\frac{(1-\eta)^2}{2c^{2\eta} d^2 \eta^3}\, n^{-2\eta} \right]^{1/(1-2\eta)} \tag{6}$$

One remarks that $k_{opt} = O(n^\xi)$ where $\xi = \dfrac{1}{1-\frac{1}{2\eta}}$.

For the high quantile estimation, we recall the classical semi-parametric Weissman-type estimator of x_{p_X} (Weissman [31])

$$\widehat{x}_{p_X} = X_{n-k,n} \left(\frac{np}{k} \right)^{-\widehat{\alpha}} \tag{7}$$

Since the parameters η, c and d in (6) are unknown, the k_{opt} cannot be used directly to determine the optimal number of order statistics for a given data set. To solve this problem several procedures are available, see e.g. Cheng and Peng [4], and Neves and Fraga Alves [22].

The reinsurance companies need to calculate the premium $\Pi_{\rho,R}$ for covering such excess claims, which is usually very large. In this case, the probabilities of extreme cases happening are relatively larger than are predicted under the classical normal distribution assumption. Therefore, the large insurance losses can be modelled by heavy-tailed distributions, for that Necir and Boukhetala [19], Vandewalle and Beirlant [28] and Necir et al. [20], Rassoul [25] have proposed different asymptotically normal estimators for $\Pi_{\rho,R}$ based on samples of claim amounts of reinsurance covers of heavy tailed i.i.d. risks. However, in economics and other fields of applications real data sets are most often dependent. As the complexity of situations that can be considered is enormous some kind of dependence must be assumed. The classical limiting theory for maxima can be applied, with small changes, provided long range dependence of extremes is sufficiently weak. Some important dependent sequences have been studied and the limit distributions of their order statistics under some conditions are then known. Stationary sequences are examples of those sequences and are realistic for many real problems, it is interesting to analysing the properties of the estimator of $\Pi_{\rho,R}$ under heavy tailed stationary sequences. The rest of this paper is organized as follows. In Section 2, we introduce the extremal index which is the key parameter for extending extreme value theory results from i.i.d. to stationary sequences. In Section 3, we construct a reinsurance premium estimation for positive strictly stationary sequence with heavy-tailed marginals which is the main result. In Section 4, we compute confidence bounds for $\Pi_{\rho,R}$ by some simulations. Section 5 is devoted to the proofs.

2 Extremal index

Many applications as in insurance and finance, telecommunication and other areas of technical risk, usually exhibit a dependence structure. Leadbetter *et al.* [17] put a mixing condition $D(u_n)$ based on the probability of exceedances of a high threshold u_n, it limits the degree of long-term dependence of the sequence, providing asymptotic independence between far apart extreme observations.

Definition 1($D(u_n)$ condition). *A strictly stationary sequence $\{X_i\}$, whose marginal distribution F has upper support point $x_F = sup\{x : F(x) < 1\}$, is said to satisfy $D(u_n)$ if, for any integers $i_1 < \ldots < i_p < j_1 < \ldots < j_q$ with $j_1 - i_p > l$,*

$$\left|P\left\{X_{i_1} \le u_n, \ldots, X_{i_p} \le u_n, X_{j_1} \le u_n, \ldots, X_{j_q} \le u_n\right\}\right.$$
$$\left. - P\left\{X_{i_1} \le u_n, \ldots, X_{i_p} \le u_n\right\} P\left\{X_{j_1} \le u_n, \ldots, X_{j_q} \le u_n\right\}\right| \le \delta(n,l),$$

where $\delta(n,l_n) \to 0$ for some sequences $l_n = o(n)$ and $u_n \to x_F$ as $n \to \infty$.

Theorem 1(Leadbetter et. al. [17]). *Let $X_1, \ldots, X_n$ be a strictly stationary sequence with marginal distribution F , and $\tilde{X}_1, \ldots, \tilde{X}_n$ an i.i.d. sequence of random variables with the same distribution F, define the following quantities $M_n = \max(X_1, \ldots, X_n)$ and $\tilde{M}_n = \max(\tilde{X}_1, \ldots, \tilde{X}_n)$. Under the $D(u_n)$ condition, with $u_n = a_n x + b_n$, if*

$$\mathbf{P}[a_n^{-1}(\tilde{M}_n - b_n) \le x] \to G(x), \text{ as } n \to \infty, \tag{8}$$

for normalizing sequences $a_n > 0$ and $b_n \in \mathbf{R}$, it follows that,

$$\mathbf{P}[a_n^{-1}(M_n - b_n) \le x] \to [G(x)]^{\theta}, \text{ as } n \to \infty, \tag{9}$$

where $G(x)$ is a non-degenerate extreme value distribution, and $\theta \in (0,1]$ is the extremal index, this parameter characterizes the short-range dependence of the maxima. An informal interpretation of θ is given in Leadbetter *et al.* [17], namely $\theta \approx$(mean cluster size)$^{-1}$. Theoretical properties of the extremal index have been studied fairly extensively; (O'Brien [23]), Hsing et al. [15], and references therein). The problem of estimating θ has also received some attention in the literature (see Smith and Weissman [27], Weissman and Novak [30] and Ferro and Segers [11]).

Let $(X_1, \ldots, X_n)$ be a positive strictly stationary sequence with extremal index θ and heavy tailed marginals $1-F$, and let $(\tilde{X}_1, \ldots, \tilde{X}_n)$ be independent variables with the same heavy tailed marginals $1-F$ as the sequence $\{X_i\}_{1\le i\le n}$. Using the relationship (8) we find that the marginal quantiles of $\{\tilde{X}_i\}_{1\le i\le n}$ are approximately by

$$Q_{\tilde{X}}(1-p) \approx G^{-1}(1-p)^n,$$

and thus

$$Q_{\tilde{X}}(1-p)^{\theta} \approx G^{-1}(1-p)^{n\theta} \tag{10}$$

Using (9), we find the approximation of the marginal quantiles of $\{X_i\}_{1\le i\le n}$ as

$$Q_X(1-p) \approx G^{-1}(1-p)^{n\theta} \tag{11}$$

From (10) and (11), we have

$$Q_X(1-p) = Q_{\widetilde{X}}(1-p)^{\theta}.$$

By the use of the Taylor formula $(1-p)^{\theta} \sim 1-\theta p, p \to 0$, one obtains

$$Q_X(1-p) = Q_{\widetilde{X}}(1-\theta p).$$

Reading the convergence (3) as an approximation for $p \to 0, \theta \in (0,1]$, one obtains

$$Q_{\widetilde{X}}(1-\theta p) \approx \theta^{-1/\alpha} Q_{\widetilde{X}}(1-p) \quad (12)$$

therefore

$$Q_X(1-p) \approx \theta^{-1/\alpha} Q_{\widetilde{X}}(1-p)$$

Using the Weissman estimator (7), therefore an estimator of $Q_X(1-p)$ is

$$\widetilde{X}_{n-k,n} \left(\frac{n\widehat{\theta}p}{k} \right)^{-1/\widehat{\alpha}}. \quad (13)$$

The extremal index θ needs to be adequately estimated not only by itself, but also because its influence in the estimation of other important parameters of rare events, such as, a high quantile in (13), various estimators of θ are proposed in the literature. The first estimator is based on the characterization of the extremal index given by O'Brien [23]. In this characterization, θ is expressed as the limiting probability that an exceedance is followed by a run of observations below a high threshold u_n:

$$\theta = \lim_{n\to\infty} \mathbf{P}(\max\{X_2, \dots, X_{r_n}\} < u_n \mid X_1 > u_n),$$

where $r_n = o(n)$ is the length of runs of values of the process falling below the threshold given that an exceedance has occurred. This characterization motivates the definition of the runs estimator for a fixed high threshold u and a specified runs length r:

$$\widehat{\theta}(u_n, r_n) = \frac{\sum_{i=1}^{n-r_n} \mathbf{1}(X_i > u_n, M_{i,i+r_n} \le u_n)}{\sum_{i=1}^{n} \mathbf{1}(X_i > u_n)},$$

where where $\mathbf{1}(.)$ is the indicator function and $M_{i,i+r_n} = \max\{X_i \dots, X_{i+r_n}\}$. The runs estimator is asymptotically normal and consistent. See Weissman and Novak [30] and references therein for additional information.

The second estimator is due to Ferro and Segers [11]. An interesting aspect of this estimator is that it does not require an auxiliary parameter (run length in the case of the runs estimator). However, one still has to choose the threshold. Using a point process approach, Ferro and Segers show that the inter-exceedance times (time differences between successive values above a threshold) of the extreme values normalized by $1-F(u_n)$ converge in distribution to a random variable T_θ with a mass of $1-\theta$ at $t=0$ and an exponential distribution with rate equal to θ on $t>0$. Using a moment estimator, they first obtain:

$$\widehat{\theta}_1 = \frac{2[\sum_{i=1}^{N-1} T_i]^2}{(N-1)\sum_{i=1}^{N-1} T_i^2},$$

where T_i are the inter-exceedance times and N is the number of exceedances of a fixed high threshold u. A bias corrected version gives,

$$\widehat{\theta}_2 = \frac{2[\sum_{i=1}^{N-1}(T_i-1)]^2}{(N-1)\sum_{i=1}^{N-1}(T_i-1)(T_i-2)}.$$

To obtain the final form of the estimator, a further adjustment is made to ensure that the values of the estimator lie between 0 and 1:

$$\widehat{\theta} = \begin{cases} 1 \wedge \widehat{\theta}_1 \ if \ \max\{T_i : 1 \le i \le N-1\} \le 2 \\ 1 \wedge \widehat{\theta}_2 \ if \ \max\{T_i : 1 \le i \le N-1\} > 2 \end{cases} \quad (14)$$

Next, we discuss the max-autoregressive (ARMAX) process, used in the simulation study, for which the extremal index is given in closed form.

3 Defining the estimator and main results

To estimate the risk measure $\Pi_{\rho,R}(X)$, given in (1), when X is a positive stationary process, and $R = Q_X(1-k/n)$. Let $k = k_n$ be sequence of integer satisfying $1 < k < n, k \to \infty, k/n \to 0$. We present now our risk measure $\Pi_{\rho,R}(X)$ as

$$\Pi_{\rho,R}(X) = -\int_0^{k/n} (\theta s)^{1/\rho} dQ_{\tilde{X}}(1-\theta s). \tag{15}$$

We derive the estimator for $Q_X(1-s)$ in (12), and after an integration, we obtain the following estimator for $\Pi_{\rho,R}(X)$

$$\widehat{\Pi}_{\rho,R}(X) = \frac{\rho (k/n)^{1/\rho}\widehat{\theta}^{1/\rho - 1/\widehat{\alpha}}}{\widehat{\alpha}-\rho}\tilde{X}_{n-k,n}, \tag{16}$$

Theorem 2. *Fix $\rho \geq 1$, if X is a positive strictly stationary sequence with extremal index θ, and assume that (4) holds with $t^{-1/\rho}Q_{\tilde{X}}(1-1/t) \to 0$ as $t \to \infty$, and $k = k_n$ be such that $k \to \infty$, $k/n \to 0$, and $\sqrt{n}A(k/n) \to 0$ as $n \to \infty$, Then, for $0 < 1/\alpha < 1/\rho$, we have*

$$\frac{(k/n)^{-1/\rho}k^{1/2}}{\tilde{X}_{n-k,n}}[\widehat{\Pi}_{\rho,R}(X) - \Pi_{\rho,R}(X)] \xrightarrow{D} \mathcal{N}(0, \sigma^2(\rho,\theta,\alpha)), \quad as\ n \to \infty,$$

$$\sigma^2(\rho,\theta,\alpha) = \theta^{2(1/\rho-1/\alpha)}\frac{\rho\alpha^2 - 2\rho^2\alpha + \rho^3 + \rho\alpha^4}{\alpha^3(\alpha-\rho)^2}.$$

4 Constructing Confidence Interval for $\Pi_{\rho,R}(X)$

4.1 Optimal choice of the number of upper order statistics

Extreme value based estimators rely essentially on the number k of upper order statistics involved in estimate computation. Hill's estimator has, in general, a substantial variance for small values of k and a considerable bias for large values of k. Hence, one has to look for a k value, denoted by k_{opt}, that balances between these two vices. The choice of this optimal value k_{opt} represents the main hurdle in the process of estimating the tail index. To solve this problem several procedures are available, see e.g. Hall and Welsh [13], Cheng and Peng [4], and Neves and Fraga Alves [22]. In our simulation study, we use the method proposed by Hall and Welsh [13], they minimized the asymptotic mean squared error for the Hill estimator such that:

$$k_{opt} = \arg\min_k E(\widehat{\alpha}-\alpha)^2 \tag{17}$$

4.2 Simulation study

To obtain confidence intervals for our estimator $\widehat{\Pi}_{\rho,R}(X)$, first we fix the the distortion parameter $\rho = 1$, and $\rho = 1.01$, then we generate 100 replications of the time series $(X_0, X_1, \ldots, X_n)$ for different sample sizes $(1000, 3000, 5000)$, where X_t is an ARMAX process satisfying

$$X_t = max(\beta X_{t-1}, Z_t),\ 0 < \beta < 1,\ t \geq 1, \tag{18}$$

where $\beta = 0.3$, and $\{Z_t\}_{t\geq 1}$ are independent and identically distributed, with tail distribution $1 - F_Z(x) = 1 - \exp(-(1-\beta^\alpha)x^{-\alpha})$, we use two tail indices $\alpha = 1.2$ and $\alpha = 1.8$, note that we estimate $\theta = 1-\beta^\alpha$ by the Ferro and Segers estimator in (14), and we use (17) for compute the values of the optimal fraction integer k_{opt}. The simulation results are presented in the following Tables, where lb and ub stand respectively for lower bound and upper bound of the confidence interval. The overall true premium Π and estimated premium $\widehat{\Pi}$ is then taken as the empirical mean of the values in the 100 repetitions.

Table 1: 95% confidence intervals for the premium, with tail index $\alpha = 1.2$, and distortion parameter $\rho = 1$.

n	Π	$\widehat{\Pi}$	lb	ub	length
1000	3.337356	3.366814	2.319555	4.414072	2.094517
3000	3.158891	3.046677	2.447528	3.645826	1.198299
5000	3.005196	2.939038	2.470065	3.408010	0.937944

Table 2: 95% confidence intervals for the premium, with tail index $\alpha = 1.2$, and distortion parameter $\rho = 1.01$.

n	Π	$\widehat{\Pi}$	lb	ub	length
1000	3.672269	3.80352	2.841751	4.765289	1.923537
3000	3.379407	3.276043	2.632976	3.91911	1.286134
5000	3.343542	3.215753	2.684792	3.746715	1.061923

Table 3: 95% confidence intervals for the premium, with tail index $\alpha = 1.8$, and distortion parameter $\rho = 1$.

n	Π	$\widehat{\Pi}$	lb	ub	length
1000	0.4350412	0.4165584	0.2946764	0.5384404	0.2437639
3000	0.4112903	0.3785951	0.3098901	0.4473002	0.1374101
5000	0.3603734	0.3361326	0.2819626	0.3903026	0.1083400

Table 4: 95% confidence intervals for the premium, with tail index $\alpha = 1.8$, and distortion parameter $\rho = 1.01$.

n	Π	$\widehat{\Pi}$	lb	ub	length
1000	0.4698994	0.4344532	0.3079989	0.5609075	0.2529086
3000	0.4071919	0.3828242	0.3095364	0.4561119	0.1465756
5000	0.3563748	0.3393401	0.2796627	0.3990175	0.1193549

5 Proof

Denoting

$$H_1 = \rho(k/n)^{1/\rho}\widehat{\theta}^{1/\rho-1/\widehat{\alpha}}\widetilde{X}_{n-k,n}\left\{\frac{1}{\widehat{\alpha}-\rho} - \frac{1}{\alpha-\rho}\right\}$$

$$H_2 = \frac{\rho(k/n)^{1/\rho}\widehat{\theta}^{1/\rho-1/\widehat{\alpha}}Q_{\widetilde{X}}(1-k/n)}{\alpha-\rho}\left\{\frac{\widetilde{X}_{n-k,n}}{Q_{\widetilde{X}}(1-k/n)} - 1\right\},$$

$$H_3 = \frac{\rho(k/n)^{1/\rho}\widehat{\theta}^{1/\rho-1/\widehat{\alpha}}Q_{\widetilde{X}}(1-k/n)}{\alpha-\rho} - \int_{Q_X(1-k/n)}^{\infty}(S_X(x))^{1/\rho}dx.$$

Then, we can verifies easily that

$$\widehat{\Pi}_{\rho,R}(X) - \Pi_{\rho,R}(X) = H_1 + H_2 + H_3.$$

H_1 can be written also

$$H_1 = \frac{\rho\widehat{\alpha}\alpha\widehat{\theta}^{1/\rho-1/\widehat{\alpha}}(k/n)^{1/\rho}\widetilde{X}_{n-k,n}}{(\widehat{\alpha}-\rho)(\alpha-\rho)}\left\{\frac{1}{\widehat{\alpha}} - \frac{1}{\alpha}\right\}$$

Since $\widehat{\alpha}$ is a consistent estimator for α (see Mason[18]), and $\widehat{\theta}$ is a consistent estimator of θ (see Weissman and Novak [30]), then for all large n

$$H_1 = (1+o_P(1))\frac{\rho\alpha^2\theta^{1\rho-1/\alpha}(k/n)^{1/\rho}Q_{\widetilde{X}}(1-k/n)}{(\alpha-\rho)^2}\left\{\frac{1}{\widehat{\alpha}} - \frac{1}{\alpha}\right\}$$

and

$$H_2 = (1+o_P(1))\frac{\rho(k/n)^{1/\rho}\theta^{1/\rho-1/\alpha}Q_{\widetilde{X}}(1-k/n)}{\alpha-\rho}\left\{\frac{\widetilde{X}_{n-k,n}}{Q_{\widetilde{X}}(1-k/n)} - 1\right\}$$

and

$$H_3 = (1+o_P(1))\frac{\rho(k/n)^{1/\rho}\theta^{1/\rho-1/\alpha}Q_{\widetilde{X}}(1-k/n)}{\alpha-\rho} - \int_{Q_X(1-k/n)}^{\infty}(S_X(x))^{1/\rho}dx$$

In view of Theorems 2.3 and 2.4 of Csörgő and Mason [5], Peng [24], and Necir et al. [20] has been shown that under the second-order condition (4) and for all large n

$$\sqrt{k}\alpha\left(\frac{1}{\widehat{\alpha}}-\frac{1}{\alpha}\right)=\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)-\sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds+o_P(1),$$

$$\sqrt{k}\left(\frac{\widetilde{X}_{n-k,n-1}}{Q_{\widetilde{X}}(1-k/n)}-1\right)=-\alpha^{-1}\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)+o_P(1),$$

and

$$\frac{\widetilde{X}_{n-k,n}}{Q_{\widetilde{X}}(1-k/n)}=1+o_P(1),$$

where $\{B_n(s), 0\leq s\leq 1, n=1,2,\dots\}$ is the sequence of Brownian bridges. This implies that for all large n

$$H_1=(1+o_P(1))\frac{\rho\alpha\theta^{1/\rho-1/\alpha}(k/n)^{1/\rho}Q_{\widetilde{X}}(1-k/n)}{k^{1/2}(\alpha-\rho)^2}\left(\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)\right.$$
$$\left.-\sqrt{\frac{n}{k}}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds+o_P(1)\right)$$

$$H_2=(1+o_P(1))\frac{\rho\theta^{1/\rho-1/\alpha}(k/n)^{1/\rho}Q_{\widetilde{X}}(1-k/n)}{k^{1/2}\alpha(\alpha-\rho)}\left(-\sqrt{\frac{n}{k}}B_n\left(1-\frac{k}{n}\right)+o_P(1)\right).$$

We can write H_3 as

$$H_3=(1+o_P(1))\frac{\rho(k/n)^{1/\rho}\theta^{1/\rho}Q_{\widetilde{X}}(1-\theta k/n)}{\alpha-\rho}-\int_{Q_{\widetilde{X}}(1-\theta k/n)}^{\infty}(S_{\widetilde{X}}(x))^{1/\rho}dx$$

Recall, from Karamata's Theorem (see de Haan and Ferreira [12]), that

$$\frac{\int_{Q_{\widetilde{X}}(1-\theta k/n)}^{\infty}(S_{\widetilde{X}}(x))^{1/\rho}dx}{(\theta k/n)^{1/\rho}Q_{\widetilde{X}}(1-\theta k/n)}\to\frac{1}{\alpha/\rho-1}=\frac{\rho}{\alpha-\rho},\ as\ n\to\infty,$$

since $(S_X(x))^{1/\rho}$ is regular varying with index $-\frac{\alpha}{\rho}<-1$ and $Q_X(1-k/n)\to\infty$ as $n\to\infty$. Hence

$$H_3\to 0,\ as\ n\to\infty.$$

From Necir et al [20] and Necir et al [21], we have

$$\frac{(k/n)^{-1/\rho}k^{1/2}}{Q_{\widetilde{X}}(1-k/n)}(H_1+H_2)=\Delta_n+o_P(1),$$

with

$$\Delta_n=\theta^{1/\rho-1/\alpha}\left[\frac{\rho\alpha}{(\alpha-\rho)^2}\left(\frac{\rho}{\alpha^2}-\frac{1}{\alpha}+1\right)(n/k)^{1/2}B_n(1-k/n)-\frac{\rho\alpha}{(\alpha-\rho)^2}(n/k)^{1/2}\int_{1-k/n}^{1}\frac{B_n(s)}{1-s}ds\right],$$

then the asymptotic variance of $\frac{(k/n)^{-1/\rho}k^{1/2}}{Q_{\widetilde{X}}(1-k/n)}(\widehat{\Pi}_{\rho,R}(X)-\Pi_{\rho,R}(X))$ will be computed by

$$\sigma^2(\rho,\theta,\alpha)=\lim_{n\to\infty}E(\Delta_n)^2=\theta^{2(1/\rho-1/\alpha)}\frac{\rho\alpha^2-2\rho^2\alpha+\rho^3+\rho\alpha^4}{\alpha^3(\alpha-\rho)^2}.$$

This completes the proof of Theorem 3.1.

Acknowledgements

The authors would like to thank the referee for careful reading and for their comments which greatly improved the paper.

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