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Generalized Incomplete 2D Hermite Polynomials and Their Generating Relations

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Abstract: The object of this paper is to derive generating relations for the generalized Incomplete 2D Hermite polynomials $\phi_{m,n}(x, y;\tau)$ by giving suitable interpretations to the indices (m) and (n) through Weisner's method.

Keywords: Generalized incomplete 2D Hermite Polynomials, Generating relations.

1. Introduction

The generalized incomplete 2D Hermite polynomials discussed in the present paper are characterized by two indices, two variables and one parameter.

These polynomials are defined [2, Eq.(12)]

$$\phi_{m,n}(x, y;\tau) = \frac{1}{\sqrt{m!n!}} \frac{h_{m,n}(x, y;\tau)}{(2\tau)^{\left(\frac{m+n}{2}\right)}} e^{\frac{xy}{2\tau}} \tag{1.1}$$

where the Incomplete 2D Hermite polynomials $h_{m,n}(x, y, \tau)$ are explicitly provided by the series [2,Eq.(1a)]

$$h_{m,n}(x, y, \tau) = m!n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r x^{m-r} y^{n-r}}{r!(m-n)!(n-r)!} \tag{1.2}$$

These polynomials defined by (1.1) satisfy the following simultaneous partial differential equations

$$\tau \frac{\partial^2 w}{\partial y^2} - \frac{x^2 w}{4\tau} - mw = 0 \tag{1.3}$$

and

$$\tau \frac{\partial^2 w}{\partial x^2} - \frac{y^2 w}{4\tau} - nw = 0 \tag{1.4}$$

Dattoli et al. [2, 3,4] introduced and discussed a theory of Incomplete 2D Hermite polynomials. They discussed the properties of a new family of multi index Lucas type polynomials , which are often encountered in problems of intracavity photon statistics . They develop an approach based on the integral representation method and show that this class of polynomials can be derived from recently introduced multi-index Hermite like polynomials.

Wunnsche [7] introduced Hermite 2D polynomials and discussed their properties and their explicit representations. Recently, Subuhi et al.[5,6] derived some generating relations involving Hermite 2D and some implicit summations formulae for Incomplete 2D Hermite polynomials by using different analytical means on their respective generating functions.

In this paper, we have obtained new generating relations for the generalized Incomplete 2D Hermite Polynomials by constructing a Lie algebra with the help of Weisner's [1] method by giving suitable interpretations to the indices (m) and (n) of the polynomials under consideration. The principal interest in the given results lies in the fact that a number of special cases listed in section 3 would yield many new results of the theory of special functions.

2. Group-theoretic method

Replacing n by $p\frac{\partial}{\partial p}$ and m by $s\frac{\partial}{\partial s}$ in (1.3) and (1.4) respectively, we get

$$\tau\frac{\partial^2 u}{\partial y^2}-s\frac{\partial u}{\partial s}-\frac{x^2 u}{4\tau}=0 \;, \tag{2.1}$$

$$\tau\frac{\partial^2 u}{\partial x^2}-p\frac{\partial u}{\partial p}-\frac{y^2 u}{4\tau}=0 \tag{2.2}$$

We see that $u(x,y,p,s;\tau)=\phi_{m,n}(x,y;\tau)p^n s^m$ is a solution of (2.1) and (2.2), since $\phi_{m,n}(x,y,\tau)$ is a solution of (1.3) and (1.4).

We first consider the following first order linear differential operators

$$A_1 = p\frac{\partial}{\partial p}$$

$$A_2 = s\frac{\partial}{\partial s}$$

$$A_3 = s^{-1}\left(\frac{-2}{\sqrt{2\tau}}\left(\frac{x}{2}-\tau\frac{\partial}{\partial y}\right)\right)$$

$$A_4 = s\left(\frac{1}{\sqrt{2\tau}}\left(\frac{x}{2}+\tau\frac{\partial}{\partial y}\right)\right)$$

$$A_5 = p^{-1}\left(\frac{-2}{\sqrt{2\tau}}\left(\frac{y}{2}-\tau\frac{\partial}{\partial x}\right)\right)$$

$$A_6 = p\left(\frac{1}{\sqrt{2\tau}}\left(\frac{y}{2}+\tau\frac{\partial}{\partial x}\right)\right)$$

such that

$$A_1[\phi_{m,n}(x,y;\tau)p^n s^m]=n\phi_{m,n}(x,y;\tau)p^n s^m$$

$$A_2[\phi_{m,n}(x,y;\tau)p^n s^m]=m\phi_{m,n}(x,y;\tau)p^n s^m$$

$$A_3[\phi_{m,n}(x,y;\tau)p^n s^m]=\sqrt{m}\phi_{m-1,n}(x,y;\tau)p^n s^{m-1}$$

$$A_4[\phi_{m,n}(x,y;\tau)p^n s^m]=\sqrt{m+1}\phi_{m+1,n}(x,y;\tau)p^n s^{m+1}$$

$$A_5[\phi_{m,n}(x,y;\tau)p^n s^m]=\sqrt{n}\phi_{m,n-1}(x,y;\tau)p^{n-1} s^m$$

$$A_6[\phi_{m,n}(x,y;\tau)p^n s^m]=\sqrt{n+1}\phi_{m,n+1}(x,y;\tau)p^{n+1} s^m$$

where the operators $\{A_i : i=1,2,3,4,5,6\}$ satisfy the following commutation relations

$[A_1,A_2]=0 \quad [A_2,A_3]=-A_3$

$[A_3,A_4]=0 \quad [A_4,A_5]=0 \quad [A_5,A_6]=0$

$[A_1,A_3]=0 \quad [A_2,A_4]=A_4$

$[A_3,A_5]=0 \quad [A_4,A_6]=0 \quad [A_1,A_4]=0$

$[A_2,A_5]=0 \quad [A_3,A_6]=0$

$[A_1,A_5]=-A_5 \quad [A_2,A_6]=0 \quad [A_1,A_6]=A_6$

where $[A,B]=AB-BA$.

The above commutation relations show that the set of operators $\{A_i : i=1,2,3,4,5,6\}$ generate a Lie-algebra λ and the sets of operators $\{A_1,A_5,A_6\}$ and $\{A_2,A_3,A_4\}$ form a sub algebras of λ.

It is clear that the differential operators

$$L_1=\tau\frac{\partial^2}{\partial y^2}-s\frac{\partial}{\partial s}-\frac{x^2}{4\tau},$$

$$L_2=\tau\frac{\partial^2}{\partial x^2}-p\frac{\partial}{\partial p}-\frac{y^2}{4\tau},$$

which can be expressed as: $L_1=A_3A_4-m$ and $L_2=A_5A_6-n$

commutes with $\{A_i : i=1,2,3,4,5,6\}$ that is

$$\begin{cases}[L_1,A_i]=0, i=1,2,3,4,5,6\\ [L_2,A_i]=0, i=1,2,3,4,5,6\end{cases} \tag{2.3}$$

The extended form of the groups generated by $\{A_i : i=1,2,3,4,5,6\}$ are as follows:

$$e^{a_1A_1}f(x,y,p,s;\tau)=f(x,y,pe^{a_1},s;\tau),$$

$$e^{a_2A_2}f(x,y,p,s;\tau)=f(x,y,p,se^{a_2};\tau),$$

$$e^{a_3A_3}f(x,y,p,s;\tau)=\exp\left(\frac{-a_3 y}{s\sqrt{2\tau}}\right)f(x,y+\frac{a_3\sqrt{2\tau}}{s},p,s;\tau)$$

$$e^{a_4A_4}f(x,y,p,s;\tau)=\exp\left(\frac{a_4 sx}{2\sqrt{2\tau}}\right), f(x,y+\frac{a_4 s\tau}{\sqrt{2\tau}},p,s;\tau)$$

$$e^{a_5A_5}f(x,y,p,s;\tau)=\exp\left(\frac{-a_5 y}{p\sqrt{2\tau}}\right), f(x+\frac{a_5\sqrt{2\tau}}{s},y,p,s;\tau)$$

$$e^{a_6A_6}f(x,y,p,s;\tau)=\exp\left(\frac{a_6 py}{2\sqrt{2\tau}}\right) f(x+\frac{a_6 py}{\sqrt{2\tau}},y,p,s;\tau)$$

where $f(x,y,p,s;\tau)$ is an arbitrary function. Then we have

$$e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}f(x,y,p,s;\tau)= \exp\left(\frac{a_6py}{2\sqrt{2\tau}}\right)\exp\left(\frac{-a_5y}{p\sqrt{2\tau}}\right)\exp\left(\frac{a_4sx}{2\sqrt{2\tau}}\right)\exp\left(\frac{-a_3y}{s\sqrt{2\tau}}\right) .f\left(\begin{array}{l} x+\frac{a_5\sqrt{2\tau}}{p}+\frac{a_6p\tau}{\sqrt{2\tau}},y \\ +\frac{a_3\sqrt{2\tau}}{s}+\frac{a_4s\tau}{\sqrt{2\tau}},pe^{a_1},se^{a_2};\tau\end{array}\right) \quad (2.4)$$

3. Generating functions

From the above discussion, we see that $u(x,y,p,s;\tau)=\phi_{m,n}(x,y;\tau)p^ns^m$ is a solution of the following systems

$$\begin{cases} L_1u=0 \\ (A_3A_4-m)u=0 \end{cases}$$

$$\begin{cases} L_2u=0 \\ (A_5A_6-n)u=0 \end{cases}$$

From (2.3) , we easily see that

$$\mathrm{S}=e^{a_6A_6}e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}$$

$$SL_1(\phi_{m,n}(x,y;\tau)p^ns^m)=L_1S(\phi_{m,n}(x,y;\tau)p^ns^m),$$

and $SL_2(\phi_{m,n}(x,y;\tau)p^ns^m)=L_2S(\phi_{m,n}(x,y;\tau)p^ns^m)$, where

Therefore, the transformation $S(\phi_{m,n}(x,y;\tau)p^ns^m)$ is also annulled by L_1 and L_2.

By setting $\{a_i=0:i=1,2,3,4;a_5=a,a_6=b\}$ and writing $f(x,y,p,s;\tau)=\phi_{m,n}(x,y;\tau)p^ns^m$ in (2.4), we get

$$e^{bA_6}e^{aA_5}(\phi_{m,n}(x,y;\tau)p^ns^m) =\exp\left(\frac{-ay}{p\sqrt{2\tau}}\right)\exp\left(\frac{bpy}{2\sqrt{2\tau}}\right) .\phi_{m,n}(x+\frac{a\sqrt{2\tau}}{p}+\frac{bp\tau}{\sqrt{2\tau}},y;\tau)p^ns^m \quad (3.1)$$

but

$$e^{bA_6}e^{aA_5}(\phi_{m,n}(x,y;\tau)p^ns^m)= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{b^l}{l!}\frac{a^k}{k!}\sqrt{(n+1)(n+2)...(n+k)} \sqrt{(n+k)(n+k-1)...(n+k-l+1)} .\phi_{m,n+k-l}(x,y;\tau)p^{n+k-l}s^m \quad (3.2)$$

Combining the above two relations (3.1) and (3.2), we get

$$\exp\left(\frac{-ay}{p\sqrt{2\tau}}\right)\exp\left(\frac{bpy}{2\sqrt{2\tau}}\right).\phi_{m,n}(x+\frac{a\sqrt{2\tau}}{p}+\frac{bp\tau}{\sqrt{2\tau}},y;\tau)= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{a^l}{l!}\frac{b^k}{k!}\sqrt{(n+1)(n+2)...(n+k)} \sqrt{(n+k)(n+k-1)...(n+k-l+1)} .\phi_{m,n+k-l}(x,y;\tau)p^{k-l} \quad (3.3)$$

where $|b|<\infty$ and $|a|<\infty$.

If we put $a=0,\ p=1$ in equation (3.3), we get

$$\exp\left(\frac{by}{2\sqrt{2\tau}}\right).\phi_{m,n}(x+\frac{b\tau}{\sqrt{2\tau}},y;\tau)= \sum_{k=0}^{\infty}\frac{b^k}{k!}\sqrt{(n+1)(n+2)...(n+k)}.\phi_{m,n+k}(x,y;\tau) \quad (3.4)$$

where $|b|<\infty$.

If we put $b=0,\ p=1$ in equation (3.3), we get

$$\exp\left(\frac{-ay}{\sqrt{2\tau}}\right).\phi_{m,n}(x+a\sqrt{2\tau},y;\tau)= \sum_{l=0}^{\infty}\frac{a^l}{l!}\sqrt{n(n-1)(n-2)...(n-l+1)}.\phi_{m,n-l}(x,y;\tau) \quad (3.5)$$

where $|a|<\infty$. Again by setting $\{a_i=0:i=1,2,5,6;a_3=c,a_4=d\}$ and writing $f(x,y,p,s;\tau)=\phi_{m,n}(x,y;\tau)p^ns^m$ in (2.4) we get

$$e^{dA_4}e^{cA_3}(\phi_{m,n}(x,y;\tau)p^ns^m) =\exp\left(\frac{-cy}{s\sqrt{2\tau}}\right)\exp\left(\frac{dsx}{2\sqrt{2\tau}}\right) .\phi_{m,n}(x,y+\frac{c\sqrt{2\tau}}{s}+\frac{ds\tau}{\sqrt{2\tau}};\tau)p^ns^m \quad (3.6)$$

but

$$e^{dA_4}e^{cA_3}(\phi_{m,n}(x,y;\tau)p^n s^m) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^l}{l!}\frac{d^k}{k!}\sqrt{(m+1)(m+2)...(m+k)}\sqrt{(m+k)(m+k-1)...(m+k-l+1)}\,.\phi_{m+k-l,n}(x,y;\tau)p^n s^{m+k-l} \quad (3.7)$$

Combining the above two relations (3.6) and (3.7), we get

$$\exp\left(\frac{-cy}{s\sqrt{2\tau}}\right)\exp\left(\frac{dsx}{2\sqrt{2\tau}}\right).\phi_{m,n}\left(x, y+\frac{c\sqrt{2\tau}}{s}+\frac{ds\tau}{\sqrt{2\tau}};\tau\right) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^l}{l!}\frac{d^k}{k!}\sqrt{(m+1)(m+2)...(m+k)}\sqrt{(m+k)(m+k-1)...(m+k-l+1)}\,.\phi_{m+k-l,n}(x,y;\tau)s^{k-l} \quad (3.8)$$

where $|c|<\infty, |d|<\infty$.

If we put $c=0, s=1$ in equation (3.8), we get

$$\exp\left(\frac{dx}{2\sqrt{2\tau}}\right).\phi_{m,n}\left(x, y+\frac{d\tau}{\sqrt{2\tau}}, y;\tau\right) = \sum_{k=0}^{\infty}\frac{d^k}{k!}\sqrt{(m+1)(m+2)...(m+k)}\,.\phi_{m+k,n}(x,y;\tau) \quad (3.9)$$

where $|d|<\infty$. If we put $d=0, s=1$ in equation (3.8), we get

$$\exp\left(\frac{-cy}{\sqrt{2\tau}}\right).\phi_{m,n}(x, y+c\sqrt{2\tau};\tau) = \sum_{l=0}^{\infty}\frac{c^l}{l!}\sqrt{m(m-1)(m-2)...(m-l+1))}\,.\phi_{m-l,n}(x,y;\tau) \quad (3.10)$$

where $|c|<\infty$.

4. Conclusion

We have seen that Weisner's group theoretic method is a power full tool in getting generating functions. It is also interesting to define a new function which forms generalization for the generalized incomplete 2D Hermite Polynomials under consideration and then by using Lie theoretic technique, we can obtain generating functions. We will deal with this aspect in the subsequent communication.

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