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About a King-type operator

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Abstract: In this paper we discuss some properties of a King-type operator. We give an approximation theorem and a Voronovskaja type theorem for this operator.

Keywords: Bernstein's polynomials, King-type operator, Voronovskaja theorem.

1. Introduction

In [6], J.P. King defined linear positive operators which generalize the classical Bernstein operators. These operators reproduce the test functions e_0 and e_2. In the papers [1], [2], [4], [5] and [7], operators of King's type are studied. In this paper, we define new King-type operators which reproduce the test functions e_1 and e_2. After studying its approximation properties, we give a Voronovkaja-type theorem.

2. Preliminaries

Let N be the set of positive integers and $N_0 = N \cup \{0\}$.
In this section, we recall some notions and results which we will use in this article. Following [8], we consider $I \subset R$, I an interval and we shall use the function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f | f : I \to R,\ f$ bounded on $I\}$, $C(I) = \{f | f : I \to R,\ f$ continuous on $I\}$ and $C_B(I) = B(I) \cap C(I)$. For $x \in I$, consider the function $\psi_x : I \to R$, $\psi_x(t) = t - x$, for any $t \in I$. Let a, b, a', b' be real numbers, $I \subset R$ interval, $a < b, a' < b'$, $[a, b] \subset I, [a', b'] \subset I$, and $[a, b] \cap [a', b'] \neq \phi$. For any $m \in N$, consider the functions $\varphi_{m,k} : I \to R$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a', b']$, for any $k \in \{0, 1, 2, ..., m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \to R$, for any $k \in \{0, 1, 2, ..., m\}$. For $m \in N$, define the operator: $L^*_m : E([a, b]) \to F(I)$ by

$$(L^*_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{1}$$

for any $f \in E([a, b])$, for any $x \in I$ and for $i \in N_0$, define $T^*_{m,i}$ by

$$\begin{aligned}(T^*_{m,i} L^*_m)(x) &= m^i (L^*_m \psi^i_x)(x) \\ &= m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi^i_x),\end{aligned} \tag{2}$$

for any $x \in [a, b] \cap [a', b']$. In the following, let s be a fixed natural number, s even and we suppose that the operators $(L^*_m)_{m\geq 1}$ verify the condition: there exists the smallest $\alpha_s, \alpha_{s+1} \in [0; \infty)$ so that

$$\lim_{m\to\infty} \frac{(T^*_{m,j} L^*_m)(x)}{m^{\alpha_j}} = B_j(x) \in R, \tag{3}$$

for any $x \in [a, b] \cap [a', b'], j \in \{s, s + 2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{4}$$

If $I \subset R$ is a given interval and $f \in C_B(I)$, then, the first order modulus of smoothness of f is the function $\omega(f; .) : [0; \infty) \to R$ defined for any $\delta \geq 0$ by

$$\begin{aligned}\omega(f; \delta) = sup\{&|f(x') - f(x'')| \\ &: x', x'' \in I; |x' - x''| \leq \delta\}.\end{aligned}$$

*Remark.*For $m \in N$, the L^*_m operators are linear and positive.

Theorem 1.[8] *Let $f : [a, b] \to R$ be a function. If $x \in [a, b] \cap [a', b']$ and f is a s times derivable function in x, the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left[(L^*_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T^*_{m,i} L^*_m)(x) \right] = 0. \tag{5}$$

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If f is a s times derivable function on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exists $m(s)\in N$ and $k_j\in R$ so that for any natural number $m, m\geq m(s)$ and for any $x\in[a,b]\cap[a',b']$ we have

$$\frac{(T^*_{m,j}L^*_m)(x)}{m^{\alpha_j}}\leq k_j, \tag{6}$$

where $j\in\{s,s+2\}$, then the convergence given in (2.5) is uniform on $[a,b]\cap[a',b']$ and

$$m^{s-\alpha_s}\left|(L^*_mf)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}(T^*_{m,i}L^*_m)(x)\right|\leq \tag{7}$$

$\leq \frac{1}{s!}(k_s+k_{s+2})\omega\left(f^{(s)};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)$, *for any $x\in[a,b]\cap[a',b']$, for any natural number $m, m\geq m(s)$.*

3. The study the convergence and a Voronovskaja-type theorem

In the following, we consider a fixed number $m_0\in N$, $m_0>2$. For the function $f:[0;1]\to R$, we define the sequence of operators $(B^*_mf)_{m\geq m_0}$ by

$$(B^*_mf)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$$
$$\sum_{k=0}^{m}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k f\left(\frac{k}{m}\right), \tag{8}$$

for any $m\geq m_0$ and any $x\in\left[\frac{1}{m_0-1},1\right]$. For fixed m_0, from $m\geq m_0$ it results that $m>m_0-1$. Then, from $x\geq\frac{1}{m_0-1}$ we have $mx-1\geq\frac{m-m_0+1}{m_0-1}>0$, so $mx-1\neq 0$ for any $x\in\left[\frac{1}{m_0-1},1\right]$.

In the following, we use the construction and the results from the first section. For the operator above, we have $\varphi_{m,k}(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$ and $A_{m,k}=f\left(\frac{k}{m}\right)$, for any $m\geq m_0$, any $k\in\{0,1,2,...,m\}$ and any $x\in\left[\frac{1}{m_0-1},1\right]$.

If $m\in N$, $m\geq m_0$, then the operator B^*_m is linear and positive.

The verify is immediate.

Lemma 1. *The identities*

$$(B^*_me_0)(x)=\frac{(m-1)x}{mx-1}, \tag{9}$$

$$(B^*_me_1)(x)=x, \tag{10}$$

$$(B^*_me_2)(x)=x^2, \tag{11}$$

$$(T^*_{m,0}B^*_m)(x)=\frac{(m-1)x}{mx-1}, \tag{12}$$

$$(T^*_{m,1}B^*_m)(x)=\frac{mx(x-1)}{mx-1}, \tag{13}$$

$$(T^*_{m,2}B^*_m)(x)=\frac{m^2x^2(1-x)}{mx-1} \tag{14}$$

hold for any $m\in N, m\geq m_0$ and any $x\in\left[\frac{1}{m_0-1},1\right]$.

Proof. We have that

$(\mathrm{B}^*_me_0)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$$\sum_{k=0}^{m}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$$
$$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(1-x+x-\frac{1}{m}\right)^m$$
$$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(1-\frac{1}{m}\right)^m$$
$$=\frac{(m-1)x}{mx-1},$$
$(\mathrm{B}^*_me_1)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$$\sum_{k=0}^{m}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k\frac{k}{m}$$
$$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(x-\frac{1}{m}\right)$$
$$\sum_{k=1}^{m}m-1k-1(1-x)^{(m-1)-(k-1)}\left(x-\frac{1}{m}\right)^{k-1}$$
$$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$$
$$\frac{mx-1}{m}\left(1-x+x-\frac{1}{m}\right)^{m-1}=x,$$
$$(B^*_me_2)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$$
$$\sum_{k=0}^{m}\frac{k^2}{m^2}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$$
$$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\cdot$$
$$\left(\frac{m-1}{m}\left(x-\frac{1}{m}\right)^2\sum_{k=2}^{m}m-2k-2(1-x)^{m-k}\right.$$
$$\left(x-\frac{1}{m}\right)^{k-2}+\frac{1}{m}\left(x-\frac{1}{m}\right)\cdot$$
$$\left.\sum_{k=1}^{m}C^{m-1}_{k-1}(1-x)^{m-k}\left(x-\frac{1}{m}\right)^{k-1}\right)=x^2,$$
which means that (3.2)-(3.4) hold.

By using the relations (3.2)-(3.4) we have that $(\mathrm{T}^*_{m,0}B^*_m)(x)=(B^*_me_0)(x)=\frac{(m-1)x}{mx-1}$, $(T^*_{m,1}B^*_m)(x)=m(B^*_m\psi_x)(x)=m((B^*_me_1)(x)-x(B^*_me_0)(x))$
$=\frac{mx(x-1)}{mx-1}$,
$(T^*_{m,2}B^*_m)(x)=m^2(B^*_m\psi^2_x)(x)$
$=m^2((B^*_me_2)(x)-2x(B^*_me_1)(x)+x^2(B^*_me_0)(x))$, from where (3.5)-(3.7) follows.

Lemma 2. *We have that*

$$B_0(x) = \lim_{m\to\infty} (T^*_{m,0}B^*_m)(x) = 1 \quad (15)$$

$$B_2(x) = \lim_{m\to\infty} \frac{(T^*_{m,2}B^*_m)(x)}{m} = x(1-x) \quad (16)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and*

$$(T^*_{m,0}B^*_m)(x) \le m_0 - 1 = k_0 \quad (17)$$

$$\frac{(T^*_{m,2}B^*_m)(x)}{m} \le \frac{m_0}{4} = k_2 \quad (18)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$.

Proof. The relations (3.8)-(3.9) results immediately from Lemma 1. The function $\frac{x}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1}, 1\right]$, so the maximum is obtained for $x = \frac{1}{m_0-1}$. On the other hand, the inequality $x(1-x) \le \frac{1}{4}$ holds, for any $x \in \left[\frac{1}{m_0-1}, 1\right]$. Then $\frac{(T^*_{m,2}B^*_m)(x)}{m} = m\frac{x}{mx-1}\, x(1-x) \le \frac{m}{4(m-m_0+1)}$, and because $\frac{m}{m-m_0+1} \le m_0$, for any $m \in N, m \ge m_0$, inequality (3.11) is obtained. Similarly, we have that $(T_{m,0}B^*_m)(x) = (m-1)\frac{x}{mx-1} \le \frac{m-1}{m-m_0+1} \le m_0 - 1$, so (3.10) results.

Theorem 2. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *Then*

$|(B^*_m f)(x) - f(x)| \le |f(x)|\frac{1-x}{mx-1} + \frac{x}{mx-1}$
$\left(m - 1 + \frac{1}{\delta}\sqrt{(m-1)x(1-x)}\right)\omega(f;\delta)$,
$|(B^*_m f)(x) - f(x)| \le \frac{(m_0-2)M}{m-m_0+1} + \frac{2(m-1)}{m-m_0+1}\omega$
$\left(f; \frac{1}{2\sqrt{m-1}}\right)$ (18) for any $\delta > 0, m \in N, m \ge m_0$ and $x \in \left[\frac{1}{m_0-1}, 1\right]$, where M= $\sup\left\{|f(x)| : x \in \left[\frac{1}{m_0-1}, 1\right]\right\}$.

Proof. We have $(B^*_m\psi_x^2)(x) = \frac{(T^*_{m,2}B^*_m)(x)}{m^2}$ and by taking (3.7) into account we obtain $(B^*_m\psi_x^2)(x) = \frac{x^2(1-x)}{mx-1}$. Now, by using Shisha-Mond Theorem (see [3]), we obtain inequality (3.12). We take that $\frac{1-x}{mx-1} \le \frac{m_0-2}{m-m_0+1}$, $\frac{x}{mx-1} \le \frac{1}{m-m_0+1}$ and $x(1-x) \le \frac{1}{4}$ for any $x \in \left[\frac{1}{m_0-1}, 1\right]$, any $m \in N, m \ge m_0$ and we consider $\delta = \frac{1}{2\sqrt{m-1}}$. Then, from (3.12), (3.13) follows.

Lemma 3. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$.

There exists $m(0)$ *so that*

$$\left|(B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right| \quad (19)$$

$$\le \frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right),$$

$$|(B^*_m f)(x) - f(x)| \le |f(x)|\,\frac{1-x}{mx-1} \quad (20)$$

$$+\frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and any* $m \in N$, $m \ge m(0)$, $m(0)$ *introduced in Theorem 1.*

Proof. The relations (3.14) results from Theorem 1 for $s = 0$, Lemma 1 and Lemma 2. By using the inequality $|a-c| - |b-c| \le |a-b|$, where $a, b, c \in R$, we have that

$$|(B^*_m f)(x) - f(x)| - \left|\frac{(m-1)x}{mx-1}f(x) - f(x)\right|$$

$$\le \left|(B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right|,$$

and taking (3.14) into account, the inequality (3.15) is obtained.

Theorem 3. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *Then*

$$\lim_{m\to\infty}(B^*_m f)(x) = f(x) \quad (21)$$

uniformly on $\left[\frac{1}{m_0-1}, 1\right]$. *There exists* $m(0)$ *so that*

$$|(B^*_m f)(x) - f(x)| \le \frac{(m_0-2)M}{m-m_0+1} \quad (22)$$

$$+\frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and any* $m \in N$, $m \ge m(0)$.

Proof. By using the inequality $\frac{1-x}{mx-1} \le \frac{m_0-2}{m-m_0+1}$ demonstrated in Theorem 2 and taking (3.15) into account, the inequality (3.17) is obtained. The relation (3.16) results from (3.17).

Theorem 4. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *If* $x \in \left[\frac{1}{m_0-1}, 1\right]$, f *is two times differentiable in* x *and* $f^{(2)}$ *is continuous in* x, *then*

$$\lim_{m\to\infty} m\left((B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right)$$
$$= (x-1)f^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x),$$
$$\lim_{m\to\infty} m\left((B^*_m f)(x) - f(x)\right)$$
$$= \frac{1-x}{x}f(x) + (x-1)f^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x).$$

Proof. Relation (3.18) results from Theorem 1 for $s = 2$, Lemma 1 and Lemma 2. From (3.18), it results (3.19).

Remark. Theorem 4 is a Voronovskaja's type theorem.

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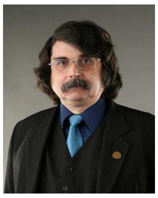

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