

A-Statistical Summability of Fourier Series and Walsh-Fourier Series

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Abstract: In this paper we study A -statistical summability of conjugate Fourier series, derived Fourier series and Walsh-Fourier series. At the end of the paper it is shown that A -statistical summability version is stronger than A -summability.

Keywords: Statistical convergence; A -statistical convergence; Fourier series; Walsh-Fourier series.

1. Introduction

Let $K = \{k_i\}$ be an index set and let φ^K be the characteristic sequence of K , i.e. $\varphi^K = (\varphi_j^K)$ with

$$\varphi_j^K = \begin{cases} 1, & \text{for } j \in K; \\ 0, & \text{otherwise.} \end{cases}$$

If φ^K is C_1 -summable then the limit

$$\lim_n \sum_{j=1}^n \varphi_j^K$$

is called the *asymptotic density* of K and is denoted by $\delta(K)$. A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to L if $\delta(K_\epsilon) = 0$ for every $\epsilon > 0$, where $K_\epsilon := \{k \in N : |x_k - L| \geq \epsilon\}$ (cf. [4], [7]). In this case L is called the *st-limit* of x .

Let c and l_∞ denote the set of all convergent and bounded sequences, respectively. Let X and Y be two sequence spaces and $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of real or complex numbers. We write $Ax = (A_n(x))$ provided $A_n(x) = \sum_k a_{nk}x_k$ converges for each n . A sequence $x = (x_k)$ is said to be A -summable to L if $\lim_n A_n(x) = L$. If $x = (x_k) \in X$ implies that $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and by (X, Y) we denote the class of such matrices. If X and Y are equipped with the limits X -lim and Y -lim, respectively, $A \in (X, Y)$ and Y -lim $_n A_n(x) = X$ -lim $_k x_k$ for

all $x \in X$, then we say that A is a regular map from X into Y and in this case we write $A \in (X, Y)_{reg}$. The matrices $A \in (c, c)_{reg}$ are called *regular*.

For a non-negative regular matrix $A = (a_{nk})_{n,k=1}^\infty$, following Freedman and Sember [6], an index set $K = \{k_i\}$ is said to have *A-density*

$$\delta_A(K) = \lim_n A_n \varphi^K$$

if $A\varphi^K \in c$. Thus

$$\delta_A(K) = \lim_n \sum_{k \in K} a_{nk} = \lim_n \sum_i a_{n, k_i}.$$

A sequence x is said to be *A-statistically convergent* to L if $\delta_A(K_\epsilon) = 0$ for every $\epsilon > 0$. In this case we write st_A -lim $x = L$, and $x_k \xrightarrow{st_A} L$. By the symbol st_A we denote the set of all A -statistically convergent sequences. Note that $c \subset st_A$.

The following characterization is due to Kolk [9, Corollary 4] which will be needed in our results.

Lemma 11 $B = (b_{nk})_{n,k=1}^\infty \in (c, st_A \cap l_\infty)_{reg}$ if and only if

- (i) $\|B\| = \sum_k |b_{nk}| < \infty$, and there exists $N = \{n_i\}$ so that $\delta_A(N) = 1$ and
- (ii) $\lim_i b_{n_i k} = 0$ ($k \in N$), i.e. st_A -lim $b_{nk} = 0$ for each k ,

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(iii) $\lim_i \sum_k b_{n_i k} = 1$, i.e. $st_A\text{-}\lim \sum_k b_{nk} = 1$.

Zygmund proved some theorems on the statistical convergence of Fourier series in the first edition of his book ([12], pp 181-188). Recently, *A*-statistical convergence have been used for some approximating operators in [1], [3] and [8]; and for Fourier integrals [10]. In this paper, we prove some results on statistical summability of conjugate Fourier series, derived Fourier series and Walsh-Fourier series.

2. Preliminaries

2.1. Let f be L -integrable and periodic with period 2π , and let the Fourier series of f be

$$\frac{1}{a_0} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{2.1}$$

Then the series conjugate to it is

$$\sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx), \tag{2.2}$$

with partial sum $\tilde{S}_n(x)$; and the derived series is

$$\sum_{k=1}^{\infty} k (b_k \cos kx - a_k \sin kx), \tag{2.3}$$

with partial sum $S'_n(x)$. We write

$$\psi_x(t) = \psi(f, t)$$

$$\begin{cases} = f(x+t) - f(x-t), & \text{for } 0 < t \leq \pi; \\ g(x), & \text{for } t = 0; \end{cases} \tag{2.4}$$

and

$$\beta_x(t) = \frac{\psi_x(t)}{4 \sin \frac{1}{2}t}, \tag{2.5}$$

where $g(x) = f(x+0) - f(x-0)$.

We shall need the following lemma which is statistical version of the Banach Weak Convergence Theorem [2].

Lemma 21 $st_A\text{-}\lim_{n \rightarrow \infty} \int_0^\pi g_n dh_x = 0$ for all $h_x \in BV[0, \pi]$, if and only if $\|g_n\| < \infty$ for all n and $st_A\text{-}\lim_{n \rightarrow \infty} g_n = 0$; where $BV[0, \pi]$ denotes the set of all functions of bounded variations on $[0, \pi]$.

Parallel version also holds if $[0, \pi]$ is replaced by $[0, 1]$.

2.2. Let us define a sequence of functions

$$h_0(x), h_1(x), \dots, h_n(x)$$

which satisfy the following conditions:

$$h_0(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq \frac{1}{2}; \\ -1, & \text{for } \frac{1}{2} \leq x < 1; \end{cases} \tag{2.6}$$

$h_0(x+1) = h_0(x)$ and $h_n(x) = h_0(2^n x)$, $n = 1, 2, \dots$. The functions $h_n(x)$ are called the *Rademacher's functions*.

The *Walsh functions* are defined by $\phi_0(x) = 1$,

$$\phi_n(x) = h_{n_1}(x)h_{n_2}(x)\dots h_{n_r}(x), \quad 0 \leq x \leq 1, \tag{2.7}$$

for $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_r}$; where the integers n_i are uniquely determined by $n_{i+1} < n_i$.

Let f be L -integrable and periodic with period 1, and let the Walsh-Fourier series of f be

$$\sum_{n=1}^{\infty} c_n \phi_n(x), \tag{2.8}$$

where

$$c_n = \int_0^1 f(x) \phi_n(x) dx, \tag{2.9}$$

are called the Walsh-Fourier coefficients of f .

Let us recall some basic properties of Walsh functions (see [4]).

For each fixed $x \in [0, 1]$ and for all $t \in [0, 1]$

- (i) $\phi_n(x+t) = \phi_n(x)\phi_n(t)$,
- (ii) $\int_0^1 f(x+t) dt = \int_0^1 f(t) dt$, and
- (iii) $\int_0^1 f(t) \phi_n(x+t) dt = \int_0^1 f(x+t) \phi_n(t) dt$;

where $+$ denotes the operation in the dyadic group, the set of all sequences $s = (s_n)$, $s_n = 0, 1$ for $n = 1, 2, \dots$, is addition modulo 2 in each coordinate.

Let for $x \in [0, 1]$,

$$J_k(x) = \int_0^x \phi_k(t) dt, \quad k = 0, 1, 2, \dots \tag{2.10}$$

It is easy to see that $J_k(x) = 0$ for $x = 0, 1$.

3. Main results

We prove the following theorems:

Theorem 31 Let $B = (b_{nk})_{n,k=1}^\infty \in (c, st_A \cap l_\infty)_{reg}$. Then for every $x \in [-\pi, \pi]$ for which $\beta_x(t) \in BV[0, \pi]$,

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} S'_n(x) = \beta_x(0+) \tag{3.1.1}$$

if and only if

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} \sin(n + \frac{1}{2})t = 0 \text{ for all } t \in [0, \pi]. \tag{3.1.2}$$

Theorem 32 Let $B \in (c, st_A \cap l_\infty)_{reg}$. Then for every $x \in [-\pi, \pi]$ for which $\beta_x(t) \in BV[0, \pi]$,

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} n b_{mn} \tilde{S}_n(x) = \frac{1}{\pi} g(x) \tag{3.2.1}$$

if and only if

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} \cos nt = 0 \text{ for all } t \in [0, \pi]. \quad (3.2.2)$$

Theorem 33 Let $B \in (c, st_A \cap l_{\infty})_{reg}$. Let $z_k(x) = c_k \phi_k(x)$ for an L -integrable function $f \in BV[0, 1)$. Then for every $x \in [0, 1)$

$$st_A\text{-}\lim_n \sum_{k=1}^{\infty} b_{nk} z_k(x) = 0 \quad (3.3.1)$$

if and only if

$$st_A\text{-}\lim_n \sum_{k=1}^{\infty} b_{nk} J_k(x) = 0, \quad (3.3.2)$$

where x is a point at which $f(x)$ is of bounded variation.

4. Proofs

Proof(Proof of Theorem 3.1.). We have

$$\begin{aligned} S'_n(x) &= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \left(\sum_{k=1}^n k \sin kt \right) dt \\ &= -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{d}{dt} \left(\frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{t}{2}} \right) dt, \\ &= I_n + \frac{2}{\pi} \int_0^{\pi} \sin(n + \frac{1}{2})t d\beta_x(t), \end{aligned} \quad (3.1.3)$$

where

$$I_n = \frac{1}{\pi} \int_0^{\pi} \beta_x(t) \cos \frac{t}{2} \left(\frac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} \right) dt. \quad (3.1.4)$$

Then

$$\sum_{n=1}^{\infty} b_{mn} S'_n(x) = \sum_{n=1}^{\infty} b_{mn} I_n + \frac{2}{\pi} \int_0^{\pi} L_m(t) d\beta_x(t), \quad (3.1.5)$$

where

$$L_m(t) = \sum_{n=1}^{\infty} b_{mn} \sin(n + \frac{1}{2})t. \quad (3.1.6)$$

Since $\beta_x(t)$ is of bounded variation on $[0, \pi]$ and $\beta_x(t) \rightarrow \beta_x(0+)$ as $t \rightarrow 0$, $\beta_x(t) \cos \frac{t}{2}$ has also the same property. Hence by Jordan's convergence criterion for Fourier series $I_n \rightarrow \beta_x(0+)$ as $n \rightarrow \infty$.

Since $B \in (c, st_A \cap l_{\infty})_{reg}$, by the conditions of Lemma 1.1, we get

$$st_A\text{-}\lim_m \sum_{n=1}^{\infty} b_{mn} I_n = \beta_x(0+). \quad (3.1.7)$$

Now, it is enough to show that (3.1.2) holds if and only if

$$st_A\text{-}\lim_m \int_0^{\pi} L_m(t) d\beta_x(t) = 0. \quad (3.1.8)$$

Hence, by Lemma 2.1, it follows that (2.1.8) holds if and only if

$$\|L_m(t)\| \leq M \text{ for all } m \text{ and for all } t \in [0, \pi], \quad (3.1.9)$$

and (3.1.2) holds. Since (3.1.9) is satisfied by Lemma 1.1(i), it follows that (3.1.8) holds if and only if (3.1.2) holds. Hence the result follows immediately.

Proof(Proof of Theorem 3.2). We have

We have

$$\begin{aligned} \tilde{S}_n(x) &= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sin nt dt, \\ &= \frac{g(x)}{n\pi} + \frac{1}{n\pi} \int_0^{\pi} \cos nt d\psi_x(t). \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} nb_{mn} \tilde{S}_n(x) = \frac{g(x)}{\pi} \sum_{n=1}^{\infty} b_{mn} + \frac{1}{\pi} \int_0^{\pi} K_m(t) d\psi_x(t), \quad (3.2.3)$$

where

$$K_m(t) = \sum_{n=1}^{\infty} b_{mn} \cos nt.$$

Now, taking st_A -lim on both sides of (3.2.3) and using Lemmas 1.1 and 2.1 as in the proof of Theorem 3.1, we get the required result.

Proof(Proof of Theorem 3.3). We have

$$\begin{aligned} z_k(x) &= c_k \phi_k(x) = \int_0^1 f(t) \phi_k(t) \phi_k(x) dt, \\ &= \int_0^1 f(t) \phi_k(x \dot{+} t) dt = \int_0^1 f(x \dot{+} t) \phi_k(t) dt, \end{aligned}$$

where $x \dot{+} t$ belongs to the set Ω of dyadic rationals in $[0, 1)$, in particular each element of Ω has the form $p/2^n$ for some non-negative integers p and n , $0 \leq p < 2^n$. Now, on integration by parts, we obtain

$$z_k(x) = [f(x \dot{+} t) J_k(t)]_0^1 - \int_0^1 J_k(t) df(x \dot{+} t),$$

$$= - \int_0^1 J_k(t) df(x \dot{+} t), \text{ since } J_k(x) = 0 \text{ for } x = 0, 1.$$

Hence

$$\sum_{k=1}^{\infty} b_{nk} z_k(x) = - \int_0^1 D_n(t) dh_x(t), \quad (3.3.3)$$

where

$$D_n(t) = \sum_{k=1}^{\infty} b_{nk} J_k(t), \quad (3.3.4)$$

and $h_x(t) = f(x+t)$. Write, for any $t \in R$, $g_n = (D_n(t))$.

Since $B \in (c, st_A \cap l_\infty)_{reg}$, it follows by Lemma 1.1 that $\|g_n\| < \infty$ for all n , and $g_n \xrightarrow{st_A} 0$. Hence by Lemma 2.1,

$$\int_0^1 D_n(t) dh_x(t) \xrightarrow{st_A} 0.$$

Now, taking st_A -lim in (3.3.3) and (3.3.4) and using Lemma 2.1, we get the desired result.

Remark. Theorem 3.3 generalizes the result of Mursaleen [11] and the following example shows that this theorem is stronger than that of [11].

Example. Let A be a $(C, 1)$ matrix, $B = I$ (identity matrix). Let us define a sequence $u = (u_k)$ by

$$u_k := \begin{cases} 1, & \text{if } k = m^2, m \in N; \\ 0, & \text{otherwise.} \end{cases}$$

Then u is A -statistically convergent to 0 but not A -summable. Now write $\hat{z}_k(x) = (1+u_k)z_k(x)$ and $\hat{J}_k(x) = (1+u_k)J_k(x)$, it is easy to see that Theorem 3.3 holds if we replace $z_k(x)$ by $\hat{z}_k(x)$ and $J_k(x)$ by $\hat{J}_k(x)$ but Theorem 3.1 of [11] does not hold.

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