Applied Mathematics & Information Sciences
An International Journal

Well-Posedness of the Modified Crank-Nicholson Difference Schemes in $C_{\tau}^{\beta,\gamma}(E)$ and $\widetilde{C}_{\tau}^{\beta,\gamma}(E)$ Spaces

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Received: April 12, 2011; Revised Oct. 24, 2011; Accepted Oct. 26, 2011
Published online: 1 Sep. 2012

Abstract: In the present paper the modified Crank-Nicholson difference schemes for the approximate solutions of the nonlocal boundary value problem

$$v^{'}(t) + Av(t) = f(t)(0 \leq t \leq 1), v(0) = v(\lambda) + \mu, 0 < \lambda \leq 1$$

for differential equations in an arbitrary Banach space E with the strongly positive operator A are considered. The well-posedness of these difference schemes in $C_{\tau}^{\beta,\gamma}(E)$ and $\widetilde{C}\ _{\tau}^{\beta,\gamma}(E)$ spaces is established. In applications, the coercive stability estimates for the solutions of difference schemes of the second order of accuracy over time and of an arbitrary order of accuracy over space variables in the case of the nonlocal boundary value problem for the $2m$th-order multidimensional parabolic equation are obtained.

Keywords: Abstract parabolic problem, Banach spaces, coercive inequality, well-posedness, r-modified Crank-Nicholson difference schemes.

1. Introduction

It is known that (see, e.g., [1]- [5] and the references given therein) many applied problems in fluid mechanics, other areas of physics and mathematical biology were formulated into nonlocal mathematical models. However, such problems were not well investigated in general.

In the paper [6] the well-posedness in the spaces of smooth functions of the nonlocal boundary value problem

$$v^{'}(t) + Av(t) = f(t)(0 \leq t \leq 1), \tag{1}$$
$$v(0) = v(\lambda) + \mu,\ 0 < \lambda \leq 1$$

for differential equations in an arbitrary Banach space E with the strongly positive operator A was established.

For the construction of difference schemes we consider a uniform grid space

$$[0,1]_{\tau} = \{t_k = k\tau, 0 \leq k \leq N, N\tau = 1\}.$$

Assume that $2\tau \leq \lambda$. We consider the first order of accuracy implicit Rothe difference scheme

$$\frac{u_k - u_{k-1}}{\tau} + Au_k = \varphi_k, \varphi_k = f(t_k), t_k = k\tau, \tag{2}$$
$$1 \leq k \leq N, u_0 = u_{[\frac{\lambda}{\tau}]} + \mu,$$

and the second order of accuracy implicit difference scheme

$$\frac{u_k - u_{k-1}}{\tau} + A(I + \frac{\tau A}{2})u_k = (I + \frac{\tau A}{2})\varphi_k, \tag{3}$$
$$\varphi_k = f(t_k - \frac{\tau}{2}), t_k = k\tau, 1 \leq k \leq N,$$
$$u_0 = Du_{[\frac{\lambda}{\tau}]} + \mu + (\lambda - [\frac{\lambda}{\tau}]\tau)\varphi_{[\frac{\lambda}{\tau}]},$$

and the second order of accuracy r-modified Crank-Nicholson difference schemes

$$\frac{u_k - u_{k-1}}{\tau} + Au_k = \varphi_k,\ \varphi_k = f(t_k - \frac{\tau}{2}), \tag{4}$$
$$t_k = k\tau, 1 \leq k \leq r,$$
$$\frac{u_k - u_{k-1}}{\tau} + \frac{A}{2}(u_k + u_{k-1}) = \varphi_k,\ \varphi_k = f(t_k - \frac{\tau}{2}),$$
$$t_k = k\tau, r + 1 \leq k \leq N,$$
$$u_0 = Du_{[\frac{\lambda}{\tau}]} + \mu + (\lambda - [\frac{\lambda}{\tau}]\tau)\varphi_{[\frac{\lambda}{\tau}]}, r\tau \geq \lambda,$$

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$$u_0 = u_{\frac{\lambda}{\tau}} + \mu,\ r\tau < \lambda, \frac{\lambda}{\tau} \in Z^+,$$

$$u_0 = D_1 \frac{1}{2}(u_{[\frac{\lambda}{\tau}]} + u_{[\frac{\lambda}{\tau}]+1})$$

$$+\mu + (\lambda - [\frac{\lambda}{\tau}]\tau - \frac{\tau}{2})\varphi_{[\frac{\lambda}{\tau}]}, r\tau < \lambda, \frac{\lambda}{\tau} \notin Z^+$$

approximately solving the boundary value problem (1). Here and in future $Z^+ = \{2, 3, \cdots\}$, $D = (I-(\lambda-[\frac{\lambda}{\tau}]\tau)A)$ and $D_1 = (I - (\lambda - [\frac{\lambda}{\tau}]\tau - \frac{\tau}{2})A)$.

Let $F_\tau(E)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Banach space E. Next on $F_\tau(E)$ we introduce the Banach spaces
$C_\tau(E) = C([0,1]_\tau, E)$,
$C_\tau^{\beta,\gamma}(E) = C^{\beta,\gamma}([0,1]_\tau, E)(0 \le \gamma \le \beta < 1)$,
$\widetilde{C}_\tau^{\beta,\gamma}(E) = \widetilde{C}^{\beta,\gamma}([0,1]_\tau, E)(0 \le \gamma \le \beta < 1)$ with the norms

$$\| \varphi^\tau \|_{C_\tau(E)} = \max_{1\le k\le N} \| \varphi_k \|_E,$$

$$\| \varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)} = \| \varphi^\tau \|_{C_\tau(E)}$$

$$+ \sup_{1\le k<k+r\le N} \| \varphi_{k+r} - \varphi_k \|_E \frac{((k+r)\tau)^\gamma}{(r\tau)^\beta},$$

$$\| \varphi^\tau \|_{\widetilde{C}_\tau^{\beta,\gamma}(E)} = \| \varphi^\tau \|_{C_\tau(E)}$$

$$+ \sup_{1\le k<k+2r\le N} \| \varphi_{k+2r} - \varphi_k \|_E \frac{((k+2r)\tau)^\gamma}{(2r\tau)^\beta}.$$

Note that the Banach space $E_\alpha = E_\alpha(E, A)(0 < \alpha < 1)$ consists of those $v \in E$ for which the norm

$$\| v \|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \| A(\lambda + A)^{-1} v \|_E$$

is finite.

The difference scheme (2) or (3) or (4) is said to be coercively stable (well posed) in $F_\tau(E)$ if we have the coercive inequality

$$\| \{\tau^{-1}(u_k - u_{k-1})\}_1^N \|_{F_\tau(E)}$$

$$\le M[\| A\mu \|_{E'} + \| \varphi^\tau \|_{F_\tau(E)}], E' \subset E,$$

where M is independent not only of φ^τ, μ but also of τ.

In this paper M represent general positive constant having different means in different cases.

In the papers [7], [8] the stability and coercive stability of difference schemes (2), (3) and (4) for $r = 1$ in $C_\tau^{\alpha,\alpha}(E)$, $C_\tau(E_\alpha)$ and $\widetilde{C}_\tau^{\alpha,\alpha}(E)$ $(0 < \alpha < 1)$ spaces and almost coercive stability (with multiplier $\min\left\{\ln \frac{1}{\tau}, 1 + |\ln \| A \|_{E\to E}|\right\}$)of difference schemes (2), (3) and (4) for $r = 1$ in $C_\tau(E)$ spaces were established.

In general, we have not been able to obtain the coercive stability estimates for the solution of Crank-Nicholson difference scheme

$$\begin{cases} \frac{u_k - u_{k-1}}{\tau} + \frac{A}{2}(u_k + u_{k-1}) = \varphi_k, \varphi_k = f(t_k - \frac{\tau}{2}), \\ t_k = k\tau, 1 \le k \le N,\ u_0 = u_{\frac{\lambda}{\tau}} + \mu, \frac{\lambda}{\tau} \in Z^+, \\ u_0 = D_1 \frac{1}{2}(u_{[\frac{\lambda}{\tau}]} + u_{[\frac{\lambda}{\tau}]+1}) \\ +\mu + (\lambda - [\frac{\lambda}{\tau}]\tau - \frac{\tau}{2})\varphi_{[\frac{\lambda}{\tau}]}, \frac{\lambda}{\tau} \notin Z^+ \end{cases}$$

for the approximate solution of problem (1). Note that the stability and coercive stability of Crank-Nicholson difference scheme of the initial value problem for evolution differential equations have been developed extensively during long time (see [14]- [25] and references given therein).

In the paper [35] the well-posedness of modified Crank-Nicholson difference schemes (4) in $L_{p,\tau}(E)$ spaces under the assumption that the operator $-A$ generates an analytic semigroup $\exp\{-tA\}$ $(t \ge 0)$ with exponentially decreasing norm, when $t \to +\infty$

$$\| exp\{-tA\} \|_{E\to E} \le Me^{-\delta t}, \tag{5}$$

$$\| Aexp\{-tA\} \|_{E\to E} \le \frac{M}{t}, t > 0,\ \delta, M > 0,$$

are established.

In the present paper the well-posedness of r-modified Crank-Nicholson difference schemes (4) in $C_\tau^{\beta,\gamma}(E)$ $(0 \le \gamma \le \beta < 1)$, $\widetilde{C}_\tau^{\beta,\gamma}(E)$ $(0 \le \gamma \le \beta < 1)$, and $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ $(0 \le \gamma \le \beta \le \alpha < 1)$ spaces under the assumption (5) is established. In applications this abstract result permit us to obtain the coercive stability estimates for the solutions of difference schemes of the second order of accuracy over time and of an arbitrary order of accuracy over space variables in the case of the nonlocal boundary value problem for the $2m$th-order multidimensional parabolic equation.

Finally, well-posedness and methods for numerical solutions of the evolution differential equations have been studied extensively by many researchers (see [11]-[12], [26]-[42], and the references therein).

2. Well-posedness of (4) in $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ spaces

Initially, the following necessary lemmas that will be given.

Lemma 2.1 [9],[13]. For any $k \ge 1$ the following estimates hold:

$$\| R^k \|_{E\longrightarrow E} \le M(1 + \delta\tau)^{-k}, \tag{6}$$

$$\| k\tau AR^k \|_{E\longrightarrow E} \le M.$$

Here and in the future $R = (I + \tau A)^{-1}$.

Lemma 2.2 [16]. For any $1 \le k \le N$, one has the estimates

$$\| k\tau AB^k C^2 \|_{E\longrightarrow E} \le M, \tag{7}$$

$$\| B^k C \|_{E\longrightarrow E} \le M. \tag{8}$$

Here and in the future $B=(I-\frac{\tau A}{2})C, C=(I+\frac{\tau A}{2})^{-1}$.
Lemma 2.3 [35]. The operators

$$I-DR^{[\frac{\lambda}{\tau}]}, r\tau\geq\lambda,$$

$$I-B^{[\frac{\lambda}{\tau}]-r}R^r, r\tau<\lambda, \frac{\lambda}{\tau}\in Z^+,$$

$$I-D_1B^{[\frac{\lambda}{\tau}]-r}R^rC, r\tau<\lambda, \frac{\lambda}{\tau}\notin Z^+$$

have inverses

$$T_\tau=(I-DR^{[\frac{\lambda}{\tau}]})^{-1} \text{ if } \quad r\tau\geq\lambda,$$

$$T_\tau=(I-B^{[\frac{\lambda}{\tau}]-r}R^r)^{-1} \text{ if } \quad r\tau<\lambda, \frac{\lambda}{\tau}\in Z^+,$$

$$T_\tau=(I-D_1B^{[\frac{\lambda}{\tau}]-r}R^rC)^{-1}$$

$$\text{if } \quad r\tau<\lambda, \frac{\lambda}{\tau}\notin Z^+$$

and the estimate holds:

$$\parallel T_\tau\parallel_{E\to E}\leq M(\delta,\lambda). \tag{9}$$

Lemma 2.4 [35]. For any $\varphi_k,\ 1\leq k\leq N$ the solution of the problem (1) exists and the following formula holds

$$u_k=\begin{cases} R^ku_0+\sum\limits_{r=1}^{k}R^{k-j+1}\varphi_j\tau, k=1,\cdots,r,\\ B^{k-r}R^ru_0+\sum\limits_{j=1}^{r}B^{k-r}R^{r-j+1}\varphi_j\tau\\ +\sum\limits_{j=r+1}^{k}B^{k-j}C\varphi_j\tau, k=r+1,\cdots,N,\\ T_\tau\{D_1[\sum\limits_{j=1}^{r}R^{r-j+1}\\ \times B^{[\frac{\lambda}{\tau}]-r}C\varphi_j\tau+\sum\limits_{j=r+1}^{[\frac{\lambda}{\tau}]-1}B^{[\frac{\lambda}{\tau}]-j}C^2\varphi_j\tau\\ +\frac{\tau C}{2}\varphi_{[\frac{\lambda}{\tau}]+1}]\\ +\mu+\{(I+\frac{(\tau A)^2}{4})\left(\lambda-[\frac{\lambda}{\tau}]\tau-\frac{\tau}{2}\right)\\ +\tau I\}C^2\varphi_{[\frac{\lambda}{\tau}]}\}, r\tau<\lambda, \frac{\lambda}{\tau}\notin Z^+, k=0,\\ T_\tau\{[\sum\limits_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}\varphi_j\tau\\ +\sum\limits_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C\varphi_j\tau]+\mu\},\\ r\tau<\lambda, \frac{\lambda}{\tau}\in Z^+, k=0,\\ T_\tau\{D\sum\limits_{j=1}^{[\frac{\lambda}{\tau}]-1}R^{[\frac{\lambda}{\tau}]-j+1}\varphi_j\tau+\mu\\ +\left(\lambda-[\frac{\lambda}{\tau}]\tau+\tau\right)R\varphi_{[\frac{\lambda}{\tau}]}\},\\ r\tau\geq\lambda, k=0. \end{cases} \tag{10}$$

Theorem 2.1. Let τ be a sufficiently small positive number. Then the solutions of difference schemes (4) in $C_\tau^{\beta,\gamma}(E_{\alpha-\beta})$ $(0\leq\gamma\leq\beta\leq\alpha, 0<\alpha<1)$ satisfy the following coercivity inequalities

$$\parallel\{\tau^{-1}(u_k-u_{k-1})\}_1^N\parallel_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})} \tag{11}$$

$$\leq\frac{M_1}{\alpha(1-\alpha)}\parallel\varphi^\tau\parallel_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}$$

$$+M_1|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|_{1+\alpha-\beta}^{\beta,\gamma},$$

where M_1 does not depend on $\varphi^\tau,\mu,\alpha,\beta,\gamma$, and τ.

Here, the space of traces $\widetilde{E}_{1+\alpha-\beta}^{\beta,\gamma}=\widetilde{E_1}^{\beta,\gamma}(E_{\alpha-\beta})$ which consist of the elements $w\in E$ for which the norm

$$|w|_{1+\alpha-\beta}^{\beta,\gamma}$$

$$=\sup_{0<\tau\leq\tau_0}\{\max\{a(\tau),b(\tau)\}+\max\{c(\tau),d(\tau),e(\tau)\}\}$$

is finite. Here

$$a(\tau)=\max_{1\leq i\leq r}||AR^iw||_{E_{\alpha-\beta}},$$

$$b(\tau)=\max_{r+1\leq i\leq N}||AB^{i-r-1}CR^rw||_{E_{\alpha-\beta}},$$

$$c(\tau)=\sup_{1\leq i<i+l\leq r\leq N}p||A(R^{i+l}-R^i)w||_{E_{\alpha-\beta}},$$

$$d(\tau)$$

$$=\sup_{1\leq i\leq r<i+l\leq N}p||AR^i(B^{i+l-r-1}CR^{r-i}-I)w||_{E_{\alpha-\beta}},$$

$$e(\tau)=\sup_{1\leq r<i<i+l\leq N}p||AB^{i-r-1}CR^r(B^l-I)w||_{E_{\alpha-\beta}},$$

Here and in the future $p=(l\tau)^{-\beta}((i+l)\tau)^\gamma$.

Proof. By [17],

$$||\{\tau^{-1}(u_k-u_{k-1})\}_1^N\parallel_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}$$

$$\leq M[|u_0-A^{-1}\varphi_1|_{1+\alpha-\beta}^{\beta,\gamma}+\frac{1}{\alpha(1-\alpha)}\parallel\varphi^\tau\parallel_{C_\tau^{\beta,\gamma}(E_{\alpha-\beta})}]$$

for the solution of the r-modified Crank-Nicholson difference schemes

$$\begin{cases}\frac{u_k-u_{k-1}}{\tau}+Au_k=\varphi_k,\ \varphi_k=f(t_k-\frac{\tau}{2}),\\ t_k=k\tau, 1\leq k\leq r,\\ \frac{u_k-u_{k-1}}{\tau}+\frac{A}{2}(u_k+u_{k-1})=\varphi_k,\\ \varphi_k=f(t_k-\frac{\tau}{2}), t_k=k\tau, r+1\leq k\leq N,\\ u_0 \text{ is given.}\end{cases} \tag{12}$$

for the approximate solutions of Cauchy problem

$$u'(t)+Au(t)=f(t)(0\leq t\leq 1), u(0) \text{ is given.}$$

The proof of estimate (11) for difference schemes (4) is based on the estimate (2) and the following estimates

$$\max_{1\le i\le r}||AR^i(u_0-A^{-1}\varphi_1)||_{E_{\alpha-\beta}}\le L$$

$$\max_{r+1\le i\le N}||AB^{i-r-1}CR^r(u_0-A^{-1}\varphi_1)||_{E_{\alpha-\beta}}\le L$$

$$\sup_{1\le i<i+l\le r\le N}p||A(R^{i+l}-R^i)$$
$$(u_0-A^{-1}\varphi_1)||_{E_{\alpha-\beta}}\le L$$

$$\sup_{1\le i\le r<i+l\le N}p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$
$$\times(u_0-A^{-1}\varphi_1)||_{E_{\alpha-\beta}}\le L$$

$$\sup_{1\le r<i<i+l\le N}p||AB^{i-r-1}CR^r(B^l-I)$$
$$\times(u_0-A^{-1}\varphi_1)||_{E_{\alpha-\beta}}\le L,$$

$$L\le M_1[|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|^{\beta,\gamma}_{1+\alpha-\beta} \tag{13}$$
$$+\frac{M}{\alpha(1-\alpha)}\|\varphi^\tau\|_{C^{\beta,\gamma}_\tau(E_{\alpha-\beta})}]$$

for the solution of problem (4).

Let $r\tau\ge\lambda$. Using formula (10), we can write

$$u_0-A^{-1}\varphi_1=T_\tau\{D\sum_{j=1}^{[\frac{\lambda}{\tau}]-1}R^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau \tag{14}$$

$$-DA^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]}\}.$$

Then, using formula (14) and estimate (9), we obtain estimate (13).

Let $r\tau<\lambda$ and $\frac{\lambda}{\tau}\in Z^+$. Using formula (10), we can write

$$u_0-A^{-1}\varphi_1=T_\tau\{\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau \tag{15}$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^r\varphi_{[\frac{\lambda}{\tau}]}+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]}\}.$$

Then, using formula (15) and estimate (9), we obtain estimate (13).

Let $r\tau<\lambda$ and $\frac{\lambda}{\tau}\notin Z^+$. Using formula (10), we can write

$$u_0-A^{-1}\varphi_1 \tag{16}$$

$$=T_\tau\{D_1[\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1}B^{[\frac{\lambda}{\tau}]-j}C^2(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^rC\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]}.$$

Then, using formula (16) and estimate (9), we obtain estimate (13). Theorem 2.1 is proved.

Note that the spaces $C^{\beta,\gamma}_\tau(E_{\alpha-\beta})$ of grid functions, in which coercive solvability has been established, depend on the parameters α, β and γ. However, the constants in the coercive inequalities depend only on α. Hence, we can be choose the parameters β and γ freely, which increases the number of spaces of grid functions in which difference schemes (4) are well posed. In particular, Theorem 2.1 implies the well-posedness theorem in $C_\tau(E_\alpha)$ established in [7].

3. Well-posedness of (4) in $C^{\beta,\gamma}_\tau(E)$ and $\widetilde{C}^{\beta,\gamma}_\tau(E)$ spaces

Theorem 3.1. Let τ be a sufficiently small positive number. Then the solutions of difference schemes (4) in $C^{\beta,\gamma}_\tau(E)$ $(0\le\gamma\le\beta, 0<\beta<1)$ satisfy the following coercivity inequalities

$$\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\|_{C^{\beta,\gamma}_\tau(E)} \tag{17}$$
$$+\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\|_{C_\tau(\widetilde{E}^{\beta,\gamma}_1)}$$
$$\le M[\frac{1}{\beta(1-\beta)}\|(I+\frac{\tau A}{2})\varphi^\tau\|_{C^{\beta,\gamma}_\tau(E)}$$
$$+|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|^{\beta,\gamma}_1],$$

where M does not depend on $\varphi^\tau,\mu,\beta,\gamma$, and τ.

Proof. By [17],

$$\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\|_{C^{\beta,\gamma}_\tau(E)} \tag{18}$$
$$+\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\|_{C_\tau(\widetilde{E}^{\beta,\gamma}_1)}$$
$$\le M[\frac{1}{\beta(1-\beta)}\|(I+\frac{\tau A}{2})\varphi^\tau\|_{C^{\beta,\gamma}_\tau(E)}$$
$$+|u_0-A^{-1}\varphi_1|^{\beta,\gamma}_1]$$

for the solution of the r-modified Crank-Nicholson difference schemes (12). The proof of estimate (17) for difference schemes (4) is based on the estimate (18) and the following estimates

$$\max_{1\le i\le r}||AR^i(u_0-A^{-1}\varphi_1)||_E\le K,$$

$$\max_{r+1\le i\le N}||AB^{i-r-1}CR^r(u_0-A^{-1}\varphi_1)||_E\}\le K,$$

$$\sup_{1\le i<i+l\le r\le N} p||AR^i(R^l-I)(u_0-A^{-1}\varphi_1)||_E \le K,$$

$$\sup_{1\le i\le r<i+l\le N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)(u_0-A^{-1}\varphi_1)||_E$$

$$\le K,$$

$$\sup_{1\le r<i<i+l\le N} p||AB^{i-r-1}CR^r(B^l-I)(u_0-A^{-1}\varphi_1)||_E\}$$

$$\le K, K \le M_1[|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|_1^{\beta,\gamma}$$
$$+\frac{M}{\beta(1-\beta)}\| (I+\frac{\tau A}{2})\varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)}] \tag{19}$$

for the solution of problem (4).

Let $r\tau \ge \lambda$. Then, using formula (14) and estimate (9), we obtain

$$\max_{1\le i\le r} ||AR^i(u_0-A^{-1}\varphi_1)||_E$$

$$\le M(\lambda)(\max_{1\le i\le r} ||AR^iD\sum_{j=1}^{[\frac{\lambda}{\tau}]-1} R^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$
$$-DA^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}$$
$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E$$
$$\le M(\lambda)(|\mu+A^{-1}(-\varphi_1+\varphi_{[\frac{\lambda}{\tau}]})|_1^{\beta,\gamma}$$
$$+\frac{M}{\beta(1-\beta)}\| (I+\frac{\tau A}{2})\varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)}), \tag{20}$$

$$\max_{r+1\le i\le N} ||AB^{i-r-1}CR^r(u_0-A^{-1}\varphi_1)||_E$$
$$\le M(\lambda)\max_{1\le i\le r} ||AB^{i-r-1}CR^rD$$
$$\times(\sum_{j=1}^{[\frac{\lambda}{\tau}]-1} R^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$
$$-DA^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}$$
$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E$$
$$\le M(\lambda)(|\mu+A^{-1}(-\varphi_1+\varphi_{[\frac{\lambda}{\tau}]})|_1^{\beta,\gamma}$$
$$+\frac{M}{\beta(1-\beta)}\| (I+\frac{\tau A}{2})\varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)}), \tag{21}$$

$$\sup_{1\le i\le r<i+l\le N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$
$$\times(u_0-A^{-1}\varphi_1)||_E$$

$$\le M(\lambda)(\sup_{1\le i\le r<i+l\le N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$
$$\times(D\sum_{j=1}^{[\frac{\lambda}{\tau}]-1} R^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$
$$-DA^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}$$
$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E$$
$$\le M(\lambda)(|\mu+A^{-1}(-\varphi_1+\varphi_{[\frac{\lambda}{\tau}]})|_1^{\beta,\gamma}$$
$$+\frac{M}{\beta(1-\beta)}\| (I+\frac{\tau A}{2})\varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)}), \tag{22}$$

$$\sup_{1\le i\le r<i+l\le N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$
$$\times(u_0-A^{-1}\varphi_1)||_E$$

$$\le M(\lambda)(\sup_{1\le i\le r<i+l\le N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$
$$\times(DR^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$
$$-(I-(\lambda-[\frac{\lambda}{\tau}]\tau)A)A^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}$$
$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E$$
$$\le M(\lambda)(|\mu+A^{-1}(-\varphi_1+\varphi_{[\frac{\lambda}{\tau}]})|_1^{\beta,\gamma}$$
$$+\frac{M}{\beta(1-\beta)}\| (I+\frac{\tau A}{2})\varphi^\tau \|_{C_\tau^{\beta,\gamma}(E)}), \tag{23}$$

$$\sup_{1\le r<i<i+l\le N} p||AB^{i-r-1}CR^r(B^l-I)$$
$$\times(u_0-A^{-1}\varphi_1)||_E$$

$$\le M(\lambda)(\sup_{1\le r<i<i+l\le N} p||AB^{i-r-1}CR^r(B^l-I)$$

$$\times(D\sum_{j=1}^{[\frac{\lambda}{\tau}]-1} R^{[\frac{\lambda}{\tau}]-j+1}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-DA^{-1}R^{[\frac{\lambda}{\tau}]}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E$$

$$\leq M(\lambda)(|\mu+A^{-1}(-\varphi_1+\varphi_{[\frac{\lambda}{\tau}]})|_1^{\beta,\gamma}$$

$$+\frac{M}{\beta(1-\beta)}\|(I+\frac{\tau A}{2})\varphi^\tau\|_{C_\tau^{\beta,\gamma}(E)}). \tag{24}$$

From estimates (20)-(24) it follows estimate (19).

Let $r\tau<\lambda$ and $\frac{\lambda}{\tau}\in Z^+$. Then, using formula (15) and estimate (9), we obtain

$$\max\{\max_{1\leq i\leq r}||AR^i(u_0-A^{-1}\varphi_1)||_E,$$

$$\max_{r+1\leq i\leq N}||AB^{i-r-1}CR^r(u_0-A^{-1}\varphi_1)||_E\}$$

$$+\max\{\sup_{\substack{1\leq i<i+l\leq N\\ i+l\leq r}} p||AR^i(R^l-I)(u_0-A^{-1}\varphi_1)||_E,$$

$$\sup_{1\leq i\leq r<i+l\leq N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)(u_0-A^{-1}\varphi_1)||_E,$$

$$\sup_{1\leq r<i<i+l\leq N} p||AB^{i-r-1}CR^r(B^l-I)(u_0-A^{-1}\varphi_1)||_E\}$$

$$\leq M(\lambda)[\max\{\max_{1\leq i\leq r}||AR^i(\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\max_{r+1\leq i\leq N}||AB^{i-r-1}CR^r(\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E\}$$

$$+\max\{\sup_{1\leq i<i+l\leq r\leq N} p||AR^i(R^l-I)$$

$$\times(\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_{E_{\alpha-\beta}},$$

$$\sup_{1\leq i\leq r<i+l\leq N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$

$$\times(\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\sup_{1\leq i\leq r<i+l\leq N} p||AB^{i-r-1}CR^r(B^l-I)$$

$$\times(\sum_{j=1}^{r}R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]}B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E\}].$$

By [19],

$$\|(B^{k+n}-B^k)C\|_{E\to E}\leq\frac{M\tau^\alpha}{t_k^\alpha}, \tag{25}$$

$$\|A(B^{k+n}-B^k)C^3\|_{E\to E}\leq\frac{M\tau^\alpha}{t_k^{1+\alpha}} \tag{26}$$

for any $, 1 \leq \ k < k+n \leq N, 0 \leq \alpha \leq 1$. Similarly to the way the estimate (18) was obtained and using estimates (2), (7), (8), (25) and (26), we can show that

$$\max\{\max_{1\leq i\leq r} ||AR^i(\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]} B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\max_{r+1\leq i\leq N} ||AB^{i-r-1}CR^r(\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}$$

$$\times(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]} B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j$$

$$-\varphi_{[\frac{\lambda}{\tau}]})\tau -A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]})||_E\}$$

$$+\max\{\sup_{1\leq i<i+l\leq r\leq N} p||AR^i(R^l-I)$$

$$\times(\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]} B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\sup_{1\leq i\leq r<i+l\leq N} p||AR^i(B^{i+l-r-1}CR^{r-i}-I)$$

$$\times(\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]} B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\sup_{1\leq r<i<i+l\leq N} p||AB^{i-r-1}CR^r(B^l-I)$$

$$\times(\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]} B^{[\frac{\lambda}{\tau}]-j}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}B^{[\frac{\lambda}{\tau}]-r}R^{r}\varphi_{[\frac{\lambda}{\tau}]})||_E\}]$$

$$\leq \frac{M}{\beta(1-\beta)}\|\ (I+\frac{\tau A}{2})\varphi^\tau\ \|_{C_\tau^{\beta,\gamma}(E)}.$$

Applying the triangle inequality and this estimate and (20), we get estimate (19).

Let $r\tau<\lambda$ and $\frac{\lambda}{\tau}\notin Z^+$. Then, using formula (16) and estimate (9), we obtain

$$\max\{\max_{1\leq i\leq r}||AR^i(u_0-A^{-1}\varphi_1)||_E,$$

$$\max_{r+1\leq i\leq N}||AB^{i-r-1}CR^r(u_0-A^{-1}\varphi_1)||_E\}$$

$$+\max\{\sup_{1\leq i<i+l\leq r\leq N} p||AR^i(R^l-I)(u_0-A^{-1}\varphi_1)||_E,$$

$$\sup_{1\leq ii\leq r<i+l\leq N} p||AR^i(B^{i+l-r-1}CR^{r-\imath}-I)(u_0-A^{-1}\varphi_1)||_E,$$

$$\sup_{1\leq r<i<i+l\leq N} p||AB^{i-r-1}CR^r(B^l-I)(u_0-A^{-1}\varphi_1)||_E\}$$

$$\leq M(\lambda)[\max\{\max_{1\leq i\leq r}||AR^i((I-(\lambda-[\frac{\lambda}{\tau}]\tau-\frac{\tau}{2})A)$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^2(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^rC\varphi_{[\frac{\lambda}{\tau}]}+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\max_{r+1\leq i\leq N}||AB^{i-r-1}CR^r((I-(\lambda-[\frac{\lambda}{\tau}]\tau-\frac{\tau}{2})A)$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^2(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^rC\varphi_{[\frac{\lambda}{\tau}]}+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]$$

$$+\mu-A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E\}$$

$$+\max\{\sup_{1\le i<i+l\le r\le N} p||AR^{i}(R^{l}-I)(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]$$

$$+\mu - A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\sup_{1\le i\le r<i+l\le N} p||A(B^{i+l-r-1}CR^{r}-R^{i})(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]$$

$$+\mu - A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E,$$

$$\sup_{1\le r<i<i+l\le N} p||AB^{i-r-1}CR^{r}(B^{l}-I)(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})]$$

$$+\mu - A^{-1}\varphi_1+A^{-1}\varphi_{[\frac{\lambda}{\tau}]})||_E\}.$$

Similarly to the way the estimate (18) was obtained and using estimates (2), (7), (8), (25) and (26), we can show that

$$\max\{\max_{1\le i\le r}||AR^{i}(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})])||_E,$$

$$\max_{r+1\le i\le N}||AB^{i-r-1}CR^{r}(D$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})])||_E\}$$

$$+\max\{\sup_{\substack{1\le i<i+l\le N\\ i+l\le r}} p||A(R^{i+l}-R^{i})(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})])||_E,$$

$$\sup_{1\le i\le r<i+l\le N} p||AR^{i}(B^{i+l-r-1}CR^{r-i}-I)(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^{2}(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^{r}C\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})])||_E,$$

$$\sup_{1\le r<i<i+l\le N} p||AB^{i-r-1}CR^r(B^l-I)(D_1$$

$$\times[\sum_{j=1}^{r} R^{r-j+1}B^{[\frac{\lambda}{\tau}]-r}C(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$+\sum_{j=r+1}^{[\frac{\lambda}{\tau}]-1} B^{[\frac{\lambda}{\tau}]-j}C^2(\varphi_j-\varphi_{[\frac{\lambda}{\tau}]})\tau$$

$$-A^{-1}D_1B^{[\frac{\lambda}{\tau}]-r}R^rC\varphi_{[\frac{\lambda}{\tau}]}$$

$$+\frac{\tau C}{2}(\varphi_{[\frac{\lambda}{\tau}]+1}-\varphi_{[\frac{\lambda}{\tau}]})])||_{E_{\alpha-\beta}}\}$$

$$\le \frac{M}{\beta(1-\beta)}\|\ (I+\frac{\tau A}{2})\varphi^\tau\ \|_{C_\tau^{\beta,\gamma}(E)}.$$

Applying the triangle inequality and this estimate and (20), we get estimate (19). Theorem 3.1 is proved.

Note that the coercive stability estimate (17) is weaker than respective an estimate for the solution of difference schemes (2) and (3) in $C_\tau^{\beta,\gamma}(E)$. However, obtaining this type of estimate is important for the applications. We denote by $a^\tau=\{a_k\}_1^N$ the mesh function of approximation. Then $\|(I+\frac{\tau A}{2})a^\tau\|_{C_\tau^{\beta,\gamma}(E)}\sim\|a^\tau\|_{C_\tau^{\beta,\gamma}(E)}=o(\tau^2)$ if we assume that $\tau\|Aa^\tau\|_{C_\tau^{\beta,\gamma}(E)}$ tends to 0 as $\tau\to 0$ not slower than $\|a^\tau\|_{C_\tau^{\beta,\gamma}(E)}$. It takes place in applications by supplementary restriction of the smooth property of the data of space variables.

Nevertheless, we have the following the smoothness estimates

$$\|\ B^{k+2n}-B^k\ \|_{E\to E}\le\frac{M\tau}{t_k}, \tag{27}$$

$$\|\ A(B^{k+2n}-B^k)C^2\ \|_{E\to E}\le\frac{M\tau^\alpha}{t_k^{1+\alpha}} \tag{28}$$

for all $1\le\ k<k+2n\le N$ and $0\le\alpha\le 1$. The estimates (27) and (28) were established in [23]. This result permit us to obtain the coercive stability estimates for the solutions of difference schemes (4) in $\widetilde{C}_\tau^{\beta,\gamma}(E)$ spaces.

Theorem 3.2. Let τ be a sufficiently small positive number. If the Crank-Nicolson difference scheme of the initial value problem for homogeneous parabolic equations is stable in $C_\tau(E)$, then the solutions of difference schemes (4) in $\tilde{C}_\tau^{\beta,\gamma}(E)$ $(0\le\gamma\le\beta, 0<\beta<1)$ satisfy the following coercivity inequalities

$$\|\ \{\tau^{-1}(u_k-u_{k-1})\}_1^N\ \|_{\widetilde{C}_\tau^{\beta,\gamma}(E)} \tag{29}$$

$$\le M[\frac{1}{\beta(1-\beta)}\|\ \varphi^\tau\ \|_{C_\tau^{\beta,\gamma}(E)}$$

$$+|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|_{*,1}^{\beta,\gamma}],$$

$$\|\ \{\tau^{-1}(u_k-u_{k-1})\}_1^N\ \|_{\widetilde{C}_\tau^{\beta,\gamma}(E)} \tag{30}$$

$$\le M[\frac{1}{\beta(1-\beta)}\|\ \varphi^\tau\ \|_{\widetilde{C}_\tau^{\beta,\gamma}(E)}$$

$$+|A^{-1}(I+\frac{\tau A}{2})(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_{[\frac{\lambda}{\tau}]-1})|_{*,1}^{\beta,\gamma}$$

$$+|\mu+A^{-1}(\varphi_{[\frac{\lambda}{\tau}]}-\varphi_1)|_{*,1}^{\beta,\gamma}],$$

where M does not depend on $\varphi^\tau,\mu\ ,\beta,\gamma,$ and τ. Here, the space of traces $\widetilde{E}_{*,1}^{\beta,\gamma}=\widetilde{E_{*,1}}^{\beta,\gamma}(E)$ which consist of the elements $w\in E$ for which the norm

$$|w|_{*,1}^{\beta,\gamma}=\sup_{0<\tau\le\tau_0}\{\max\{\max_{1\le i\le r}||AR^iw||_E,$$

$$\max_{r+1\le i\le N}||AB^{i-r-1}CR^rw||_E\}$$

$$+\max\{\sup_{1\le i<i+2l\le r\le N}q||AR^i(R^{2l}-I)w||_E,$$

$$\sup_{1\le i\le r<i+2l\le N}q||AR^i(B^{i+2l-r-1}CR^{r-i}-I)w||_E,$$

$$\sup_{1\le r<i<i+2l\le N}q||AB^{i-r-1}CR^r(B^{2l}-I)w||_E\}\}$$

is finite. Here $q=(2l\tau)^{-\beta}((i+2l)\tau)^\gamma$.

4. Applications

We consider the nonlocal boundary value problem on the range $\{0\le t\le 1, x\in \mathrm{R}^n\}$ for the $2m$th-order multidimensional differential equation of parabolic type

$$\begin{cases}\frac{\partial v(t,x)}{\partial t}+\sum\limits_{|\tau|=2m}a_\tau(x)\frac{\partial^{|\tau|}v(t,x)}{\partial x_1^{\tau_1}...\partial x_n^{\tau_n}}+\sigma v(t,x)\\ =f(t,x), 0<t<1, |\ \tau\ |=\tau_1+...+\tau_n,\\ v(0,x)=v(\lambda,x)+\mu(x), 0<\lambda\le 1, x\in\mathrm{R}^n,\end{cases} \tag{31}$$

where $a_r(x),\mu(x)$ and $f(t,x)$ are given sufficiently smooth functions. In this paper σ is the sufficiently large positive constant.

We will assume that the symbol

$$B^x(\xi)=\sum_{|r|=2m}a_r(x)\,(i\xi_1)^{r_1}...(i\xi_n)^{r_n},$$

$$\xi=(\xi_1,\cdot\cdot\cdot,\xi_n)\in\mathrm{R}^n$$

of the differential operator of the form

$$B^x=\sum_{|r|=2m}a_r(x)\frac{\partial^{|r|}}{\partial x_1^{r_1}\dots\partial x_n^{r_n}} \tag{32}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \leq (-1)^m B^x(\xi) \leq M_2|\xi|^{2m} < \infty$$

for $\xi \neq 0$.

Now, the abstract theorems given above are applied in the investigation of difference schemes for approximate solution of the problem (31). The discretization of problem (31) is carried out in two steps. Let us define the grid space R_h^n $(0 < h \leq h_0)$ as the set of all points of the Euclidean space R^n whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm 1, \pm 2, \cdots, k = 1, \cdots, n.$$

To the differential operator $A^x = B^x + \sigma I$ defined by (32), we assign the difference operator $A_h^x = B_h^x + \sigma I_h$. The operator

$$B_h^x = h^{-2m} \sum_{2m \leq |s| \leq S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \ldots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}}, \quad (33)$$

which acts on functions defined on the entire space R_h^n. Here $s \in \mathrm{R}^{\,2n}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm} f^h(x) = \pm \left(f^h(x \pm e_k h) - f^h(x) \right),$$

and here and in future e_k is the unit vector of the axis x_k.

An infinitely differentiable function of the continuous argument $y \in \mathrm{R}^n$ that is continuous and bounded together with all its derivatives is said to be smooth function. We say that the differences operator A_h^x is a λ-th order $(\lambda > 0)$ approximation of the differential operator A^x if the inequality

$$\sup_{x \in R_h^n} |A_h^x \varphi(x) - A^x \varphi(x)| \leq M(\varphi) h^\lambda$$

holds for any smooth function $\varphi(y)$. The coefficients b_s^x are chosen in such a way that the operator A_h^x approximates in a specified way the operator A^x.We shall assume that the operator A_h^x approximates the differential operator A^x with any prescribed order [44].

The function $A^x(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ in the right-hand side of equality (33) with the expression $\pm(\exp\{\pm i\xi_k h\} - 1)$, respectively, and is called the symbol of the difference operator B_h^x.

We shall assume that for $|\xi_k h| \leq \pi$ and fixed x the symbol $A^x(\xi h, h)$ of the operator $B_h^x = A_h^x - \sigma I_h$ satisfies the inequalities

$$\begin{cases} (-1)^m A^x(\xi h, h) \geq M|\xi|^{2m}, \\ |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2}. \end{cases} \quad (34)$$

Suppose that the coefficient b_s^x of the operator $B_h^x = A_h^x - \sigma I_h$ is bounded and satisfies the inequalities

$$|b_s^{x+e_k h} - b_s^x| \leq M h^\epsilon, x \in \mathrm{R}_h^n, \epsilon \in (0, 1]. \quad (35)$$

With the help of A_h^x we arrive at the nonlocal boundary-value problem

$$\frac{dv^h(t,x)}{dt} + A_h^x v^h(t,x) = f^h(t,x), 0 \leq t \leq 1, \quad (36)$$

$$v^h(0,x) = v^h(\lambda, x) + \mu^h(x), x \in \mathrm{R}_h^n,$$

for an infinite system of ordinary differential equations.

In the second step we replace problem (36) by the difference schemes

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + A_h^x u_k^h(x) = \varphi_k^h(x), \\ \varphi_k(x) = f^h(t_k - \frac{\tau}{2}, x), t_k = k\tau, \\ 1 \leq k \leq r, N\tau = 1, x \in \mathrm{R}_h^n, \\ \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + \frac{A_h^x}{2}(u_k^h(x) + u_{k-1}^h(x)) \\ = \varphi_k^h(x), \varphi_k^h(x) = f^h(t_k - \frac{\tau}{2}, x), \\ t_k = k\tau, r+1, \ x \in \mathrm{R}_h^n, \\ u_0^h(x) = (I - (\lambda - [\frac{\lambda}{\tau}]\tau - \frac{\tau}{2})A_h^x)\frac{1}{2}(u_{[\frac{\lambda}{\tau}]}^h(x) \\ + u_{[\frac{\lambda}{\tau}]+1}^h(x)) + \mu^h(x) \\ + (\lambda[\frac{\lambda}{\tau}]\tau - \frac{\tau}{2})\varphi_{[\frac{\lambda}{\tau}]}^h(x), \\ r\tau < \lambda, \frac{\lambda}{\tau} \notin Z^+, x \in \mathrm{R}_h^n, \\ u_0^h(x) = (I - (\lambda - [\frac{\lambda}{\tau}]\tau)A_h^x))u_{[\frac{\lambda}{\tau}]}^h(x) \\ + \mu^h(x) + (\lambda - [\frac{\lambda}{\tau}]\tau)\varphi_{[\frac{\lambda}{\tau}]}^h(x), r\tau \geq \lambda, x \in \mathrm{R}_h^n, \\ u_0^h(x) = u_{\frac{\lambda}{\tau}}^h(x) + \mu^h(x), \\ \\ r\tau < \lambda, \frac{\lambda}{\tau} \in Z^+, x \in \mathrm{R}_h^n. \end{cases} \quad (37)$$

Let us give a number of corollaries of the abstract theorems given in the above. To formulate our result we need to introduce the spaces $C_h = C(R_h^n)$ and $C_h^\beta = C^\beta(\mathrm{R}_h^n)$ of all bounded grid functions $u^h(x)$ defined on R_h^n, equipped with the norms

$$||u^h||_{C_h} = \sup_{x \varepsilon \mathrm{R}_h^n} |u^h(x)|,$$

$$||u^h||_{C_h^\beta} = \sup_{x \varepsilon \mathrm{R}_h^n} |u^h(x)| + \sup_{x, y \in \mathrm{R}_h^n} \frac{|u^h(x) - u^h(x+y)|}{|y|^\beta}.$$

Theorem 4.1. Suppose that assumptions (34) and (35) for the operator A_h^x hold. Then the solutions of difference schemes (37) satisfy the coercivity estimates:

$$\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_1^N \|_{C_\tau^{\beta,\gamma}\left(C_h^{2m(\alpha-\beta)+\nu}\right)}$$

$$\leq M(\alpha, \beta, \gamma, \nu)(\| \varphi^{\tau,h} \|_{C_\tau^{\beta,\gamma}\left(C_h^{2m(\alpha-\beta)+\nu}\right)}$$

$$+ \sum_{2m \leq |s| \leq S} h^{-2m} ||\Delta_{1-}^{s_1} \Delta_{1+}^{s_2} \ldots \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} \mu^h \|_{C_h^{2m(\alpha-\gamma)+\nu}}$$

$$+ ||\varphi_1^h - \varphi_{[\frac{\lambda}{\tau}]}^h \|_{C_h^{2m(\alpha-\gamma)+\nu}}),$$

$$0 \leq \gamma \leq \beta \leq \alpha, 0 < \nu + 2m(\alpha - \gamma) < 1,$$

$$\| \{\tau^{-1}(u_k^h - u_{k-1}^h)\}_1^N \|_{\widetilde{C}_\tau^{\beta,\gamma}(C_h^\nu)}$$

$$\leq M(\beta,\gamma,\nu)(\| \varphi^{\tau,h} \|_{C_\tau^{\beta,\gamma}(C_h^\nu)}$$

$$+\sum_{2m\leq|s|\leq S} h^{-2m}||\Delta_{1-}^{s_1}\Delta_{1+}^{s_2}...\Delta_{n-}^{s_{2n-1}}\Delta_{n+}^{s_{2n}}\mu^h \|_{C_h^{2m(\beta-\gamma)+\nu}}$$

$$+||\varphi_1^h-\varphi_{[\frac{\lambda}{\tau}]}^h \|_{C_h^{2m(\beta-\gamma)+\nu}}),$$

$$0\leq\gamma\leq\beta, 0<\nu+2m(\beta-\gamma)<1,$$

where $M(\alpha,\beta,\gamma,\nu)$ and $M(\beta,\gamma,\nu)$ do not depend on $\varphi^{\tau,h}$, μ^h, h and τ.

The proof of Theorem 4.1 is based on the abstract Theorems 2.1 and 3.1 and the positivity of the operator A_h^x in C_h [44] and on the following two theorems on the coercivity inequality for the solution of the elliptic difference equation in C_h^β and on the structure of the spaces $E_\beta(C_h, A_h^x)$.

Theorem 4.2 [10], [43]. Suppose that assumptions (34) and (35) for the operator A_h^x hold. Then for the solutions of the elliptic difference equation

$$A_h^x u^h(x)=\omega^h(x), x\in \mathrm{R}_h^n \qquad (38)$$

the estimates

$$\sum_{2m\leq|s|\leq S} h^{-2m}||\Delta_{1-}^{s_1}\Delta_{1+}^{s_2}...\Delta_{n-}^{s_{2n-1}}\Delta_{n+}^{s_{2n}}u^h||_{C_h^\beta}$$

$$\leq M(\sigma,\beta)||\omega^h||_{C_h^\beta}$$

are valid.

Theorem 4.3 [10], [43]. Suppose that assumptions (34) and (35) for the operator A_h^x hold. Then for any $0<\beta<\frac{1}{2m}$ the norms in the spaces $E_\beta(C_h, A_h^x)$ and $C_h^{2m\beta}$ are equivalent uniformly in h.

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