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$(\in, \in \vee q)$-Fuzzy Prime Ideals of BCK-Algebras

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Abstract: The aim of this paper is $(\in, \in \vee q)$-fuzzification of the concept of prime ideals in commutative BCK-algebras. We state and prove some theorems in $(\in, \in \vee q)$-fuzzy prime ideals in commutative BCK-algebras also we study relation between an $(\in, \in \vee q)$-fuzzy prime ideal and an $(\in \vee q)$-fuzzy level prime ideals.

Keywords: BCK-algebra, Prime ideal, Fuzzy prime ideal, $(\in, \in \vee q)$-fuzzy prime ideal.
AMS Subject Classification: 06F35, 03E72, 03G25, 11R44.

1 Introduction

The concept of fuzzy sets was first initiated by Zadeh([21]) in 1965. Since then these ideas have been applied to other algebraic structures such as group, semigroup, ring,vector spaces etc. Imai and Iseki ([11])introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebra is a proper sub class of the class of BCI-algebras. Iseki([12]) introduced the concept of prime ideal in commutative BCK-algebras. In ([2]) Ahsan, Deeba and Thaheem have studied the theory of ideals, in particular, prime ideals of a commutative BCK-algebras. In([18,19]) Jun and Xin have studied fuzzy prime ideals and invertible fuzzy ideals inBCK-algebras. Abdullah ([1]) introduced the notion of intuitionistic fuzzy prime ideals of commutative BCK-algebras. Murali ([16])proposed a definition of a fuzzy point belonging to fuzzy subset under natural equivalence on fuzzy subset . Bhakat and Das ([5,6])used the relation of "belongs to" and "quasi-coincident" between fuzzy point and fuzzy set to introduced the concept of $(\in, \in \vee q)$-fuzzy subgroup, $(\in, \in \vee q)$-fuzzy subring and $(\in \vee q)$-level subset. Basnet and Singh ([7])introduced $(\in, \in \vee q)$-fuzzy ideals of BG-algebra in 2011. Dhanani and Pawar([10]) introduced the concept of $(\in, \in \vee q)$-fuzzy ideals(prime) of lattice. It is now natural to investigate similar type of generalisation of the existing fuzzy subsystem with other algebraic structure. In this paper, we introduced the notion of $(\in, \in \vee q)$- fuzzy prime ideals of commutative BCK-algebras and got some interesting result.

2 Preliminaries

Definition 2.1([1,18,19]) *An algebra $(X, *, 0)$ of type (2, 0) is called a BCK-algebra if it satisfies the following axioms:*
(i) $((x*y)*(x*z))*(z*y) = 0$
(ii) $(x*(x*y))*y = 0$
(iii) $x*x = 0$
(iv) $0*x = 0$
(v) $x*y = 0$ *and* $y*x = 0 \Rightarrow x = y$ *for all* $x,y,z \in X$.
We can define a partial ordering " $\leq$ *" on X by* $x \leq y$ *iff* $x*y = 0$.

Definition 2.2([1,18,19]) *A BCK-algebra X is said to be commutative if it satisfies the identity* $x \wedge y = y \wedge x$ *where* $x \wedge y = y*(y*x) \forall x,y \in X$. *In a commutative BCK-algebra, it is known that* $x \wedge y$ *is the greatest lower bound of x and y.*
In a BCK-algebra X, the following hold:
(i) $x*0 = x$
(ii) $(x*y)*z = (x*z)*y$
(iii) $x*y \leq x$
(iv) $(x*y)*z \leq (x*z)*(y*z)$
(v) $x \leq y$ *implies* $x*z \leq y*z$ *and* $z*y \leq z*x$.

Definition 2.3([20]) *A nonempty subset I of a BCK-algebra X is called an ideal of X if*
(i) $0 \in I$
(ii) $x*y \in I$ *and* $y \in I \Rightarrow x \in I$ *for all* $x,y \in X$.

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Definition 2.4([20]) *A fuzzy set μ in BCK-algebra X is called a fuzzy ideal of X if it satisfies*
(i) $\mu(0) \geq \mu(x)$
(ii) $\mu(x) \geq min\{\mu(x*y), \mu(y)\}$ *for all* $x, y \in X$.

Definition 2.5([1]) *An ideal I of a commutative BCK-algebra X is said to be prime if* $x \wedge y \in I \Rightarrow x \in I$ *or* $y \in I$

Definition 2.6([19]) *A non constant fuzzy ideal μ of a commutative BCK-algebra X is said to be fuzzy prime if* $\mu(x \wedge y) \leq max\{\mu(x), \mu(y)\}$ *for all* $x, y \in X$ *Since* $x \wedge y \leq x, y$ *and μ is order reversing, it follows that* $\mu(x) \leq \mu(x \wedge y)$ *and* $\mu(y) \leq \mu(x \wedge y)$ *therefore a non constant fuzzy ideal μ of a commutative BCK-algebra X is fuzzy prime iff* $\mu(x \wedge y) = max\{\mu(x), \mu(y)\}$ *for all* $x, y \in X$ *or equivalently* $\mu(x \wedge y) = \mu(x)$ *or* $\mu(y)$ *for all* $x, y \in X$.

Definition 2.7*Let λ and μ be two fuzzy sets, then their union $\lambda \cup \mu$ and their intersection $\lambda \cap \mu$ are defined by* $(\lambda \cup \mu)(x) = max\{\lambda(x), \mu(x)\}$ *and* $(\lambda \cap \mu)(x) = min\{\lambda(x), \mu(x)\}$.

3 $(\in, \in \vee q)$-fuzzy prime ideals of BCK-algebra

In what follows, let X denote a commutative BCK-algebra unless otherwise stated.

Definition 3.1([5]) *A fuzzy set μ of the form*

$$\mu(y) = \begin{cases} t, & \text{if } y = x,\ t \in (0,1], \\ 0, & \text{if } y \neq x, \end{cases}$$

is called a fuzzy point with support x and value t and it is denoted by x_t.

Definition 3.2([5]) *Let μ be a fuzzy set in X and x_t be a fuzzy point then*
(i) If $\mu(x) \geq t$ *then we say x_t belongs to μ and write* $x_t \in \mu$
(ii)If $\mu(x) + t > 1$ *then we say x_t quasi coincident with μ and write* $x_t q \mu$
(iii)If $x_t \in \vee q \mu \Leftrightarrow x_t \in \mu$ *or* $x_t q \mu$
(iv) If $x_t \in \wedge q \mu \Leftrightarrow x_t \in \mu$ *and* $x_t q \mu$
The symbol $x_t \overline{\alpha} \mu$ *means* $x_t \alpha \mu$ *does not hold and* $\overline{\in \wedge q}$ *means* $\overline{\in} \vee \overline{q}$
For a fuzzy point x_t. and a fuzzy set μ in set X, Pu and Liu gave meaning to the symbol $x_t \alpha \mu$ *where* $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$

Definition 3.3[3] *Let μ be a fuzzy set in BCK-algebra X and* $t \in (0\ 1]$, *let*
$\mu_t = \{x \in X | x_t \in \mu\} = \{x \in X | \mu(x) \geq t\}$
$< \mu >_t = \{x \in X | x_t q \mu\} = \{x \in X | \mu(x) + t > 1\}$
$[\mu]_t = \{x \in X | x_t \in \vee q \mu\} = \{x \in X | \mu(x) \geq t\, or\, \mu(x) + t > 1\}$
Here μ_t is called t level set of μ , $< \mu >_t$ is called q level set of μ and $\overline{[\mu]}_t$ is called $(\in \vee q)$ level set of μ
Clearly $[\mu]_t = < \mu >_t \cup \mu_t$

Definition 3.4[20] *A fuzzy set μ of a BCK-algebra X is said to be (α, β)-fuzzy ideal of X, where* $\alpha \neq \in \wedge q$ *if*
(i) $x_t \alpha \mu \Rightarrow 0_t \beta \mu$
(ii) $(x*y)_t \alpha \mu, y_s \alpha \mu \Rightarrow x_{m(t,s)} \beta \mu$ *for all* $x, y \in X$
where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}; t, s \in (0,1]$ *and* $m(t,s) = min(t,s)$.

Example 3.5*Consider BCK-algebra* $X = \{0, x, y, z\}$ *with the following cayley table.*

*	0	x	y	z	w
0	0	0	0	0	0
x	x	0	x	0	x
y	y	y	0	y	0
z	z	x	z	0	z
w	w	w	y	w	0

Define a map $\mu : X \rightarrow [0,1]$ *by* $\mu(0) = 0.7, \mu(x) = \mu(z) = 0.3, \mu(y) = \mu(w) = 0.2$, *then μ is an $(\in, \in \vee q)$-fuzzy ideal of X.*

Definition 3.6*An (α, β)-fuzzy ideal μ of a BCK-algebra X is said to be an (α, β)-fuzzy prime ideal of X if*
$(x \wedge y)_t \alpha \mu \Rightarrow x_t \beta \mu$ *or* $y_t \beta \mu$, *for all* $x, y \in X$.
where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}; \alpha \neq \in \wedge q$ *and* $t \in (0,1]$.

Theorem 3.7*A fuzzy set μ of a BCK-algebra X is a fuzzy prime ideal if and only if μ is an $(\in, \in)$-fuzzy prime ideal.*

*Proof.*Let μ be a fuzzy prime ideal, therefore $\mu(x \wedge y) \leq max\{\mu(x), \mu(y)\}$
Let $(x \wedge y)_t \in \mu \Rightarrow \mu(x \wedge y) \geq t$
$\Rightarrow max\{\mu(x), \mu(y)\} \geq \mu(x \wedge y) \geq t$
$\Rightarrow \mu(x) \geq t$ or $\mu(y) \geq t$
$\Rightarrow x_t \in \mu$ or $y_t \in \mu$
Therefore μ is an $(\in, \in)$-fuzzy prime ideal.
Conversely, Let μ be an $(\in, \in)$-fuzzy prime ideal.
Let $x, y \in X$ and $\mu(x \wedge y) = t$ where $t \in [0,1]$
then $\mu(x \wedge y) \geq t \Rightarrow (x \wedge y)_t \in \mu$
$\Rightarrow x_t \in \mu$ or $y_t \in \mu$
$\Rightarrow \mu(x) \geq t$ or $\mu(y) \geq t$
$\Rightarrow max\{\mu(x), \mu(y)\} \geq t = \mu(x \wedge y)$
Therefore μ is a fuzzy prime ideal.

Theorem 3.8*μ is a (q, q)-fuzzy prime ideal if and only if μ is an $(\in, \in)$-fuzzy prime ideal.*

*Proof.*Let μ be a (q,q)-fuzzy ideal of a BCK-algebra X.
Let $x,y \in X$ such that $(x\wedge y)_t \in \mu$
$\Rightarrow \mu(x\wedge y) \geq t$
$\Rightarrow \mu(x\wedge y)+\delta > t$, for some $\delta > 0$
$\Rightarrow \mu(x\wedge y)+\delta - t + 1 > 1$
$\Rightarrow (x\wedge y)_{\delta-t+1} q\mu$
Since μ is a (q,q) fuzzy prime ideal X.
Therefore we have $x_{\delta-t+1} q\mu$ or $y_{\delta-t+1} q\mu$
$\Rightarrow \mu(x)+\delta-t+1 > 1$ or $\mu(y)+\delta-t+1 > 1$
$\Rightarrow \mu(x)+\delta > t$ or $\mu(y)+\delta > t$
$\Rightarrow \mu(x) \geq t$ or $\mu(y) \geq t$
$\Rightarrow x_t \in \mu$ or $y_t \in \mu$
therefore $(x\wedge y)_t \in \mu \Rightarrow x_t \in \mu$ or $y_t \in \mu$
Hence μ is an $(\in,\in)$-fuzzy prime ideal of X.
Conversely,
Assume μ is an $(\in,\in)$-fuzzy prime ideal of X.
Let $(x\wedge y)_t q\mu$
$\Rightarrow \mu(x\wedge y)+t > 1$
$\Rightarrow \mu(x\wedge y) > 1-t$
$\Rightarrow \mu(x\wedge y) \geq \delta-t+1 > 1-t$ for some $\delta > 0$
$\Rightarrow (x\wedge y)_{\delta-t+1} \in \mu$
Since μ is an $(\in,\in)$ fuzzy prime ideal X.
Therefore we have $x_{\delta-t+1} \in \mu$ or $y_{\delta-t+1} \in \mu$
$\Rightarrow \mu(x) \geq \delta-t+1 > 1-t$ or $\mu(y) \geq \delta-t+1 > 1-t$
$\Rightarrow \mu(x) > 1-t$ or $\mu(y) > 1-t$
$\Rightarrow \mu(x)+t > 1$ or $\mu(y)+t > 1$
$\Rightarrow x_t q\mu$ or $y_t q\mu$
therefore $(x\wedge y)_t q\mu \Rightarrow x_t q\mu$ or $y_t q\mu$
Hence μ is a (q,q)-fuzzy prime ideal of X.

Theorem 3.9*An $(\in,\in\vee q)$-fuzzy ideal μ of a BCK-algebra X is an $(\in,\in\vee q)$-fuzzy prime ideal of X iff*

$$max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\} \quad \forall x,y \in X$$

*Proof.*First let μ be an $(\in,\in\vee q)$-fuzzy prime ideal of X. To prove

$$max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\} \tag{3.1}$$

Assume that(3.1) is not valid,then there exists $x,y \in X$such that

$$max\{\mu(x),\mu(y)\} < min\{\mu(x\wedge y),0.5\}$$

choose a real number t such that

$$max\{\mu(x),\mu(y)\} < t < min\{\mu(x\wedge y),0.5\}$$

then $t \in (0,0.5]$ and $\mu(x\wedge y) > t \Rightarrow (x\wedge y)_t \in \mu$
and also $\mu(x) < t$ and $\mu(y) < t$ i.e. $x_t \overline{\in} \mu, y_t \overline{\in} \mu$
also $\mu(x)+t < 2t < 2\times 0.5 = 1$ i.e. $x_t \overline{\in \vee q} \mu$, $y_t \overline{\in \vee q} \mu$ which is a contradiction, since μ is an $(\in,\in\vee q)$-fuzzy prime ideal. Hence we must have

$$max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\}$$

Conversely, Suppose

$$max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\} \tag{3.2}$$

Let $x,y \in X$ such that $(x\wedge y)_t \in \mu$ where $t \in (0,1]$
i.e. $\mu(x \wedge y) \geq t$. Now (3.2)$\Rightarrow max\{\mu(x),\mu(y)\} \geq min\{t,0.5\}$.
Now we have
Case I: $t \leq 0.5$
then $max\{\mu(x),\mu(y)\} \geq t \Rightarrow \mu(x) \geq t$ or $\mu(y) \geq t \Rightarrow x_t \in \mu$ or $y_t \in \mu$
Case II: $t > 0.5$
then $max\{\mu(x),\mu(y)\} \geq 0.5 \Rightarrow \mu(x) \geq 0.5$ or $\mu(y) \geq 0.5$
$\Rightarrow \mu(x)+t \geq 0.5+t > 0.5+0.5 = 1$ or $\mu(y)+t \geq 0.5+t > 0.5+0.5 = 1$
$\Rightarrow x_t q\mu$ or $y_t q\mu$ combining case I and case II $x_t \in \vee q\mu$ or $y_t \in \vee q\mu$
Hence $(x\wedge y)_t \in \mu \Rightarrow x_t \in \vee q\mu$ or $y_t \in \vee q\mu$
that is μ is an $(\in,\in\vee q))$-fuzzy prime ideal of X.

Corollary 3.10*Every fuzzy prime ideal is an $(\in,\in\vee q)$-fuzzy prime ideal.*

*Note 1.*Converse of above is not true as seen from the following Example.

Example 3.11*Consider a BCK-algebra $X = \{0,x,y,z\}$ with the following cayley tables.*

*	0	x	y	z
0	0	0	0	0
x	x	0	0	x
y	y	x	0	y
z	z	z	z	0

Table 1

∧	0	x	y	z
0	0	0	0	0
x	0	x	x	0
y	0	x	y	0
z	0	0	0	z

Table 2

Define a map $\mu : X \rightarrow [0,1]$ by $\mu(0) = 0.6, \mu(x) = 0.5, \mu(y) = 0.7, \mu(z) = 0.56$ then μ is an $(\in,\in\vee q)$-fuzzy prime ideal X, but not a fuzzy prime ideal of X because $\mu(x\wedge z) = \mu(0) = 0.6 > max\{\mu(x),\mu(z)\} = max\{0.5,0.56\} = 0.56$.

Theorem 3.12*If a fuzzy set μ of a BCK-algebra X is an $(\in,\in\vee q)$-fuzzy prime ideal of X and $\mu(x) < 0.5\ \forall x \in X$, then μ is also an $(\in,\in)$-fuzzy prime ideal of X.*

*Proof.*Let μ be an $(\in,\in\vee q)$-fuzzy prime ideal of X and $\mu(x) < 0.5\ \forall x \in X$ Let$(x\wedge y)_t \in \mu \Rightarrow \mu(x\wedge y) \geq t$ therefore $t \leq \mu(x\wedge y) < 0.5$ and also $\mu(x) < 0.5$, $\mu(y) < 0.5$ therefore $t < 0.5$ and also $\mu(x)+t < 0.5+0.5 = 1$ and $\mu(y)+t < 0.5+0.5 = 1$
$\Rightarrow x_t \overline{q}\mu$ and $y_t \overline{q}\mu$ Since μ is $(\in,\in\vee q)$- fuzzy prime ideal, so we must have $x_t \in \mu$ or $y_t \in \mu$ i.e. μ is an $(\in,\in)$-fuzzy prime ideal of X.

Theorem 3.13 *A fuzzy set μ in X is an $(\in,\in\vee q)$-fuzzy prime ideal of X if and only if the set $\mu_t = \{x\in X|\mu(x)\geq t\}$ is a prime ideal of X, where $\mu_t \neq \phi$ for all $t\in(0,0.5]$*

Proof. Assume that μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X. Let $t\in(0,0.5]$ and $x\wedge y\in\mu_t$. Therefore $\mu(x\wedge y)\geq t$.
It follows that
$max\{\mu(x),\mu(y)\}\geq min\{\mu(x\wedge y),0.5\}\geq min\{t,0.5\}=t$
therefore $\mu(x)\geq t$ or $\mu(y)\geq t$, that is $x\in\mu_t$ or $y\in\mu_t$
therefore μ_t is a prime ideal of X.
Conversely
Suppose that μ_t is a prime ideal of X for all $t\in(0,0.5]$ and assume (3.1) is not valid, then their exists some $a,b\in X$ such that $max\{\mu(a),\mu(b)\}<min\{\mu(a\wedge b),0.5\}$ hence we can take $t\in(0,0.5)$ such that

$$max\{\mu(a),\mu(b)\}<t<min\{\mu(a\wedge b),0.5\} \qquad (3.3)$$

(3.3)$\Rightarrow a\wedge b\in\mu_t$.Since μ_t is a prime ideal of X, it follows that $a\in\mu_t$ or $b\in\mu_t$, so that $\mu(a)\geq t$ or $\mu(b)\geq t$, which contradicts (3.3), therefore we must have $max\{\mu(x),\mu(y)\}\geq min\{\mu(x\wedge y),0.5\}$ consequently μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.

Theorem 3.14 *Let A be a non empty subset of a BCK-algebra X. Consider the fuzzy set μ_A in X defined by*

$$\mu_A(x)=\begin{cases}1, & \text{if } x\in A,\\ 0, & \text{otherwise,}\end{cases}$$

Then A is a prime ideal of X iff μ_A is an $(\in,\in\vee q)$-fuzzy prime ideal of X.

Proof. Let A be an ideal of X. Then $(\mu_A)_t=\{x\in X|\mu_A(x)\geq t\}=A$, $\forall\, t\in(0,0.5]$ which is a prime ideal. Hence by above theorem μ_A is an $(\in,\in\vee q)$-fuzzy prime ideal of X.
Conversely, assume that μ_A is an $(\in,\in\vee q)$-fuzzy prime ideal of X.
Let $x\wedge y\in A$, then $\mu_A(x\wedge y)=1$
$max\{\mu_A(x),\mu_A(y)\}\geq min\{\mu_A(x\wedge y),0.5\}=min\{1,0.5\}=0.5$
$\Rightarrow max\{\mu_A(x),\mu_A(y)\}\geq 0.5$
$\Rightarrow \mu_A(x)\geq 0.5$ or $\mu_A(y)\}\geq 0.5$
$\Rightarrow \mu_A(x)=1$ or $\mu_A(y)\}=1$
$\Rightarrow x\in A$ or $y\in A$
Therefore $x\wedge y\in A\Rightarrow x\in A$ or $y\in A$
Hence A is a prime ideal of X.

Theorem 3.15 *Let A be a prime ideal of X, then for every $t\in(0,0.5]$, their exists an $(\in,\in\vee q)$-fuzzy prime ideal μ of X, such that $\mu_t=A$.*

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x)=\begin{cases}1, & \text{if } x\in A,\\ s, & \text{otherwise,}\end{cases}$$

for all $x\in X$
where $s<t\in(0,0.5]$, $\mu_t=\{x\in X|\mu(x)\geq t>s\}=A$ hence μ_t is a prime ideal. Now if μ is not an $(\in,\in\vee q)$-fuzzy prime ideal of X, then there exists some $a,b\in X$ such that $max\{\mu(a),\mu(b)\}<min\{\mu(a\wedge b),0.5\}$ hence we can take $t\in(0,0.5)$ such that

$$max\{\mu(a),\mu(b)\}<t<min\{\mu(a\wedge b),0.5\} \qquad (3.4)$$

therefore $\mu(a\wedge b)\geq t$ for some $a,b\in X\Rightarrow a\wedge b\in\mu_t=A$, a prime ideal therefore $a\in A$ or $b\in A\Rightarrow\mu(a)=1$ or $\mu(b)=1$ which contradicts(3.4). Hence μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.

Theorem 3.16 *Let μ be a fuzzy set in BCK-algebra X.Then μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X iff $[\mu]_t$ is a prime ideal of X for all $t\in(0\ 1]$. We call $[\mu]_t$ an $(\in\vee q)$-level prime ideal of μ*

Proof. Assume that μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X, to prove $[\mu]_t$ is a prime ideal of X. Let $x\wedge y\in[\mu]_t$ for $t\in(0\ 1]$ then $(x\wedge y)_t\in\vee q\mu$ then $\mu(x\wedge y)\geq t$ or $\mu(x\wedge y)+t>1$ since μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.Therefore $max\{\mu(x),\mu(y)\}\geq min\{\mu(x\wedge y),0.5\}$. Now we have the following cases:
Case I: $\mu(x\wedge y)>t$
$max\{\mu(x),\mu(y)\}\geq min\{t,0.5\}$

Subcase I: $t>0.5$

$$\begin{aligned}&max\{\mu(x),\mu(y)\}\geq 0.5\\ &\Rightarrow \mu(x)\geq 0.5 \quad or \quad \mu(y)\geq 0.5\\ &\Rightarrow \mu(x)+t>0.5+0.5=1 \text{ or } \mu(y)+t>0.5+0.5=1\\ &\Rightarrow x_tq\mu \quad or \quad y_tq\mu\end{aligned}$$

Subcase II: $t\leq 0.5$

$$\begin{aligned}&max\{\mu(x),\mu(y)\}\geq t\\ &\Rightarrow \mu(x)\geq t \quad or \quad \mu(y)\geq t\\ &\Rightarrow x_t\in\mu \quad or \quad y_t\in\mu\\ \text{Hence}\quad (x\wedge y)_t\in\vee q\mu \quad &\Rightarrow x_t\in\vee q\mu \quad or \quad y_t\in\vee q\mu\\ \text{i.e.}\quad (x\wedge y)_t\in[\mu]_t \quad &\Rightarrow x_t\in[\mu]_t \quad or \quad y_t\in[\mu]_t\end{aligned}$$

Case II: $\mu(x\wedge y)+t>1$

$$max\{\mu(x),\mu(y)\}\geq min\{1-t,0.5\}$$

Subcase I: $t\leq 0.5$

$$\begin{aligned}&max\{\mu(x),\mu(y)\}\geq 0.5\geq t\\ &\Rightarrow \mu(x)\geq t \quad or \quad \mu(y)\geq t\\ &\Rightarrow x_t\in\mu \quad or \quad y_t\in\mu\end{aligned}$$

Subcase II: $t>0.5$

$$\begin{aligned}&max\{\mu(x),\mu(y)\}\geq 1-t\\ &\Rightarrow \mu(x)\geq 1-t \quad or \quad \mu(y)\geq 1-t\\ &\Rightarrow \mu(x)+t\geq 1 \quad or \quad \mu(y)+t\geq 1\\ &\Rightarrow x_tq\mu \quad or \quad y_tq\mu\end{aligned}$$

Hence $(x\wedge y)_t\in\vee q\mu\Rightarrow x_t\in\vee q\mu \quad or \quad y_t\in\vee q\mu$
i.e $(x\wedge y)_t\in[\mu]_t\Rightarrow x_t\in[\mu]_t$ or $y_t\in[\mu]_t$
Conversely, let μ be a fuzzy set in X and $t\in(0\ 1]$ such that $[\mu]_t$ is a prime ideal of X. To prove μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X. If μ is not an $(\in,\in\vee q)$-fuzzy prime ideal of X, then there exists $a,b\in X$ such that $max\{\mu(a),\mu(b)\}<min\{\mu(a\wedge b),0.5\}$ holds. Choose t such that

$$max\{\mu(a),\mu(b)\}<t<min\{\mu(a\wedge b),0.5\} \qquad (3.5)$$

then $\mu(a\wedge b) > t \Rightarrow a\wedge b \in \mu_t \subseteq [\mu]_t$ which is a prime ideal $\Rightarrow a \in [\mu]_t$ or $b \in [\mu]_t$
$\Rightarrow \mu(a) \geq t$ or $\mu(a)+t > 1$ or $\mu(b) \geq t$ or $\mu(b)+t > 1$ which contradicts (3.5). Hence we must have $max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\}$

Theorem 3.17*Let λ and μ be two $(\in,\in\vee q)$-fuzzy prime ideals of X then $\lambda\cup\mu$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.*

*Proof.*Here λ and μ be $(\in,\in\vee q)$-fuzzy prime ideals of X. Therefore

$$max\{\lambda(x),\lambda(y)\} \geq min\{\lambda(x\wedge y),0.5\}, \text{ and}$$
$$max\{\mu(x),\mu(y)\} \geq min\{\mu(x\wedge y),0.5\}, \quad \forall x,y\in X. \quad (3.6)$$

To prove $\lambda\cup\mu$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X. It is enough to show that

$$max\{(\lambda\cup\mu)(x),(\lambda\cup\mu)(y)\} \geq min\{(\lambda\cup\mu)(x\wedge y),0.5\} \quad \forall x,y\in X. \quad (3.7)$$

Now $max\{\lambda\cup\mu)(x),\lambda\cup\mu)(y)\}$
$= max\{max\{\lambda(x),\mu(x)\},max\{\lambda(y),\mu(y)\}\}$
$= max\{max\{\lambda(x),\lambda(y)\},max\{\mu(x),\mu(y)\}\}$
$\geq max\{min\{\lambda(x\wedge y),0.5\},min\{\mu(x\wedge y),0.5\}\}$ by (3.6)

$$\Rightarrow max\{(\lambda\cup\mu)(x),\lambda\cup\mu)(y)\} \geq max\{min\{(\lambda(x\wedge y),0.5\},min\{\mu(x\wedge y),0.5\}\} \quad (3.8)$$

Now we have the following cases:
Case I: $\lambda(x\wedge y) \leq 0.5$ and $\mu(x\wedge y) \leq 0.5$, then

$$(3.8) \Rightarrow max\{(\lambda\cup\mu)(x),(\lambda\cup\mu)(y)\} \geq max\{\lambda(x\wedge y),\mu(x\wedge y)\} \geq (\lambda\cup\mu)(x\wedge y) = min\{(\lambda\cup\mu)(x\wedge y),0.5\}$$

Case II: $\lambda(x\wedge y) \leq 0.5$ and $\mu(x\wedge y) > 0.5$, then

$$(3.8) \Rightarrow max\{(\lambda\cup\mu)(x),(\lambda\cup\mu)(y)\} \geq max\{\lambda(x\wedge y),0.5\} = 0.5 = min\{max\{\lambda(x\wedge y),\mu(x\wedge y)\},0.5\} = min\{(\lambda\cup\mu)(x\wedge y),0.5\}$$

Case III: $\lambda(x\wedge y) > 0.5$ and $\mu(x\wedge y) \leq 0.5$

$$(3.8) \Rightarrow max\{(\lambda\cup\mu)(x),(\lambda\cup\mu)(y)\} \geq max\{0.5,\mu(x\wedge y)\} = 0.5 = min\{max\{\lambda(x\wedge y),\mu(x\wedge y)\},0.5\} = min\{(\lambda\cup\mu)(x\wedge y),0.5\}$$

Case IV: $\lambda(x\wedge y) > 0.5$ and $\mu(x\wedge y) > 0.5$, then

$$(3.8) \Rightarrow max\{(\lambda\cup\mu)(x),(\lambda\cup\mu)(y)\} \geq max\{0.5,0.5\} = 0.5 = min\{max\{\lambda(x\wedge y),\mu(x\wedge y)\},0.5\} = min\{(\lambda\cup\mu)(x\wedge y),0.5\}$$

Hence from above (3.7) hold $\forall x,y\in X$.
Therefore $(\lambda\cup\mu)$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.

Theorem 3.18*Let $\{\mu_i : i\in\wedge\}$ be a family of $(\in,\in\vee q)$-fuzzy prime ideal of X, then $\mu = \cup\{\mu_i : i\in\wedge\}$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.*

Theorem 3.19*Let λ and μ be two $(\in,\in\vee q)$-fuzzy prime ideals of X, then $\lambda\cap\mu$ may not be an $(\in,\in\vee q)$-fuzzy prime ideal of X as shown in Example below.*

Example 3.20*Consider a BCK-algebra X as in Example 3.11 and fuzzy sets λ and μ defined by $\lambda(0) = 0.45,\lambda(x) = \lambda(y) = 0.46,\lambda(z) = 0.2$ and $\mu(0) = 0.4,\mu(x) = 0.32,\mu(y) = 0.35,\mu(z) = 0.45$. Then by routine calculations it can be verified that λ and μ both are $(\in,\in\vee q)$-fuzzy prime ideals X and also $(\lambda\cap\mu)(0) = 0.4,(\lambda\cap\mu)(x) = 0.32,(\lambda\cap\mu)(y) = 0.35,(\lambda\cap\mu)(z) = 0.2$. But $\lambda\cap\mu$ is not an $(\in,\in\vee q)$-fuzzy prime ideal of X because $max\{(\lambda\cap\mu)(x),(\lambda\cap\mu)(z)\} = max\{0.32,0.2\} = 0.32 \ngeq min\{(\lambda\cap\mu)(x\wedge z),0.5\} = min\{(\lambda\cap\mu)(0),0.5\} = min\{0.4,0.5\} = 0.4$.*

4 Cartesian product of BCK-algebras and their $(\in,\in\vee q)$-fuzzy prime ideals

Theorem 4.1*Let X,Y be two BCK-algebras, then their cartesian product $X\times Y = \{(x,y)|x\in X,y\in Y\}$ is also a BCK-algebra under the binary operation $*$ defined in $X\times Y$ by $(x,y)*(p,q) = (x*p,y*q)$ for all $(x,y),(p,q)\in X\times Y$.*

*Proof.*Straightforward.

Definition 4.2*Let μ_1 and μ_2 be two $(\in,\in\vee q)$-fuzzy prime ideals of a BCK-algebra X.Then their cartesian product $\mu_1\times\mu_2$ is defined by $(\mu_1\times\mu_2)(x,y) = max\{\mu_1(x),\mu_2(x)\}$ Where $(\mu_1\times\mu_2) : X\times X\to[0,1] \quad \forall x,y\in X$.*

Theorem 4.3*Let μ_1 and μ_2 be two $(\in,\in\vee q)$-fuzzy prime ideals of a BCK-algebra X.Then $\mu_1\times\mu_2$ is also an $(\in,\in\vee q)$-fuzzy prime ideal of $X\times X$.*

*Proof.*Similar to theorem 3.17, just replace $\cup$ by $\times$, λ by μ_1 and μ by μ_2.

5 Homomorphism of BCK-algebras and $(\in,\in\vee q)$-fuzzy prime ideals

Definition 5.1*Let X and X' be two commutative BCK-algebras, then a mapping $f : X\to X'$is said to be a lattice homomorphism if $f(x\wedge y) = f(x)\wedge f(y)\ \forall x,y\in X$.*

Theorem 5.2*Let X and X' be two commutative BCK-algebras and $f : X\to X'$ be a lattice homomorphism. If μ be an $(\in,\in\vee q)$-fuzzy prime ideal of X', then $f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.*

Proof.$f^{-1}(\mu)$ is defined as $f^{-1}(\mu)(x) = \mu(f(x))\forall x\in X$.
Let μ be an $(\in,\in\vee q)$-fuzzy prime ideal of X'
and $x,y\in X$ such that $(x\wedge y)_t \in f^{-1}(\mu)$
then $f^{-1}(\mu)(x\wedge y) \geq t$
$\mu(f(x\wedge y)) \geq t$

$\Rightarrow (f(x\wedge y))_t \in \mu$
$\Rightarrow (f(x)\wedge f(y))_t \in \mu$ [Since f is a lattice homomorphism.]
$\Rightarrow (f(x))_t \in \vee q\mu$ or $(f(y))_t \in \vee q\mu$ because μ be an $(\in,\in\vee q)$-fuzzy prime ideal of X'
$\Rightarrow \mu(f(x)) \geq t$ or $\mu(f(x))+t > 1$ or $\mu(f(y)) \geq t$ or $\mu(f(y))+t > 1$
$\Rightarrow f^{-1}(\mu)(x) \geq t$ or $f^{-1}(\mu)(x)+t > 1$ or $f^{-1}(\mu)(y) \geq t$ or $f^{-1}(\mu)(y)+t > 1$
$\Rightarrow x_t \in f^{-1}(\mu)$ or $x_t q f^{-1}(\mu)$ or $y_t \in f^{-1}(\mu)$ or $y_t q f^{-1}(\mu)$
$\Rightarrow x_t \in \vee q f^{-1}(\mu)$ or $y_t \in \vee q f^{-1}(\mu)$
Therefore $(x\wedge y)_t \in f^{-1}(\mu) \Rightarrow x_t \in \vee q f^{-1}(\mu)$ or $y_t \in \vee q f^{-1}(\mu)$
Hence $f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X

Theorem 5.3*Let X and X' be two commutative BCK-algebras and $f: X\rightarrow X'$ be an onto lattice homomorphism. If μ be a fuzzy subset of X' such that $f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X, then μ is also an $(\in,\in\vee q)$-fuzzy prime ideal of X'.*

*Proof.*Let $x',y' \in X'$ such that $(x'\wedge y')_t \in \mu$ where $t\in[01]$ then $\mu(x'\wedge y') \geq t$ since f is onto so there exists $x,y\in X$ such that $f(x) = x', f(y) = y'$ also f is lattice homomorphism so
$f(x\wedge y) = f(x)\wedge f(y) = x'\wedge y'$ Now $\mu(x'\wedge y') \geq t \Rightarrow \mu(f(x\wedge y)) \geq t$
$\Rightarrow f^{-1}(\mu)(x\wedge y) \geq t$
$\Rightarrow (x\wedge y)_t \in f^{-1}(\mu)$
$\Rightarrow x_t \in \vee q f^{-1}(\mu)$ or $y_t \in \vee q f^{-1}(\mu)$

[since $f^{-1}(\mu)$ is an $(\in,\in\vee q)$-fuzzy prime ideal of X.]

$\Rightarrow f^{-1}(\mu)(x) \geq t$ or $f^{-1}(\mu)(x)+t > 1$ or $f^{-1}(\mu)(y) \geq t$ or $f^{-1}(\mu)(y)+t > 1$
$\Rightarrow \mu(x') \geq t$ or $\mu(x')+t > 1$ or $\mu(y') \geq t$ or $\mu(y')+t > 1$
$\Rightarrow x'_t \in \mu$ or $x'_t q\mu$ or $y'_t \in \mu$ or $y'_t q\mu$
$\Rightarrow x'_t \in \vee q\mu$ or $y'_t \in \vee q\mu$
therefore $(x'\wedge y')_t \in \mu \Rightarrow x'_t \in \vee q\mu$ or $y'_t \in \vee q\mu$
Hence μ is an $(\in,\in\vee q)$-fuzzy prime ideal of X'.

6 Conclusion

In this paper, we have studied $(\in,\in\vee q)$-fuzzy prime ideals in commutative BCK-algebras and investigated some of their useful properties. In my opinion, these definitions and results can be extended to other algebraic systems also. In the notions of (α,β)-fuzzy ideals we can define twelve different types of ideals by three choices of α and four choices of β. we can apply $(\in,\in\vee q)$-fuzzy prime ideals is in the field of fuzzy medical diagnosis, artificial intelligence, information science, agriculture etc.

In future, the following studies may be carried out:
(1) $(\in,\in\vee q)$-weakly prime and weakly semiprime fuzzy ideals.
(2) $(\in,\in\vee q)$-fuzzy prime ideals of lattice.
(3) $(\in,\in\vee q)$-fuzzy prime and irreducible ideals in BCK-algebras.

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