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Lacunary statistical convergence in random 2-normed spaces

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Abstract: In this paper we purpose to define the notions of lacunary statistical convergence and lacunary statistically Cauchy in random 2-normed spaces and proved some interesting results for these concepts in this set up.

Keywords: t-norm; random 2-normed space; statistical convergence; lacunary statistical convergence; lacunary statistical Cauchy.

1. Introduction

An interesting and important generalization of the notion of metric space was introduced by Menger [11] under the name of statistical metric space, which is now called probabilistic metric space.

Infact the probabilistic theory has become an area of active research for the last forty years. An important family of probabilistic metric spaces are probabilistic normed spaces (briefly, PN-spaces). The notion of probabilistic normed spaces was introduced in [25] and [26] and further it was extended to random/probabilistic 2-normed spaces by Goleţ [7] using the concept of 2-norm of Gähler [6].

The concept of statistical convergence for sequences of real number was introduced by Fast [2] and Steinhauss [27] independently in the same year 1951 and since then several generalizations and applications of this concept have been investigated by various authors, e.g. [4,5,8,10,15,18, 23]. This notion has also been defined and studied in different set ups, e.g. in probabilistic normed space [9,14,20, 21,24]; in intuitionistic fuzzy normed spaces [12,16,17, 19]; and in fuzzy/random 2-normed space [13,22]. In this paper we shall study lacunary statistical convergence and lacunary statistical Cauchy in random 2-normed spaces.

We shall assume throughout this paper that the symbol $\Re$ will denote set of all real numbers. A function $f : \Re \to \Re_0^+$ is called a $distribution\ function$ if it is a non-decreasing and left continuous with $\inf_{t\in\Re} f(t) = 0$ and $\sup_{t\in\Re} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$.

If $a \in \Re_0^+$, then $H_a \in D^+$, where

$$H_a(t) = \begin{cases} 1 & \text{if } t > a; \\ 0 & \text{if } t \le a. \end{cases}$$

It is obvious that $H_0 \ge f$ for all $f \in D^+$.

A t-$norm$ is a continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c * d \ge a * b$ if $c \ge a$ and $d \ge b$ for all $a, b, c \in [0,1]$.

A $triangle\ function$ τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

The concept of 2-normed spaces was first introduced by Gähler [6].

A 2-normed space is a pair $(X, \|.,.\|)$, where X is a linear space of a dimension greater than one and $\|.,.\|$ is a real valued mapping on $X \times X$ such that the following conditions be satisfied:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, whenever $x, y \in X$ and $\alpha \in \Re$,
(iv) $\|x + y, z\| \le \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Example 1.1. Take $X = \Re^2$ being equipped with the 2-norm $\|x, y\| =$ the area of the parallelogram spanned by

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the vectors x and y, which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|, \text{ where } x=(x_1,x_2), y=(y_1,y_2).$$

In 2006 Goleţ [7] introduced the notion of random 2-normed space. Quite recently, Alotaibi and Mohiuddine [1] determined the stability of cubic functional equation in this setting.

Let X be a linear space of a dimension greater than one, τ a triangle function, and let $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathcal{D}_+$. If the following conditions are satisfied:

(i) $\mathcal{F}(x,y;t) = H_0(t)$ if x and y are linearly dependent,
(ii) $\mathcal{F}(x,y;t) \neq H_0(t)$ if x and y are linearly independent,
(iii) $\mathcal{F}(x,y;t) = F(y,x;t)$ for every x,y in X,
(iv) $\mathcal{F}(\alpha x,y;t) = \mathcal{F}(x,y;\frac{t}{|\alpha|})$ for every $t>0$, $\alpha \neq 0$ and $x,y \in X$,
(v) $\mathcal{F}(x+y,z;t) \geq \tau(\mathcal{F}(x,z;t),\mathcal{F}(y,z;t))$ whenever $x,y,z \in X$.

Then $\mathcal{F}$ is called a *probabilistic* 2-*norm* on X and $(X,\mathcal{F},\tau)$ is called a *probabilistic* 2-*normed space* (for short, PTNS). If (v) is replaced by

(v′) $\mathcal{F}(x+y,z;t_1+t_2) \geq \mathcal{F}(x,z;t_1) * \mathcal{F}(y,z;t_2)$, for all $x,y,z \in X$ and $t_1,t_2 \in \Re_0^+$.

Triple $(X,\mathcal{F},*)$ is called a *random* 2-*normed space* (for short, RTN-space).

Example 1.2. Let $(X,\|.,.\|)$ be a 2-normed space with $\|x,z\| = |x_1z_2 - x_2z_1|$, $x=(x_1,x_2)$, $z=(z_1,z_2)$ and $a*b = ab$ for $a,b \in [0,1]$. For all $x \in X$, $t>0$ and nonzero $z \in X$, consider

$$\mathcal{F}_1(x,z;t) = \begin{cases} \frac{t}{t+\|x,z\|} & \text{if } t>0 \\ 0 & \text{if } t \leq 0; \end{cases}$$

and

$$\mathcal{F}_2(x,z;t) = H_0(t - \|x,z\|).$$

Then $(X,\mathcal{F}_1,*)$ and $(X,\mathcal{F}_2,*)$ are random 2-normed spaces.

2. Main results

In this section we study the ~~concept~~ of lacunary statistically convergent and lacunary statistically Cauchy sequences in random 2-normed spaces. Before proceeding further, we recall the definition of density and related concepts which form the background of the present work.

Definition 2.1 [3]**.** Let K be a subset of $\mathcal{N}$, the set of natural numbers. Then the *asymptotic density of* K denoted by $\delta(K)$, is defined as

$$\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x=(x_k)$ is said to be *statistically convergent* (cf. [2], [27]) to the number ℓ if for each $\epsilon>0$, the set $K(\epsilon) = \{k \leq n : |x_k - \ell| > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - \ell| > \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = \ell$.

Definition 2.2. By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$, and the ratio k_r/k_{r-1} will be abbreviated by q_r.

Let $K \subseteq \mathcal{N}$. The number

$$\delta_\theta(K) = \lim_r \frac{1}{h_r}|\{k \in I_r : k \in K\}|$$

is said to be the θ-*density* of K, provided the limit exists.

Definition 2.3 [5]**.** Let θ be a lacunary sequence. Then a sequence $x=(x_k)$ is said to S_θ-convergent to the number L if for every $\epsilon>0$, the set $K(\epsilon)$ has θ-density zero, where

$$K(\epsilon) := \{k \in \mathcal{N} : |x_k - L| \geq \epsilon\}.$$

In this case we write $S_\theta\text{-}\lim x = L$ or $x_k \to L(S_\theta)$.

Let $(X,\mathcal{F},*)$ be a RTN-space. We say that a sequence $x=(x_k)$ is *convergent* in $(X,\mathcal{F},*)$ or simply $(X,\mathcal{F},*)$-*convergent* to ℓ [22] if for every $\epsilon>0$, $t \in (0,1)$ there exist a positive integer k_0 such that $\mathcal{F}(x_k - \ell, z;\epsilon) > 1-t$ whenever $k \geq k_0$ and nonzero $z \in X$. In this case we write $\mathcal{F}\text{-}\lim_k x_k = \ell$ and ℓ is called the $\mathcal{F}$-limit of $x=(x_k)$.

Now we define the S_θ-convergence in RTN-space.

Definition 2.4. Let $(X,\mathcal{F},*)$ be a RTN-space and θ be a lacunary sequence. We say that a sequence $x=(x_k)$ is said to be S_θ-*convergent* to ℓ in random 2-norm space X (for short, $S_\theta^{(RTN)}$-*convergent*) if for every $\epsilon>0$, $t \in (0,1)$ and nonzero $z \in X$

$$\delta_\theta(\{k \in \mathcal{N} : \mathcal{F}(x_k - \ell, z;\epsilon) \leq 1-t\}) = 0$$

or equivalently

$$\delta_\theta(\{k \in \mathcal{N} : \mathcal{F}(x_k - \ell, z;\epsilon) > 1-t\}) = 1.$$

In this case we write $x_k \xrightarrow{\mathcal{F}} \ell(S_\theta)$ or $S_\theta^{(RTN)}\text{-}\lim x = \ell$, and denote the set of all S_θ-convergent sequences in random normed spaces by $(S_\theta)_\mathcal{F}$.

Theorem 2.1. Let $(X,\mathcal{F},*)$ be a RTN-space and θ be a lacunary sequence. If a sequence $x=(x_k)$ is a lacunary statistically convergent in random normed space X, then $S_\theta^{(RTN)}$-limit is unique.

Proof. Suppose that $S_\theta^{(RTN)}\text{-}\lim x = \ell_1$ and $S_\theta^{(RTN)}\text{-}\lim x = \ell_2$. Let $\epsilon > 0$ and $t > 0$. Choose $s \in (0,1)$ such that $(1-s)*(1-s) \geq 1-\epsilon$. Then, for any nonzero $z \in X$, we define the following sets as

$$K_1(s,t) = \{k \in \mathcal{N} : \mathcal{F}(x_k - \ell_1, z; t) \leq 1-s\},$$
$$K_2(s,t) = \{k \in \mathcal{N} : \mathcal{F}(x_k - \ell_2, z; t) \leq 1-s\}.$$

So that we have $\delta_\theta(K_1(s,t)) = 0$ and $\delta_\theta(K_2(s,t)) = 0$ for all $t > 0$. Now let

$$K_3(s,t) = K_1(s,t) \cup K_2(s,t).$$

It follows that $\delta_\theta(K_3(s,t)) = 0$, which implies

$$\delta_\theta(\mathcal{N} \setminus K_3(s,t)) = 1.$$

If $k \in \mathcal{N} \setminus K_3(s,t)$, we have

$$\begin{aligned}\mathcal{F}(\ell_1-\ell_2, z; t) &\geq \mathcal{F}(x_k-\ell_1, z; t/2) * \mathcal{F}(x_k-\ell_2, z; t/2)\\ &> (1-s)*(1-s) \geq 1-\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we get $\mathcal{F}(\ell_1-\ell_2, z; t) = 1$ for all $t > 0$ and nonzero $z \in X$, which gives $\ell_1 = \ell_2$. Hence $S_\theta^{(RTN)}$-limit is unique.

This completes the proof of the theorem.

Theorem 2.2. Let $(X, \mathcal{F}, *)$ be a RTN-space and θ be any lacunary sequence. If $\mathcal{F}\text{-}\lim x = \ell$ then $S_\theta^{(RTN)}\text{-}\lim x = \ell$. But converse need not be true.

Proof. Let $\mathcal{F}\text{-}\lim x = \ell$. Then for every $\epsilon > 0$, $t \in (0,1)$ and nonzero $z \in X$, there is a number $k_0 \in \mathcal{N}$ such that $\mathcal{F}(x_k - \ell, z; t) > 1-\epsilon$ for all $k \geq k_0$. Hence the set $A(\epsilon) = \{k \in \mathcal{N} : \mathcal{F}(x_k-\ell, z; t) \leq 1-\epsilon\}$ has natural density zero, that is, $\delta_\theta(A(\epsilon)) = 0$. Hence $S_\theta^{(RTN)}\text{-}\lim x = \ell$.

For converse, we construct the following example:

Example 2.1. Let $X = \Re^2$ with the 2-norm $\|x,z\| = \|x_1z_2 - x_2z_1\|$, where $x = (x_1,x_2)$, $z = (z_1,z_2)$ and $a*b = ab$ for all $a,b \in [0,1]$. Let $\mathcal{F}(x,z;t) = \frac{t}{t+\|x,z\|}$, where $x \in X$, $t > 0$ and nonzero $z \in X$. In this case, we observe that $(\Re^2, \mathcal{F}, *)$ is a RTN-space. Define a sequence $x = (x_k)$ by

$$x_k = \begin{cases}(k,0)\ ;\ \text{for}\ \ k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathcal{N}\\ (0,0)\ ;\ \text{otherwise.}\end{cases}$$

Let for $\epsilon > 0, t > 0$

$$K(\epsilon,t) = \{k \in \mathcal{N} : \mathcal{F}(x_k, z; t) \leq 1-\epsilon\}.$$

Then

$$\begin{aligned}K(\epsilon,t) &= \left\{k \in \mathcal{N} : \frac{t}{t+\|x_k,z\|} \leq 1-\epsilon\right\},\\ &= \left\{k \in \mathcal{N} : \|x_k,z\| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\},\\ &= \{k \in \mathcal{N} : x_k = (k,0)\},\\ &= \{k \in \mathcal{N} : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathcal{N}\},\end{aligned}$$

and so, we get

$$\begin{aligned}\frac{1}{h_r}|K(\epsilon,t)| &\leq \frac{1}{h_r}|\{k \in \mathcal{N} : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathcal{N}\}|\\ &\leq \frac{\sqrt{h_r}}{h_r},\end{aligned}$$

which implies that $\lim_r \frac{1}{h_r}|K(\epsilon,t)| = 0$. Hence

$$\delta_\theta(K(\epsilon,t)) = \lim \frac{\sqrt{h_r}}{h_r} = 0 \text{ as } r \longrightarrow \infty$$

implies that $x_k \xrightarrow{\mathcal{F}} 0(S_\theta)$. On the other hand $x_k \not\xrightarrow{\mathcal{F}} 0$, since

$$\begin{aligned}\mathcal{F}(x_k, z; t) &= \frac{t}{t+\|x_k,z\|}\\ &= \begin{cases}\frac{t}{t+kz_2}\ ,\ \text{for}\ \ k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r (r \in \mathcal{N});\\ 1\ ,\ \text{otherwise;}\end{cases}\end{aligned}$$

and hence

$$\lim_k \mathcal{F}(x_k, z; t) = \begin{cases}0\ ,\ \text{for}\ \ k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r (r \in \mathcal{N});\\ 1\ ,\ \text{otherwise;}\end{cases}$$

This completes the proof of the theorem.

Theorem 2.3. Let $(X, \mathcal{F}, *)$ be a RTN-space. Then, for any lacunary sequence θ, $S_\theta^{(RTN)}\text{-}\lim x = \ell$ if and only if there exists a subset $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathcal{N}$ such that $\delta_\theta(K) = 1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} x_{k_n} = \ell$.

Proof. *Necessity*. Suppose that $S_\theta^{(RTN)}\text{-}\lim x = \ell$. Then, for any $t > 0$, $s \in \mathcal{N}$ and nonzero $z \in X$, let

$$K(s,t) = \left\{k \in \mathcal{N} : \mathcal{F}(x_k - \ell, z; t) \geq 1 - \frac{1}{s}\right\},$$

and

$$M(s,t) = \left\{k \in \mathcal{N} : \mathcal{F}(x_k - \ell, z; t) < \frac{1}{s}\right\}.$$

Then $\delta_\theta(K(s,t)) = 0$ and

$$M(1,t) \supset M(2,t) \supset \cdots M(i,t) \supset M(i+1,t) \supset \cdots \tag{2.3.1}$$

and

$$\delta_\theta(M(s,t)) = 1,\ s = 1,2,\cdots. \tag{2.3.2}$$

Now we have to show for $n \in M(s,t)$, $x = (x_{k_n})$ is $\mathcal{F}$-convergent to ℓ. On contrary suppose that sequence $x = (x_{k_n})$ is not $\mathcal{F}$-convergent to ℓ. Therefore there is $\epsilon > 0$ and a positive integer k_0 such $\mathcal{F}(x_{k_n} - \ell, z; t) \geq \epsilon$ for all $k \geq k_0$ and nonzero $z \in X$. Let

$$M(\epsilon,t) = \{n \in \mathcal{N} : \mathcal{F}(x_{k_n} - \ell, z; t) < \epsilon\},$$

for all $k \leq k_0$, $\epsilon > \frac{1}{s}$, $s \in \mathcal{N}$ and nonzero $z \in X$. Then $\delta_\theta(M(\epsilon,t)) = 0$ and by (2.3.1), $M(s,t) \subset M(\epsilon,t)$.

Hence $\delta_\theta(M(s,t)) = 0$, which contradicts (2.3.2). Therefore $x = (x_k)$ is $\mathcal{F}$-convergent to ℓ.

Sufficiency. Suppose that there exists a subset $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathcal{N}$ such that $\delta_\theta(K) = 1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} x_{k_n} = \ell$. Then there exists a positive integer $N \in \mathcal{N}$ such that for $k \geq N$

$$\mathcal{F}(x_k - \ell, z; t) > 1 - \epsilon.$$

Now

$$K(\epsilon,t) = \{k \in \mathcal{N} : \mathcal{F}(x_k - \ell, z; t) \leq 1 - \epsilon\}$$

and $K' = \{k_{N+1}, k_{N+2}, \cdots\}$. Then $\delta_\theta(K') = 1$ and $K(\epsilon,t) \subseteq \mathcal{N} - K'$ which implies that $\delta_\theta(K(\epsilon,t)) = 0$. Hence $S_\theta^{(RTN)}\text{-}\lim x = \ell$.

This completes the proof of the theorem.

Lastly, we define lacunary statistically Cauchy sequence in random 2-normed space.

Definition 2.5. Let $(X,\nu,*)$ be a RTN-space and θ be any lacunary sequence. Then, a sequence $x = (x_k)$ is said to be S_θ-*Cauchy* in RTN-space X if for every $\epsilon > 0$ and $t > 0$ and nonzero $z \in X$, there exist a number $N = N(\epsilon, z)$ such that for all $k, l \geq N$

$$\delta_\theta(\{k \in N : \mathcal{F}(x_k - x_l, z; \epsilon) \leq 1 - t\}) = 0.$$

Theorem 2.4. Let $(X,\nu,*)$ be a RTN-space and θ be any lacunary sequence. Then, a sequence $x = (x_k)$ is $S_\theta^{(RTN)}$-convergent if and only if it is $S_\theta^{(RTN)}$-Cauchy.

Proof. Let $x = (x_k)$ be $S_\theta^{(RTN)}$-convergent to ℓ, i.e., $S_\theta^{(RTN)}\text{-}\lim x = \ell$. Then for a given $\epsilon > 0$, choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$. Then, for $t > 0$ and nonzero $z \in X$, we have

$$\delta_\theta(A(r,t)) = \delta_\theta(\{n \in \mathcal{N} : \mathcal{F}(x_n - \ell, z; t/2) \leq 1 - r\}) = 0 \qquad (3.1.1)$$

which implies that

$$\delta_\theta(A^C(r,t)) = \delta_\theta(\{n \in \mathcal{N} : \mathcal{F}(x_n - \ell, z; t/2) > 1 - r\}) = 1.$$

Let $m \in A^C(r,t)$. Then $\mathcal{F}(x_m - \ell, z; t/2) > 1 - r$.

Now, let

$$B(\epsilon,t) = \{n \in \mathcal{N} : \mathcal{F}(x_n - x_m, z; t) \leq 1 - \epsilon\}.$$

We need to show that $B(\epsilon,t) \subset A(r,t)$. Let $n \in B(\epsilon,t)$. Then $\mathcal{F}(x_n - x_m, z; t) \leq 1-\epsilon$ and hence $\mathcal{F}(x_n - \ell, z; t/2) \leq 1 - r$, i.e. $n \in A(r,t)$. Otherwise, if $\mathcal{F}(x_n - \ell, z; t/2) > 1 - r$, then

$$\begin{aligned} 1 - \epsilon &\geq \mathcal{F}(x_n - x_m, z; t) \\ &\geq \mathcal{F}(x_n - \ell, z; t/2) * \mathcal{F}(x_m - \ell, z; t/2) \\ &> (1-r)*(1-r) > 1 - \epsilon, \end{aligned}$$

which is not possible. Hence $B(\epsilon,t) \subset A(r,t)$ which implies that $x = (x_k)$ is $S_\theta^{(RTN)}$-Cauchy.

Conversely, let $x = (x_k)$ be $S_\theta^{(RTN)}$-Cauchy but not $S_\theta^{(RTN)}$-convergent. Then there exists $M \in \mathcal{N}$ such that

$$\delta_\theta(E(\epsilon,t)) = \delta_\theta(\{n \in \mathcal{N} : \mathcal{F}(x_n - x_M, z; t) \leq 1 - \epsilon\}) = 0,$$

and

$$\delta_\theta(F(\epsilon,t)) = \delta_\theta(\{n \in \mathcal{N} : \mathcal{F}(x_n - \ell, z; t/2) > 1 - \epsilon\}) = 0.$$

This implies that $\delta_\theta(F^C(\epsilon,t)) = 1$. Since

$$\mathcal{F}(x_n - x_m, z; t) \geq 2\mathcal{F}(x_n - \ell, z; t/2) > 1 - \epsilon,$$

if $\mathcal{F}(x_n - \ell, z; t/2) > \frac{1-\epsilon}{2}$. Therefore $\delta_\theta(E^C(\epsilon,t)) = 0$, i.e. $\delta_\theta(E(\epsilon,t)) = 1$, which leads to a contradiction, since $x = (x_k)$ was $S_\theta^{(RTN)}$-Cauchy. Hence $x = (x_k)$ must be $S_\theta^{(RTN)}$-convergent.

This completes the proof of the theorem.

3. Conclusion

The idea of random 2-norm is very useful to deal with the convergence problems of sequences. In the present work, we have introduced a wider class of lacunary statistically convergent sequences in RTN-space to deal with the sequences which are not covered by Fridy and Orhan [5].

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