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# Approximare Analytical Solution for High-Order Integro-Differential Equation by Chebyshev Waveletes

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**Abstract:** In this work, we present a computational method for solving high-order integro-differential equations which is based on the use of Chebyshev Wavelets. The solution process is illustrated and various physically relevant results are obtained. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution. Comparison of the obtained results with exact solutions shows that the used method is an effective and highly promising method for various classes of high-order integro-differential equations.

**Keywords:** high-order integro-differential equations; Chebyshev wavelets; operational matrix integration

## 1 Introduction

Integro-differential equations have gained a lot of interest in many application fields, such as biological, physical and engineering problems. Integro-differential equations are important, but they are hard to solve even numerically, so the progress on how to solve them is slow. Therefore, their numerical treatment is desired. Goswami et al. [1] used wavelet on bounded interval to solve the integral equations, Lakestani et al. [2] used spline wavelets to solve the integro-differentail equations, also Nevles et al. [3] used orthogonal wavelets to solve the integral equations, Chrysafinos [4] used wavelet-Galerkin method or integro-differential equations, Abbasa et al. [5] applied multiwavelet direct method for solving integro-differential equations. Furthermore other authors used different methods for solving integro-differential equations [6,7]. Orthogonal functions and polynomials have been used by many authors for solving functional equations. The main idea of using an orthogonal basis is that the problem under study reduces to a linear or nonlinear algebraic equation. This can be done by truncated series of orthogonal basis functions for the solution of problem and using the operational matrices. In this paper Chebyshev Wavelets basis, on the interval $[0,1]$ have been used. The method has been used by many authors to handle a wide variety of scientific and engineering applications to solve various functional equations. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [8,9,10,11,12,13]. The novelty of this paper is an extension of Chebyshev wavelets method for solving high-order integro-differential equations [14,15,16,17]. This paper is arranged as follows: In Section 2, the properties of Chebyshev wavelets and the way to construct the collocation technique for this type of equation are described. In Section 3 the proposed method is applied to some types of high-order integro-differential equations, and a comparison is made with the existing analytic or exact solutions that were reported in other published works in the literature. Finally we give a brief conclusion in the last section.

## 2 Wavelets and Chebyshev Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet,[18,19,20].When the dilation parameter $a$ and the translation parameter $b$, vary continuously we have following family of continuous wavelets as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}}\psi\left(\frac{x-b}{a}\right), \quad a,b \in \mathbb{R},\ a \neq 0. \tag{1}$$

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If we take dilation and translation parameters $a^{-k}$, and $nba^{-k}$,respectively where $a > 1, b > 0$, $n$ and $k$ are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^{\frac{k}{2}} \psi(a^k x - nb). \tag{2}$$

These functions are a wavelet basis for $L^2(\mathbb{R})$ and in special case $a = 2$, and $b = 1$, the functions $\psi_{k,n}(x)$ are an orthonormal basis.

Chebyshev wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments, $n = 1,2,...,2^{k-1}$, $k$ is an arbitrary positive integer and $m$ is the order of Chebyshev polynomials of the first kind. They are defined on the interval $[0,1]$, as follows:

$$\psi_{\mathrm{n,m}}(x) = \psi(k,n,m,x)$$
$$= \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0, & otherwise \end{cases} \tag{3}$$

where

$$\tilde{T}_m(x) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(x), & m > 0. \end{cases} \tag{4}$$

and $m = 0,1,...,M-1$ and $n = 1,2,...,2^{k-1}$. $T_m(x)$ are the famous Chebyshev polynomials of the first kind of degree $m$, which are orthogonal with respect to the weight function $W(x) = \frac{1}{\sqrt{1-x^2}}$, on the interval $[-1,1]$, and satisfy the following recursive formula:

$$\begin{cases} T_0(x) = 1, T_1(x) = x, \\ T_{m+1}(x) = 2xT_m(x) - T_{m-1}(x), & m = 1,2,... \end{cases} \tag{5}$$

The set of Chebyshev wavelets is an orthogonal set with respect to the weight function $W_n(x) = W(2^k x - 2n + 1)$.

A function $f(x)$ defined on the interval $[0,1]$ may be presented as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x). \tag{6}$$

The series representation of $f(x)$ in (6) is called a wavelet series and the wavelet coefficients $c_{nm}$ are given by $c_{nm} = (f(x), \psi_{nm}(x))_{W_n(x)}$.

The convergence of the series (6), in $L^2[0,1]$, means that

$$\lim_{s1,s2\to\infty} \left\| f(x) - \sum_{n=1}^{s1} \sum_{m=0}^{s2} c_{nm} \psi_{nm}(x) \right\| = 0. \tag{7}$$

Therefore one can consider the following truncated series for series (6)

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(x) = C^T \psi(x), \tag{8}$$

where $C$ and $\psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$\begin{aligned} C &= [c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1} \\ &\quad , ..., c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \\ &= [c_1, c_2, ..., c_M, c_{M+1}, ..., c_{2^{k-1}M}]^T \end{aligned} \tag{9}$$

and

$$\begin{aligned} \psi(x) &= [\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x) \\ &\quad , ..., \psi_{2,M-1}(x), ..., \psi_{2^{k-1},0}(x), ..., \psi_{2^{k-1},M-1}(x)]^T \\ &= [\psi_1(x), \psi_2(x), ..., \psi_M(x) \\ &\quad , \psi_{M+1}(x), ..., \psi_{2^{k-1}M}(x)]^T \end{aligned} \tag{10}$$

The integration of the product of two Chebyshev wavelets vector functions with respect to the weight function $W_n(x)$, is derived as

$$\int_0^1 W_n(x) \psi(x) \psi^T(x) dx = I, \tag{11}$$

where $I$ is an identity matrix.

A function $f(x,y)$ defined on $[0,1] \times [0,1]$ can be approximated as the following

$$f(x,y) \simeq \psi^T(x) K \psi(y). \tag{12}$$

Here the entries of matrix $K = [k_{ij}]_{2^{k-1}M \times 2^{k-1}M}$ will be obtain by

$$\begin{aligned} k_{ij} &= \left( \psi_i(x), (f(x,y), \psi_j(y))_{W_n(y)} \right)_{W_n(x)}, \\ i,j &= 1,2,...,2^{k-1}M. \end{aligned} \tag{13}$$

The integration of the vector $\psi(x)$, defined in (10), can be achieved as

$$\int_0^x \psi(t) dt = P\psi(x) \tag{14}$$

where $P$ is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of integration [8,9]. This matrix is determined as follows

$$P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \dots & F \\ O & L & F & \ddots & \vdots \\ O & O & L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & F \\ O & \cdots & O & O & L \end{bmatrix} \tag{15}$$

Where $L, F$ and $O$ are $M \times M$ matrices given by

$$L = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(-1)^r(\frac{1}{r-2} - \frac{1}{r}) & \cdots & -\frac{1}{2(r-2)} & 0 & \frac{1}{2r} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\sqrt{2}}{2}(-1)^M(\frac{1}{M-2} - \frac{1}{M}) & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-2)} & 0 \end{bmatrix} \tag{16}$$

$$F=\begin{bmatrix} 2 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ -\frac{2\sqrt{2}}{3} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^r}{r}-\frac{1-(-1)^{r-2}}{r-2}\right) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\sqrt{2}}{2}\left(\frac{1-(-1)^M}{M}-\frac{1-(-1)^{M-2}}{M-2}\right) & 0 & \cdots & 0 \end{bmatrix} \quad (17)$$

$$O=\begin{bmatrix} 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad (18)$$

The property of the product of two Chebyshev wavelets vector functions will be as follows

$$\psi(x)\psi^T(x)Y \approx \tilde{Y}\psi(x) \quad (19)$$

where $Y$ is a given vector and $\tilde{Y}$ is a $2^{k-1}M\times 2^{k-1}M$ matrix. This matrix is called the operational matrix of product.

## 3 Solution of high-order integro-differential equations via Chebyshev wavelets method

To illustrate the basic ideas of this method, let us consider the following integro-differential equation

$$y^{(n)}(x)+f(x)y(x)+\int_0^x k(x,t)y(t)dt=g(x), \quad (20)$$

with initial conditions

$$y(0)=\alpha_0, y'(0)=\alpha_1,\ldots,y^{(n-1)}(0)=\alpha_{n-1}. \quad (21)$$

Let's consider the following approximation for unknown function $y^{(n)}(x)$,

$$y^{(n)}(x)=C^T\psi(x) \quad (22)$$

where $C$ is $2^{k-1}M\times 1$ matrices given by

$$\begin{aligned} C=&[c_{1,0},c_{1,1},\ldots,c_{1,M-1},c_{2,0},c_{2,1},\ldots,c_{2,M-1}\\ &,\ldots,c_{2^{k-1},0},\ldots,c_{2^{k-1},M-1}]^T\\ =&[c_1,c_2,\ldots,c_M,c_{M+1},\ldots,c_{2^{k-1}M}]^T \end{aligned} \quad (23)$$

and $\psi(x)$ is defined in (10). Use this approximation will be resulted to:

$$y(x)\approx c^TP^n\psi(x)+\sum_{j=0}^{n-1}\alpha_j\frac{x^j}{j!} \quad (24)$$

where $P$ is operational matrix of integration. Also consider the following approximations

$$\begin{aligned} &f(x)\approx f_1^T\psi(x),\\ &g(x)\approx f_2^T\psi(x),\\ &\sum_{j=0}^{n-1}\alpha_j\frac{x^j}{j!}\approx f_3^T\psi(x),\\ &k(x,t)\approx \psi^T(x)K\psi(t), \end{aligned} \quad (25)$$

where $f_1,f_2,f_3$ are the $2^{k-1}M\times 1$ matrices, and $K$ is the $2^{k-1}M\times 2^{k-1}M$ matrice.

Substitution of approximations (22), (24), (25) into the Eq.(20), will be resulted to:

$$\begin{aligned} f_2^T\psi(x)=&c^T\psi(x)+f_1^T\psi(x)(c^TP^n\psi(x)+f_3^T\psi(x))\\ &+\int_0^x\psi^T(x)K\psi(t)(c^TP^n\psi(t)+f_3^T\psi(t))dt\\ =&c^T\psi(x)+f_1^T\psi(x)(c^TP^n\psi(x)+f_3^T\psi(x))\\ &+\psi^T(x)K\int_0^x\psi(t)(c^TP^n\psi(t)+f_3^T\psi(t))dt\\ =&c^T\psi(x)+f_1^T\psi(x)c^TP^n\psi(x)+f_1^T\psi(x)f_3^T\psi(x)\\ &+\psi^T(x)K\int_0^x(\psi(t)c^TP^n\psi(t)+\psi(t)f_3^T\psi(t))dt\\ =&c^T\psi(x)+f_1^T\psi(x)c^TP^n\psi(x)\\ &+f_1^T\psi(x)f_3^T\psi(x)+\psi^T(x)K\tilde{Y}P\psi(x), \end{aligned} \quad (26)$$

where $\tilde{Y}$ is $2^{k-1}M\times 2^{k-1}M$ operational matrix for production and $P$ is the $2^{k-1}M\times 2^{k-1}M$ operational matrix of integration [8,9,10].

According to the Galerkin method by multiplying $W_n(x)\psi^T(x)$ in both sides of the Eq.(26) and then applying $\int_0^1(.)dx$ linear or non-linear equation in terms of the entries of $C$ will be obtained. The elements of vector functions $C$ can be computed by solving these equation.

### Error analysis

**Theorem 1 [21]:** Assume $p$ be the number of vanishing moments for a wavelet $\psi_{nm}(x)$ and let $f(x)\in C^P[0,1]$. Then the wavelet coefficient, $c_{nm}$, decays as follows

$$|c_{nm}|\le C_P2^{-n(p+1/2)}\max_{\xi\in[0,1]}\left|f^{(P)}(\xi)\right|, \quad (27)$$

Where $C_p$ is an independent constant from $n,m$ and $f(x)$. The above theorem implies that wavelet coefficients are exponentially decayed with respect to $P$ and by increasing $p$ the decay increases. Since the truncated Chebyshev wavelets series is approximate solution of a system, so one has an error function $error(f(x))$ for $f(x)$ as follows

$$error(f(x)) = \left| f(x) - \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) \right| \quad (28)$$

where setting $x = x_j, x_j \in [0,1]$, the absolute error value of $x_j$ can be obtained.

The error bound of the approximate solution by using Chebyshev wavelets series is given by the following theorem.

**Theorem 2 [21]:** Suppose $f(x) \in C^p[0,1]$ and $C^T\psi(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)$ is the approximate solution using Chebyshev wavelets method. Then the error bound would be obtained as follows

$$\|error(f(x))\| \leq \frac{1}{p!2^{p(k-1)}} \max_{\xi\in[0,1]} \left| f^{(p)}(\xi) \right|. \quad (29)$$

## 4 Numerical examples

In this section, some examples of high-order integro-differential equations are considered and will be solved. These examples are solved for $k = 1$ and $M = 8$.

**Example 1.** Consider the following integro-differential equation

$$\begin{aligned} &y'(x) + y(x) = 1 + 2x + \textstyle\int_0^x x(1+2x)e^{t(x-t)}y(t)dt, \\ &y(0) = 1. \end{aligned} \quad (30)$$

The exact solution is $y(x) = e^{x^2}$. Let's consider the following approximations

$$\begin{aligned} &1 \approx f_1^T\psi(x), \\ &1+2x \approx f_2^T\psi(x), \\ &y'(x) \approx c^T\psi(x), \\ &y(x) \approx c^TP\psi(x) + y(0) \\ &\qquad = c^TP\psi(x) + f_3^T\psi(x), \\ &x(1+2x)e^{t(x-t)} \approx \psi^T(x)K\psi(t). \end{aligned}$$

Substitution into the Eq. (30), lead to the following equation

$$\begin{aligned} f_2^T\psi(x) &= c^T\psi(x) + f_1^T\psi(x)(c^TP\psi(x) + f_3^T\psi(x)) \\ &- \int_0^x \psi^T(x)K\psi(t)(c^TP\psi(t) + f_3^T\psi(t))dt, \\ &= c^T\psi(x) + f_1^T\psi(x)c^TP\psi(x) + f_1^T\psi(x)f_3^T\psi(x) \\ &- \psi^T(x)K\int_0^x (\psi(t)c^TP\psi(t) + \psi(t)f_3^T\psi(t))dt \\ &= c^T\psi(x) + f_1^T\psi(x)c^TP\psi(x) + f_1^T\psi(x)f_3^T\psi(x) \\ &- \psi^T(x)K\tilde{Y}P\psi(x) \end{aligned} \quad (31)$$

Multiply $W_n(x)\psi^T(x)$, on both sides of the Eq.(31), apply $\int_0^1 (.)dx$, and then solve the equation. The elements of vector functions $C$ can be obtained as follow

$$\begin{aligned} C = [&2.459921807, 2.222729964, 0.6356289576, \\ &0.1826361012, 0.03662164886, 0.007330474902, \\ &0.001184876223, 0.0001879434420]^T \end{aligned}$$

Therefore, the following solution will result.

$$\begin{aligned} y(x) &\approx C^TPy(x) + f_3^Ty(x) \\ &= 0.3911649707x^7 - 0.6813347882x^6 \\ &+ 0.8428981630x^5 + 0.06099955548x^4 \\ &+ 0.1223416335x^3 + 0.9839702958x^2 \\ &+ 0.0008239011610x + 0.9999933725 \end{aligned}$$

Table 1 shows some values of the solutions and absolute errors at some $x, s$ and plots of the exact and approximate solutions are shown in Figure (1) and Figure (2). In this example use the Taylor expansion of $e^{t(x-t)}$ in $x = 0$.

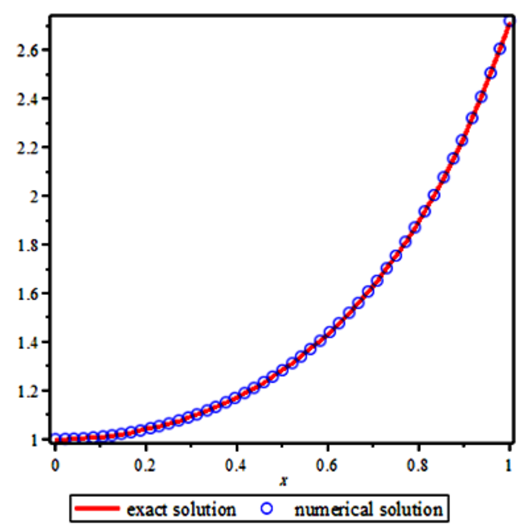


**Fig. 1:** Comparison of the exact and approximate solution of Example 1.

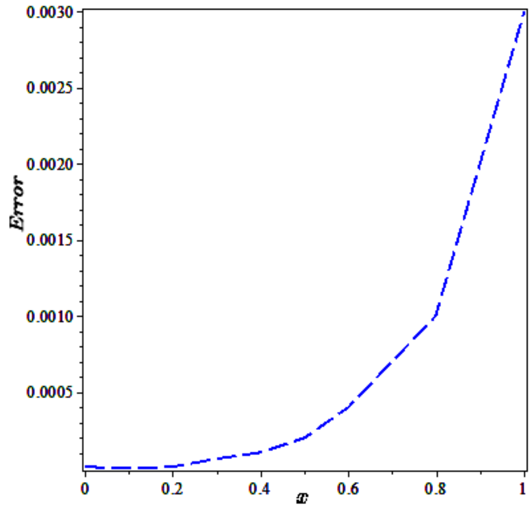


**Fig. 2:** The absolute errors of Example 1 for various $x \in [0,1]$.

**Table 1:** Numerical results of Example 1.

| $x$ | *Exact solution* | *Chebyshev wavelets* | *Absolute Error* |
|---|---|---|---|
| 0.0 | 1.0000000000 | 0.9999933725 | 0.0000066275 |
| 0.1 | 1.0100501670 | 1.0100516940 | 0.0000015270 |
| 0.2 | 1.0408107740 | 1.0408244260 | 0.0000136520 |
| 0.3 | 1.0941742840 | 1.0942322870 | 0.0000580030 |
| 0.4 | 1.1735108710 | 1.1736310480 | 0.0001201770 |
| 0.5 | 1.2840254170 | 1.2842537610 | 0.0002283440 |
| 0.6 | 1.4333294150 | 1.4337538760 | 0.0004244610 |
| 0.7 | 1.6323162200 | 1.6330463770 | 0.0007301570 |
| 0.8 | 1.8964808790 | 1.8976441010 | 0.0011632220 |
| 0.9 | 2.2479079870 | 2.2496863700 | 0.0017783830 |
| 1.0 | 2.7182818280 | 2.7208571040 | 0.0025752760 |

**Example 2.** Consider the following equation

$$y''(x) - xy(x) = g(x) + \int_0^x x^2 e^t y(t)dt, \quad y(0) = 1, y'(0) = 0, \tag{32}$$

where $g(x) = -(1+x)\cos x - \frac{x^2}{2}(e^x(\cos x + \sin x) - 1)$.

The exact solution is $y(x) = \cos x$. Let's consider the following approximations

$$\begin{aligned} x &\approx f_1^T \psi(X), \\ g(x) &\approx f_2^T \psi(x), \\ y''(x) &\approx C^T \psi(x), \\ y'(x) &\approx C^T P\psi(x) + y'(0), \\ y(x) &\approx C^T P^2 \psi(x) + xy'(0) + y(0) \\ &= C^T P^2 \psi(x) + f_3^T \psi(x), \\ x^2 e^t &\approx \psi^T(x) K \psi(t). \end{aligned}$$

The vector $C$ is computed by solving the equation of nonlinear for eight unknowns, via the Maple package, as follow

$$\begin{aligned} C = [&-1.031085599, 0.2071027476, 0.04809969975, \\ &-0.002205020154, -0.0002510004840, \\ &-0.000033910154, 6.4812730 \times 10^{-8}, \\ &-0.000005586572553]^T \end{aligned}$$

Therefore, we have the following approximate solution

$$\begin{aligned} y(x) \approx &-0.0003896079759x^7 + 0.0001546713811x^6 \\ &-0.002133764458x^5 + 0.04339755439x^4 \\ &-0.0006676702145x^3 - 0.4999395723x^2 \\ &+2.074417985 \times 10^{-7}x + 1.000000000 \end{aligned}$$

Table 2 shows some values of the solutions and absolute errors at some $x,s$ and plots of the exact and approximate solutions are shown in Figure (3) and Figure (4). In this example use the Taylor expansion of $-(1+x)\cos x - \frac{x^2}{2}(e^x(\cos x + \sin x) - 1)$, $x^2 e^t$ in $x = \frac{1}{2}$.

**Table 2:** Numerical results of Example 2.

| $x$ | *Exact solution* | *Chebyshev wavelets* | *Absolute Error* |
|---|---|---|---|
| 0.0 | 1.0000000000 | 1.0000000000 | 0.0000000000 |
| 0.1 | 0.9950041653 | 0.9950042759 | 0.0000001106 |
| 0.2 | 0.9800665778 | 0.9800658754 | 0.0000007024 |
| 0.3 | 0.9553364891 | 0.9553338363 | 0.0000026528 |
| 0.4 | 0.9210609940 | 0.9210561434 | 0.0000048506 |
| 0.5 | 0.8775825619 | 0.8775767918 | 0.0000057701 |
| 0.6 | 0.8253356149 | 0.8253323731 | 0.0000032418 |
| 0.7 | 0.7648421873 | 0.7648479860 | 0.0000057987 |
| 0.8 | 0.6967067093 | 0.6967322754 | 0.0000255691 |
| 0.9 | 0.6216099683 | 0.6216714208 | 0.0000614525 |
| 1.0 | 0.5403023059 | 0.5404218182 | 0.0001195123 |

**Example 3.** Consider the following integro-differential equation

$$y^{(4)}(x) + (x^2 - 1)y(x) + \int_0^x e^{(t-2x)} y(t)dt = xe^{-x}(x + e^{-x}) \quad y(0) = 1,\ y'(0) = -1,\ y''(0) = 1,\ y'''(0) = -1. \tag{33}$$

with the exact solution $y(x) = e^{-x}$.

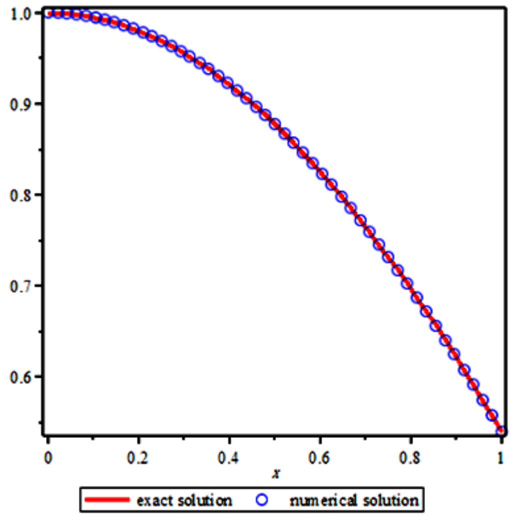


**Fig. 3:** Comparison of the exact and approximate solution of Example 2.

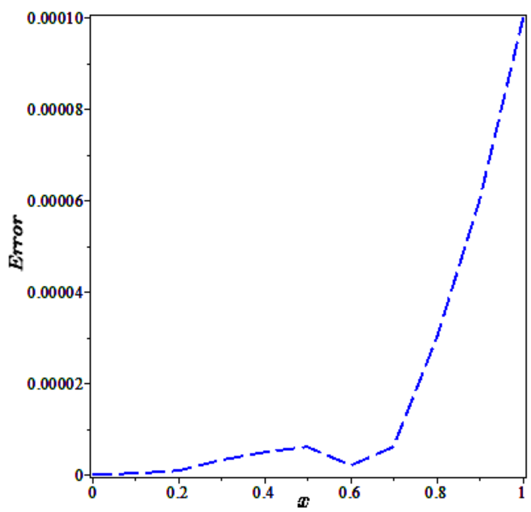


**Fig. 4:** The absolute errors of Example 2 for various $x \in [0,1]$.

Let's

$$(x^2-1) \approx f_1^T \psi(x),$$
$$xe^{-x}(x+e^{-x}) \approx f_2^T \psi(x),$$
$$y^{(4)}(x) \approx C^T \psi(x),$$
$$y(x) \approx C^T P^4 \psi(x) + \tfrac{1}{6}x^3 y'''(0) + \tfrac{1}{2}x^2 y''(0) + xy'(0) + y(0) = C^T P^4 \psi(x) + f_3^T \psi(x),$$
$$e^{(t-2x)} \approx \psi^T(x) K \psi(t).$$

By applying the Chebyshev wavelets method and solving the resulted linear equation, the following results would be achieved.

$$C = [0.5267139138, -0.5091773717, 0.01820611575, 0.01008902560, -0.002251757235, 0.0003590418624, -0.00003021554629, 7.503944587 \times 10^{-7}]^T$$

Therefore,the following solution will result.

$$\begin{aligned} y(x) &\approx C^T P^4 \psi(x) + f_3^T \psi(x) \\ &= 0.0004006501638x^7 - 0.0008180424275x^6 \\ &\quad -0.009770157671x^5 + 0.04230486056x^4 \\ &\quad -0.1668084476x^3 + 0.5000167372x^2 \\ &\quad -1.000000758x + 1.000000006 \end{aligned}$$

Some numerical results of this solution are presented in Table (3) the Taylor expansion of $e^{(t-2x)}, xe^{-x}(x+e^{-x})$ in $x = \frac{1}{2}$.

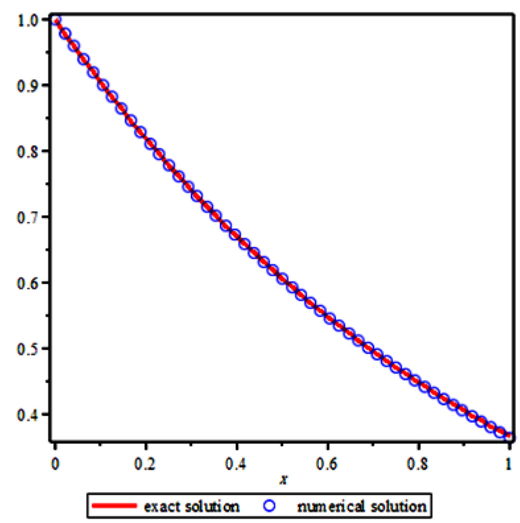


**Fig. 5:** Comparison of the exact and approximate solution of Example 3.

**Example 4.** Consider the following equation with the exact solution $y(x) = x^4 - x^3$.

$$\begin{aligned} & y^{(6)}(x) + x^6(x+5)y(x) + \int_0^x (t^4 - x^4) y(t)dt \\ & = \tfrac{1}{8}x^8 - \tfrac{229}{45}x^9 + 4x^{10} + x^{11} \\ & y(0) = y'(0) = y''(0) = 0, y'''(0) = -6, \\ & y^{(4)}(0) = 24, y^{(5)}(0) = 0. \end{aligned} \qquad (34)$$

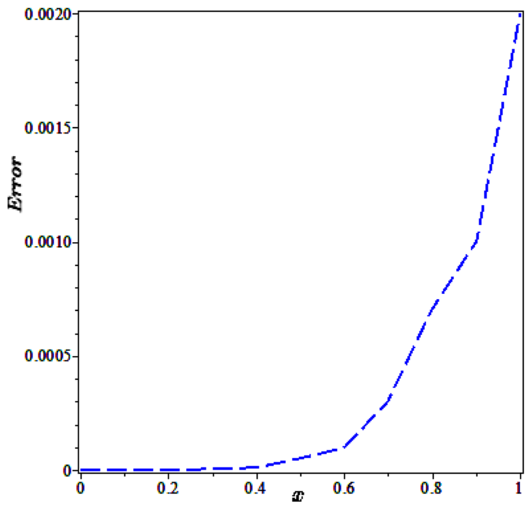


**Fig. 6:** The absolute errors of Example 3 for various $x \in [0,1]$.

**Table 3:** Numerical results of Example 3.

| $x$ | *Exact solution* | *Chebyshev wavelets* | *Absolute Error* |
|---|---|---|---|
| 0.0 | 1.0000000000 | 1.0000000006 | 0.0000000006 |
| 0.1 | 0.9048374180 | 0.9048374211 | 0.0000000031 |
| 0.2 | 0.8187307531 | 0.8187305704 | 0.0000001827 |
| 0.3 | 0.7408182207 | 0.7408158760 | 0.0000023447 |
| 0.4 | 0.6703200460 | 0.6703069038 | 0.0000131422 |
| 0.5 | 0.6065306597 | 0.6064818399 | 0.0000488198 |
| 0.6 | 0.5488116361 | 0.5486709834 | 0.0001406527 |
| 0.7 | 0.4965853038 | 0.4962444591 | 0.0003408447 |
| 0.8 | 0.4493289641 | 0.4486003494 | 0.0007286147 |
| 0.9 | 0.4065696597 | 0.4051534496 | 0.0014162101 |
| 1.0 | 0.3678794412 | 0.3653248482 | 0.0025545930 |

Let's consider the following approximations

$$
\begin{aligned}
&x^6(x+5) \approx f_1^T \psi(x),\\
&\tfrac{1}{8}x^8 - \tfrac{229}{45}x^9 + 4x^{10} + x^{11} \approx f_2^T \psi(x),\\
&y^{(6)}(x) \approx C^T \psi(x),\\
&y(x) \approx C^T P^6 \psi(x) + \tfrac{1}{120}x^5 y^{(5)}(0) + \tfrac{1}{24}x^4 y^{(4)}(0)\\
&\qquad + \tfrac{1}{6}x^3 y'''(0) + \tfrac{1}{2}x^2 y''(0) + xy'(0) + y(0)\\
&\qquad = C^T P^6 \psi(x) + f_3^T \psi(x),\\
&(t^4 - x^4) \approx \psi^T(x) K \psi(t).
\end{aligned}
$$

The vector $C$ is computed by solving the equation of linear for eight unknowns, via the Maple package, as follow

$$
\begin{aligned}
C = [&0.03414680234, 0.03666456160, 0.01488269850,\\
&0.001861353490, -0.0007517563843,\\
&-0.0003093780003, -0.00009320203923,\\
&-0.00004079964123]^T
\end{aligned}
$$

Therefore,the following solution will result.

$$
\begin{aligned}
y(x) &\approx C^T P^6 \psi(x) + f_3^T \psi(x)\\
&= 0.00001440179776x^7 - 0.00003315141043x^6\\
&+0.00003245990217x^5 + 0.9999834877x^4\\
&-0.9999955652x^3 - 5.769157089 \times 10^{-7}x^2\\
&+2.834107322 \times 10^{-8}x - 2.245887356 \times 10^{-10}
\end{aligned}
$$

Some values of exact, approximate solutions and absolute error are presented in Table (4) and the plots of exact and approximate solutions are shown in Figure (7) and Figure (8).

**Table 4:** Numerical results of Example 4.

| $x$ | *Exact solution* | *Chebyshev wavelets* | *Absolute Error* |
|---|---|---|---|
| 0.0 | 0.0000000000 | − 0.0000000002 | 0.0000000002 |
| 0.1 | − 0.0009000000 | − 0.0090000001 | 0.0000000001 |
| 0.2 | − 0.0064000000 | − 0.0064000001 | 0.0000000001 |
| 0.3 | − 0.0189000000 | − 0.0188999998 | 0.0000000002 |
| 0.4 | − 0.0384000000 | − 0.0383999999 | 0.0000000001 |
| 0.5 | − 0.0625000000 | − 0.0624999990 | 0.0000000010 |
| 0.6 | − 0.0864000000 | − 0.0863999925 | 0.0000000075 |
| 0.7 | − 0.1029000000 | − 0.1028999652 | 0.0000000348 |
| 0.8 | − 0.1024000000 | − 0.1023998733 | 0.0000001267 |
| 0.9 | − 0.0729000000 | − 0.0728996051 | 0.0000003949 |
| 1.0 | 0.0000000000 | 0.0000010840 | 0.0000010840 |

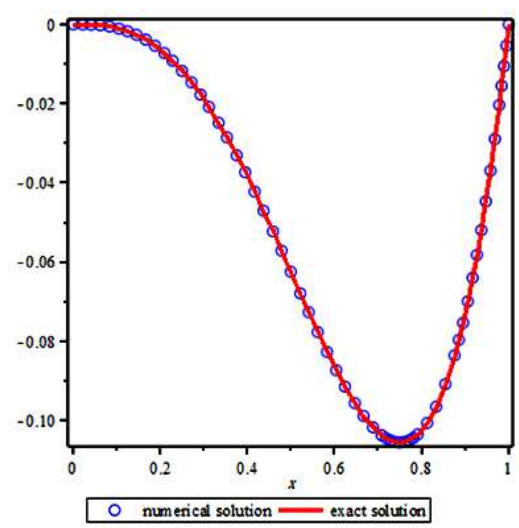


**Fig. 7:** Comparison of the exact and approximate solution of Example 4.

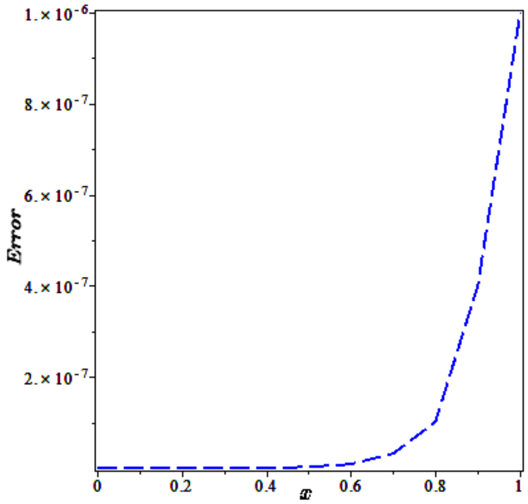


**Fig. 8:** The absolute errors of Example 4 for various $x \in [0,1]$.

## 5 Conclusion

In this work, we have used Chebyshev Wavelets method for numerical solution of linear and nonlinear high-order integro-differential equations. The main advantage of Chebyshev wavelet method for solving the equation is that after discreting the coefficients matrix of algebraic equations is spare. As shown in the four examples of this paper, the proposed method is a powerful procedure for solving the problems. The simplicity and also easy-to-apply in programming are two special features of this method. The package Maple 16 is used for computation.

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