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On n Satisfying, n Dividing $\varphi(n)\sigma(n)+1$

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Abstract

Let $\boldsymbol{\sigma(n)}$ denotes the sum of the positive divisors of the positive integer $\boldsymbol{n}$ and $\boldsymbol{\varphi}$ be the Euler's totient function([1],page25). Cleary, $\boldsymbol{n}$ divides $\boldsymbol{\varphi(n)\ \sigma(n)+1}$ if $\boldsymbol{n}$ is a prime. Then, the question is there a composite $\boldsymbol{n}$ that divides $\boldsymbol{\varphi(n)\ \sigma(n)+1}$? Considering this problem Yang-Gao Chen and Jin-Hui Fang [5] have proved that $\boldsymbol{n \in R \Rightarrow \omega(n) \geq 3}$, where $\boldsymbol{\omega(n)}$ as usual is the number of distinct prime factors of $\boldsymbol{n}$. and $\boldsymbol{R = \bigcup_{k=1}^{\infty} R_k}$, $\boldsymbol{R_k = \{n: \varphi(n)\sigma(n)+1 = kn\}}$ .We devoted the study of this problem where we prove that every $\boldsymbol{n}$ in $\boldsymbol{R}$ is odd and that $\boldsymbol{n \in R\ and\ 3 \mid n \Rightarrow n = 3}$, from which it follows that for any composite $\boldsymbol{n}$ in $\boldsymbol{R}$, the least prime factors is $\boldsymbol{\geq 5}$. Also, we obtained lower bounds for $\boldsymbol{\omega(n)}$ and $\boldsymbol{n}$ for any $\boldsymbol{n \in R = \bigcup_{k=1}^{\infty} R_k}$ with $\boldsymbol{1 \leq k \leq 672}$, which improves the result of [5] in some cases.

Key words : Euler's totient function, the sum of the positive divisor of $\boldsymbol{n}$.

1. Introduction:

Let $\sigma(n)$ denotes the sum of the positive divisors of the positive integer n and φ be the Euler's totient function([1],page25). It is easy to see that $\varphi(n)\sigma(n)+1$ is divisible by n if n is a prime number (in fact, if $n = p$, a prime, then $\varphi(p)\sigma(p)+1 = (p-1)(p+1)+1 = p^2$ divisible by n). Now the question:

(1.1) Is there a composite number n for which $n \mid \varphi(n)\sigma(n)+1$?

That is, if

(1.2) $R_k = \{n: \varphi(n)\sigma(n)+1 = kn\}$ and,

(1.3)

$$R = \bigcup_{k=1}^{\infty} R_k .$$

The question in(1.1) is seeking composite numbers in R.

Yang-Gao Chen and Jin-Hui Fang ([5] Theorem2) have proved that

(1.4) $n \in R \Rightarrow \omega(n) \geq 3$.

where $\omega(n)$ is as before the number of distinct prime factor of n.

The purpose of this study is to improve (1.4) in the case $n \in R_k$ with $k \leq 336$ and $5 \mid n$, and with $k \leq 672$, and $5 \nmid n$. Also, we find lower bounds for $n \in R_k$.

Preliminary:

In this part we prove some preliminaries lemmas. As already noted in [5], any $n \in R_k$ is squarefree. Since $\varphi(n)\sigma(n)+1$ is odd for $n > 4$, it follows that

(2.1) $n \in R_k \Rightarrow k$ and n are both odd .

Hence $R_2 = R_4 = R_6 = R_8 = \cdots = \emptyset$.

Therefore, any $n \in R$ is odd and squarefree so that it can be written as

(2.2) $n = p_1 p_2 p_3 \ldots p_r$, where $p_1 < p_2 < p_3 < \cdots < p_r$ are all odd primes

$r = \omega(n)$ and,

(2.3)

$$(p_1-1)(p_2-1)..(p_r-1)(p_1+1)(p_2+1)\ldots(p_r+1)+1 = kp_1p_2p_3\ldots p_r$$

Through this paper, we assume $n \in R$ and that it is of the form (2.2).

2.1. Lemma : $p_i^2 \not\equiv 1 (mod\ p_j)\ for\ i \neq j.$

Proof : Suppose $p_i^2 \equiv 1(mod\ p_j)$ for some $i \neq j$, then it follows from (2.3) that $p_j \mid 1$, a contradiction .Therefore, the lemma holds.

2.2. Lemma : The only element in R divisible by 3 is 3. In other words, R has no composite number divisible by 3.

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Proof : Cleary, $3 \in R$ (since $\varphi(3)\sigma(3)+1 = 2.4+1 = 9 = 3.3$). If possible $n \in R$ with $r \geq 2$ and $p_1 = 3$ .Then, by Lemma 2.1, $p_i \equiv 2(mod 3)$ for $2 \leq i \leq r$ so that reducing (2.3) to module 3 we get $0+1 \equiv 0(mod 3)$, a contradiction. Thus, $r \ngeq 2$, hence the lemma.

3 .Main result

In all that follows n denotes the composite number in R_k so that, by (2.3) the product P_n is defined by

(3.1)

$$P_n \doteq \prod_{i=1}^{r} \frac{p_i}{p_i - 1} > \frac{(p_1+1)(p_2+1)\ldots(p_r+1)}{k}.$$

Now, since $r \geq 3$, by (1.4) and $p_1 \geq 5, p_2 \geq 7, p_3 \geq 13$, it is follows that

(3.2)

$$P_n > \begin{cases} \dfrac{6 \times 8 \times 14}{k} = \dfrac{672}{k} & if \quad 5 \mid n \\ \dfrac{8 \times 12 \times 14}{k} = \dfrac{1344}{k} & if \quad 5 \nmid n \end{cases}$$

3.1. Theorem : Suppose $n \in R_k$(k is odd) and $5 \mid$n then

i) $1 \leq k \leq 129 \Rightarrow \omega(n) \geq 58246$ and $n > 10^{310360}$.

ii) $131 \leq k \leq 143 \Rightarrow \omega(n) \geq 11504$ and $n > 10^{17698}$.

iii) $145 \leq k \leq 191 \Rightarrow \omega(n) \geq 371$ and $n > 10^{1064}$.

iv) $193 \leq k \leq 287 \Rightarrow \omega(n) \geq 18$ and $n > 10^{22}$

v) $289 \leq k \leq 336 \Rightarrow \omega(n) \geq 8$ and $n > 10^{5}$.

Proof : Suppose $n \in R_k$, $5 \mid n$.Then, n is of the from (2.2) with $p_1 = 5$, so that $p_i \not\equiv 1(mod\ 5)$ for $2 \leq i \leq r$.Therefore, p_i's are from the set $A = \{p : p\ prime\ , p > 5, p \not\equiv 1(mod\ 5)\}$. Also, if $A = \{a_1, a_2, a_3, \ldots\}$ with $a_j \leq a_{j+1}$ for j = 1,2,3, ... Then, $p_i \geq a_i$ for $2 \leq i \leq r$. hence we have

(3.3)

$$P_n = \prod_{i=1}^{r} \frac{p_i}{(p_i - 1)} \leq \prod_{i=1}^{r} \frac{a_i}{(a_i - 1)} = \alpha(r).\ (say).$$

Now, comparing (3.2) and (3.3), we find that

(3.4) $\alpha(r) > \frac{672}{k}$.

i) Suppose $1 \leq k \leq 129$, then by (3.4), $\alpha(r) > 5.2093$ and hence

$\ln \alpha(r) = \ln \prod_{i=1}^{r} \frac{a_i}{(a_i-1)} = \ln \sum_{i}^{r} \frac{a_i}{(a_i-1)} > \ln(5.2093) = 1.65044549.$

A computation yields $r \geq 58246$, proving the first part of (i).

Now, by the Robin's inequality ([3], Theorem6) given by :

(3.5) $n > \left(\frac{r \ln r}{3}\right)^r$ for $n \in \mathbb{N}$ with $\omega(n) = r$.

Since $n \in R_k$ with $1 \leq k \leq 129$, we get by (3. 5) for such n that

$$n > \left(\frac{58246 . \ln(58246)}{3}\right)^{58246} > 10^{310360} .$$

ii) Suppose $129 \leq k \leq 143$, then by (3.4), $\alpha(r) > 4.6993$ and hence $\ln(4.6993) = 1.54741356$. A computation yields $r \geq 11504$, giving first part of (ii).Therefore, for $n \in R_k$ with $129 \leq k \leq 143$, we get by (3.5) that

$$n > \left(\frac{11504 . \ln 11504}{3}\right)^{11504} > 10^{17698} .$$

iii) Suppose $145 \leq k \leq 191$, then by (3.4), $\alpha(r) > 3.518324$ and hence $\ln \alpha(r) > ln(3.518324) = 1.257984912$. A computation yields $r \geq 371$, proving first part of (iii).

Therefore, for $n \in R_k$ with $145 \leq k \leq 191$, we get by (3.5) that

$$n > \left(\frac{371. \ln 371}{3}\right)^{371} > 10^{988} .$$

iv) Suppose $193 \leq k \leq 287$, then by (3.4), $\alpha(r) > 2.34146$ and hence $\ln \alpha(r) > \ln(2.34146\ ) = 0.8507761248$. A computation yields r ≥ 18.

Therefore, for $n \in R_k$ with $193 \leq k \leq 287$, we get ,by (3.5) that

$$n > \left(\frac{18 . \ln 18}{3}\right)^{18} > 10^{22} .$$

v) Suppose $289 \leq k \leq 335$.Then by (3.4), $\alpha(r) > 2.0059$ and
hence $\ln \alpha(r) > \ln(2.0059\ ) = 0.69069$.A computation yields $r \geq 371$.
Therefore, for $n \in R_k$ with$289 \leq k \leq 335$, we get by (3.5) we have that

$$n > \left(\frac{8. \ln 8}{3}\right)^{8} > 10^{5}$$

3.2 Theorem : Suppose $n \in R_k$(k is odd) and $5 \nmid n$ then
i) $1 \leq k \leq 207 \Rightarrow \omega(n) \geq 65536$ and $n > 10^{67963}$.
ii) $209 \leq k \leq 223 \Rightarrow \omega(n) \geq 27889$ and $n > 10^{138843}$.
iii) $225 \leq k \leq 267 \Rightarrow \omega(n) \geq 4188$ and $n > 10^{17028}$.
iv) $269 \leq k \leq 333 \Rightarrow \omega(n) \geq 646$ and $n > 10^{2009}$.
v) $335 \leq k \leq 445 \Rightarrow \omega(n) \geq 100$ and $n > 10^{218}$.
vi) $447 \leq k \leq 671 \Rightarrow \omega(n) \geq 14$ and $n > 10^{15}$.

Proof: Suppose $n \in R_k$, $5 \nmid n$, then n is of the form (2.2),where p_i's are from the set $D = \{p: p\ prime\ , p \geq 7\}$. Also, if $D = \{d_1, d_2, d_3, \ldots ,\}$ with $d_1 < d_2 < d_3$ ..., then $d_1 \geq 7, d_2 \geq 11, d_3 \geq 13$ and $p_i \geq d_i$ for $i \geq 1$. Therefore,
(3.6)

$$P_n \leq \delta(r), \text{where } \delta(r) = \prod_{i=1}^{r} \frac{d_i}{(d_i - 1)} .$$

Now, by (3.6) and (3.2),we get
(3.7) $\delta(r) > \frac{1344}{k}$.

i) Suppose $1 \leq k \leq 207$, then by (3.7) , $\delta(r) > 6.4927536$ and
hence $\ln \delta(r) > \ln(6.4927536\ ) = 1.8706867$. A computation yields
$r \geq 65536$, proving the first part of (i). Also, by (3.5), any $n \in T_k$ with $1 \leq k \leq 207$ is such that

$$n > \left(\frac{65536 .\ \ln(65536)}{3}\right)^{65536} > 10^{67963} .$$

ii) Suppose $209 \leq k \leq 223$.So that $\delta(r) > \frac{1344}{223} = 6.02690583$
and hence $\ln \delta(r) > \ln(6.02690583\ ) = 1.7962$. A computation yields $r \geq 27889$.
Therefore, for $n \in R_k$ *with* $209 \leq k \leq 223$, we get by (3.5) that

$$n > \left(\frac{27889 . \ln 27889}{3}\right)^{27889} > 10^{138843} .$$

iii) Suppose $225 \leq k \leq 267$. So that , $\delta(r) > \frac{1344}{267} = 5.0337808$.
and hence $\ln \delta(r) > \ln(5.0337808\ ) = 1.61615$. A computation gives $r \geq 4188$ and by (3.5) we have

$$n > \left(\frac{4188 . \ln 4188}{3}\right)^{4188} > 10^{17028}.$$

iv) Suppose $269 \leq k \leq 333$. So that $\delta(r) > \frac{1344}{333} = 4.036036$ and
hence $\ln \delta(r) > \ln(4.036036\ ) = 1.39526$ .A computation gives $r \geq 646$ and by (3.5) we have

$$n > \left(\frac{646. \ln 646}{3}\right)^{646} > 10^{2009} .$$

v) Suppose $335 \leq k \leq 445$. So that $\delta(r) > \frac{1344}{445} = 3.0202$ and
hence $\ln \delta(r) > \ln(3.0202\ ) = 1.1053$. A computation gives $r \geq 100$ and by (3.5) we have

$$n > \left(\frac{100 .\ ln\, 100}{3}\right)^{100} > 10^{218} .$$

vi) Suppose $447 \leq k \leq 671$. So that $\delta(r) > \frac{1344}{671} = 2.0029$ and
hence $\ln \delta(r) > \ln(2.0029\ ) = 0.694636$. A computation gives $r \geq 14$ and by (3.5) we have

$$n > \left(\frac{14 . \ln 14}{3}\right)^{14} > 10^{15}.$$

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العدد n الذي يقسم المقدار $\varphi(n)\sigma(n)+1$

حسين عبدالقادر العيدروس

الملخص

إذا كانت $\sigma(n)$ مجموع القواسم الموجبة للعدد الصحيح الموجب n و كانت φ دالة أويلر ، فمن السهل معرفة أنه إذا كان العدد n عددا أوليا فإنه يقسم المقدار $\varphi(n)\sigma(n)+1$. ولكن هل يوجد عدد غير أولي n يقسم المقدار $\varphi(n)\sigma(n)+1$ ؟ هذا السؤال طرحه الباحثان ينق – قو شن و جان – هيو فانج ، وتمكنا ([5] ، نظرية 2) من إثبات أنه عندما $n \in R$ فإن $\omega(n) \geq 3$ ، حيث $\omega(n)$ تمثل عدد العوامل الأولية المختلفة للعدد n والمجموعتان R و R_k معرفتان بالشكل التالي : $R = \bigcup_{k=1}^{\infty} R_k$ ، و $R_k = \{n: \varphi(n)\sigma(n)+1 = kn\}$. استطاع الباحث إثبات أنه إذا كان $n \in R$ و كان $3|n$ فإن $n=3$ ، مما يعني أن أصغر عامل أولي للعد غير الأولي $n \in R$ سيكون أكبر أو يساوي العدد 5 ، كما توصل الباحث إلى الحد الأدنى لعدد العوامل الأولية $\omega(n)$ لأي $n \in R = \bigcup_{k=1}^{\infty} R_k$ ، حيث $1 \leq k \leq 672$ ، وبهذا يكون الباحث قد حسن بعض النتائج في [5].

كلمات المفتاح : دالة أويلر، مجموع القواسم الموجبة للعدد الصحيn.