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Fifth and Eleventh-Order Iterative Methods for Roots of Nonlinear Equations

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Abstract

In this work, two iterative methods, based on Newton's method, to obtain the numerical solutions of nonlinear equations have been constructed. We proved that our methods converge in fifth and eleventh orders. Analytical investigation has been established to show that our schemes have higher efficiency indexes than some recent methods. Numerical examples are experimented to investigate the performance of the proposed schemes. Moreover, theoretical order of convergence is verified on the experiment work.

Key Words: Nonlinear equation, Iterative method, Newton's method, Convergence order.

1. Introduction:

The importance of solving nonlinear equations comes out from its applications in science and engineering [13, 15, 18, 11, 7]. Many numerical applications use high precision in their computation, so higher-order numerical methods are required [5].

Some differential equations and integral equations require solving of nonlinear equations [2, 10]. In this work, we present two new iterative methods to find a simple root λ of a nonlinear equation $f(x)=0$, where $f: I \subset R \to R$: is a scalar function on an open interval I.

One of the simple one step well-known methods for root finding of nonlinear equations is the classical Newton's method (CN)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,... \tag{1}$$

This method iteratively produces a sequence of approximations that converge quadratically to a simple zero λ of the function f.

Traub, [18], started classifying iterative methods. He suggested a third-order iterative method. Jarratt [9, 8], proposed a family of methods consisting of two points, two steps, costing one function, two derivative evaluations per iteration and one parameter to reach order of convergence four.

In recent years, some high order iterative methods, for solving nonlinear equations, have been improved and investigated see [13, 15, 1, 14, 12, 17, 6, 16] and the references therein.

In this paper we consider the newton's method

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} \end{aligned}, \tag{2}$$

which also called double Newton's method (DN), the method converges in fourth order [6].

First, we present a variant of the double-Newton's method with fifth-order convergence. Based on the new method, an eleventh-order iterative method is proposed. Finally, numerical examples are given to show the performance of the two methods.

Basic definitions:

Definition 1. [3] Let λ be a simple zero of a real function $f(x)$, let $< x_n >$ be a real sequence that converges towards λ. We say that the order of convergence of the sequence is $\alpha \in R^+$ if there exists $\beta \in R^+$ such that $\lim_{n\to\infty}\left[|x_{n+1}-\lambda| / |x_n-\lambda|^\alpha\right] = \beta,$ β is called asymptotic error constant. If

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$\alpha = 2$ or 3 the sequence is said to have quadratic convergence or cubic convergence, respectively.

Definition 2. [11] Let $\varepsilon_n = x_n - \lambda$ is the error in the n-th iteration, we call the relation

$$\varepsilon_{n+1} = \beta\varepsilon_n^{\alpha} + O(\varepsilon_n^{\alpha+1}) \tag{3}$$

as the error equation. If we can obtain the error equation for any iterative method, then the value of α is its order of convergence.

If x_{n+1}, x_n and x_{n-1} are three successive iterations closer to the root λ. Then, the computational order of convergence ρ (see [19]) is approximated by using (3) as

$$\rho = \frac{\ln|(x_{n+1}-\lambda)/(x_n-\lambda)|}{\ln|(x_n-\lambda)/(x_{n-1}-\lambda)|} . \tag{4}$$

The efficiency index is $p^{1/w}$, where p is the order of the method and w is the number of function evaluations per iteration required by the method, see [4].

2. Main Results

Firstly, we introduce new fifth-order iterative method as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}\left[1 + \frac{f(y_n)\big(f'(x_n) - f'(y_n)\big)}{2f(x_n)f'(y_n)}\right]. \end{aligned} \tag{5}$$

For the method (5), we have the following convergence result.

Theorem 1 Let λ be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I . If x_0 is sufficiently close to λ, then the method defined by (5) is of fifth-order and satisfies the error equation

$$\varepsilon_{n+1} = c_2^2(2c_2^2 - 1.5c_3)\varepsilon_n^5 + O(\varepsilon_n^6)$$

where $\varepsilon_n = \lambda - x_n, c_k = f^{(k)}(\lambda)/(k!f'(\lambda))$.

Proof.

Using Taylor expansion of $f(x_n)$ and $f'(x_n)$ about λ , we have

$$f(x_n) = f'(\lambda)[\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + O(\varepsilon_n^4)] , \tag{6}$$

$$f'(x_n) = f'(\lambda)[1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + O(\varepsilon_n^3)], \tag{7}$$

therefore

$$\begin{aligned} &f(x_n)/f'(x_n) \\ &= \varepsilon_n - c_2\varepsilon_n^2 + 2(c_2^2 - c_3)\varepsilon_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\varepsilon_n^4 \\ &\quad + (6c_3^2 + 10c_2c_4 + 8c_2^4 - 4c_5 - 20c_2^2c_3)\varepsilon_n^5 + (17c_3c_4 \\ &\quad + 13c_2c_5 + 52c_2^3c_3 - 28c_2^2c_4 - 5c_6 - 33c_2c_3^2 - 16c_2^5)\varepsilon_n^6 \\ &\quad + (16c_2c_6 - 6c_7 - 36c_2^2c_5 + 22c_3c_5 - 92c_2c_3c_4 + 70c_2^3c_4 \\ &\quad + 12c_4^2 + 126c_2^2c_3^2 - 18c_3^3 - 128c_2^4c_3 + 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8) \end{aligned} \tag{8}$$

so

$$\begin{aligned} d_n &= y_n - \lambda = \varepsilon_n - f(x_n)/f'(x_n) \\ &= c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (4c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 + (4c_5 + 20c_2^2c_3 \\ &\quad - 6c_3^2 - 10c_2c_4 - 8c_2^4)\varepsilon_n^5 + (28c_2^2c_4 + 5c_6 + 33c_2c_3^2 + 16c_2^5 \\ &\quad - 17c_3c_4 - 13c_2c_5 - 52c_2^3c_3)\varepsilon_n^6 + (6c_7 - 16c_2c_6 + 36c_2^2c_5 \\ &\quad - 22c_3c_5 + 92c_2c_3c_4 - 70c_2^3c_4 - 12c_4^2 - 126c_2^2c_3^2 + 18c_3^3 \\ &\quad + 128c_2^4c_3 - 32c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8) \end{aligned} \tag{9}$$

Taylor expansions of $f(y_n)$ and $f'(y_n)$ around λ are given as

$$f(y_n) = f'(\lambda)[d_n + c_2d_n^2 + c_3d_n^3 + O(d_n^4)] \tag{10}$$

$$f'(y_n) = f'(\lambda)[1 + 2c_2d_n + 3c_3d_n^2 + O(d_n^3)], \tag{11}$$

so by (9) we obtain

$$\begin{aligned} & f(y_n) \\ &= f'(\lambda)[c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (5c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 \\ &\quad + (24c_2^2c_3 + 4c_5 - 12c_2^4 - 6c_3^2 - 10c_2c_4)\varepsilon_n^5 + (28c_2^5 \\ &\quad + 34c_2^2c_4 + 5c_6 + 37c_2c_3^2 - 73c_2^3c_3 - 17c_3c_4 - 13c_2c_5)\varepsilon_n^6 \\ &\quad + (6c_7 - 16c_2c_6 + 44c_2^2c_5 - 22c_3c_5 + 104c_2c_3c_4 - 102c_2^3c_4 \\ &\quad - 12c_4^2 - 160c_2^2c_3^2 + 18c_3^3 + 206c_2^4c_3 - 64c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8)] \end{aligned} \tag{12}$$

and

$$\begin{aligned} f'(y_n) = f'(\lambda)[1 &+ 2c_2^2\varepsilon_n^2 + 4c_2(c_3 - c_2^2)\varepsilon_n^3 + (8c_2^4 + 6c_2c_4 \\ &- 11c_2^2c_3)\varepsilon_n^4 + (8c_2c_5 - 20c_2^2c_4 + 28c_2^3c_3 \\ &- 16c_2^5)\varepsilon_n^5 + (60c_2^3c_4 + 10c_2c_6 + 32c_2^6 - 68c_2^4c_3 \\ &- 16c_2c_3c_4 + 12c_3^3 - 26c_2^2c_5)\varepsilon_n^6 + O(\varepsilon_n^7)] \end{aligned} . \tag{13}$$

Now,

$$\begin{aligned} & f(y_n)/f'(y_n) \\ &= d_n - c_2d_n^2 + 2(c_2^2 - c_3)d_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)d_n^4 + O(d_n^5) \end{aligned} \tag{14}$$

so, by (9) we get

$$\begin{aligned} & f(y_n)/f'(y_n) \\ &= c_2\varepsilon_n^2 + 2(c_3 - c_2^2)\varepsilon_n^3 + (3c_2^3 + 3c_4 - 7c_2c_3)\varepsilon_n^4 + (4c_5 + 16c_2^2c_3 \\ &\quad - 4c_2^4 - 6c_3^2 - 10c_2c_4)\varepsilon_n^5 + (6c_2^5 - 32c_2^3c_3 + 29c_2c_3^2 + 22c_2^2c_4 \\ &\quad + 5c_6 - 17c_3c_4 - 13c_2c_5)\varepsilon_n^6 + (6c_7 - 16c_2c_6 + 28c_2^2c_5 - 12c_3c_5 \\ &\quad + 80c_2c_3c_4 - 38c_2^3c_4 - 12c_4^2 - 98c_2^2c_3^2 + 18c_3^3 \\ &\quad + 68c_2^4c_3 - 12c_2^6)\varepsilon_n^7 + O(\varepsilon_n^8) \end{aligned} . \tag{15}$$

Using (7) and (13) we attain

$$\begin{aligned} &f'(x_n)-f'(y_n) \\ &=f'(\lambda)[2c_2\varepsilon_n+(3c_3-2c_2^2)\varepsilon_n^2+4(c_4-c_2c_3+c_2^3)\varepsilon_n^3 \\ &\quad+(5c_5-8c_2^4-6c_2c_4+11c_2^2c_3)\varepsilon_n^4+(6c_6-8c_2c_5 \\ &\quad+20c_2^2c_4-28c_2^3c_3+16c_2^5)\varepsilon_n^5+O(\varepsilon_n^6)] \end{aligned}, \qquad (16)$$

therefore

$$\begin{aligned} &f(y_n)\left(f'(x_n)-f'(y_n)\right) \\ &=[f'(\lambda)]^2[2c_2^2\varepsilon_n^3+(7c_2c_3-6c_2^3)\varepsilon_n^4+(10c_2c_4-28c_2^2c_3 \\ &\quad+18c_2^4+6c_3^2)\varepsilon_n^5+(13c_2c_5-50c_2^5-40c_2^2c_4+104c_2^3c_3 \\ &\quad+17c_3c_4-41c_2c_3^2)\varepsilon_n^6+O(\varepsilon_n^7)] \end{aligned}. \qquad (17)$$

By (6) and (13), we achieve

$$\begin{aligned} &f(x_n)f'(y_n) \\ &=[f'(\lambda)]^2[\varepsilon_n+c_2\varepsilon_n^2+(c_3+2c_2^2)\varepsilon_n^3 \\ &\quad+(c_4-2c_2^3+4c_2c_3)\varepsilon_n^4 \\ &\quad+(c_5-5c_2^2c_3+4c_2^4+6c_2c_4)\varepsilon_n^5+O(\varepsilon_n^6)] \end{aligned} \qquad (18)$$

Applying (17) and (18),

$$\begin{aligned} A&=\frac{f(y_n)\left(f'(x_n)-f'(y_n)\right)}{f(x_n)f'(y_n)} \\ &=2c_2^2\varepsilon_n^2+(7c_2c_3-8c_2^3)\varepsilon_n^3+(10c_2c_4-37c_2^2c_3 \\ &\quad+22c_2^4+6c_3^2)\varepsilon_n^4+(13c_2c_5-52c_2^5-52c_2^2c_4 \\ &\quad+127c_2^3c_3+17c_3c_4+28c_2c_3^2)\varepsilon_n^5+O(\varepsilon_n^6) \end{aligned}. \qquad (19)$$

Now, from (14) and (19) we obtain

$$\begin{aligned} &\frac{f(y_n)}{f'(y_n)}A \\ &=2c_2^3\varepsilon_n^4+(11c_2^2c_3-12c_2^4)\varepsilon_n^5+(16c_2^2c_4-78c_2^3c_3 \\ &\quad+44c_2^5+20c_2c_3^2)\varepsilon_n^6+(21c_2^2c_5-128c_2^6-116c_2^3c_4 \\ &\quad+354c_2^4c_3+58c_2c_3c_4-119c_2^2c_3^2+12c_3^3)\varepsilon_n^7+O(\varepsilon_n^8) \end{aligned}, \qquad (20)$$

and from (9) and (15) we get

$$\begin{aligned} &d_n-f(y_n)/f'(y_n) \\ &=c_2^3\varepsilon_n^4+4(c_2^2c_3-c_2^4)\varepsilon_n^5+2(3c_2^2c_4-10c_2^3c_3+5c_2^5 \\ &\quad+2c_2c_3^2)\varepsilon_n^6+4(2c_2^2c_5+3c_2c_3c_4-8c_2^3c_4-7c_2^2c_3^2 \\ &\quad+15c_2^4c_3-5c_2^6)\varepsilon_n^7+O(\varepsilon_n^8) \end{aligned} \qquad (21)$$

Finally, by using (9), (20), and (21), we obtain the error relation as

$$\begin{aligned}\varepsilon_{n+1} &= x_{n+1} - \lambda \\ &= d_n - \frac{f(y_n)}{f'(y_n)} - \frac{f(y_n)}{f'(y_n)}\frac{A}{2} \\ &= (2c_2^4 - 1.5c_2^2c_3)\varepsilon_n^5 + (19c_2^3c_3 - 2c_2^4c_4 - 12c_2^5 - 6c_2c_3^2)\varepsilon_n^6 \\ &\quad + (44c_2^6 + 31.5c_2^2c_3^2 - 2.5c_2^2c_5 - 17c_2c_3c_4 - 26.5c_2^3c_4 \\ &\quad - 117c_2^4c_3 - 6c_3^3)\varepsilon_n^7 + O(\varepsilon_n^8) \\ &= c_2^2(2c_2^2 - 1.5c_3)\varepsilon_n^5 + O(\varepsilon_n^6)\end{aligned} \tag{22}$$

This means that the method defined by (5) is of the fifth-order.
Finally, we construct the eleventh-order iterative method as follows

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}\left[1 + \frac{f(y_n)\left(f'(x_n) - f'(y_n)\right)}{2f(x_n)f'(y_n)}\right] \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}\left[1 + \frac{f(z_n)\left(f'(x_n) - f'(y_n)\right)}{2f(x_n)f'(z_n)}\right]\end{aligned} \tag{23}$$

Theorem 2 Let λ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I . If x_0 is sufficiently close to λ , then the method defined by (23) is of eleventh-order and satisfies the error equation

$$\varepsilon_{n+1} = c_2^4(2c_2^2 - 1.5c_3)^3\varepsilon_n^{11} + O(\varepsilon_n^{12})$$

Proof.

By (23) and using (22)

$$\begin{aligned}\tilde{d}_n &= z_n - \lambda \\ &= (2c_2^4 - 1.5c_2^2c_3)\varepsilon_n^5 + (19c_2^3c_3 - 2c_2^4c_4 - 12c_2^5 - 6c_2c_3^2)\varepsilon_n^6 \\ &\quad + (44c_2^6 + 31.5c_2^2c_3^2 - 2.5c_2^2c_5 - 17c_2c_3c_4 - 26.5c_2^3c_4 \\ &\quad - 117c_2^4c_3 - 6c_3^3)\varepsilon_n^7 + O(\varepsilon_n^8)\end{aligned} . \tag{24}$$

Using Taylor expansion of $f(z_n)$ and $f'(z_n)$ about λ , we have

$$f(z_n) = f'(\lambda)[\tilde{d}_n + c_2\tilde{d}_n^2 + \cdots], \tag{25}$$

and

$$f'(z_n) = f'(\lambda)[1 + 2c_2\tilde{d}_n + 3c_3\tilde{d}_n^2 + \cdots] , \tag{26}$$

therefore

$$\begin{aligned} & f(z_n)/f'(z_n) \\ &= \tilde{d}_n - c_2\tilde{d}_n^2 + 2(c_2^2 - c_3)\tilde{d}_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)\tilde{d}_n^4 + O(\tilde{d}_n^5)\end{aligned} . \tag{27}$$

Using (25), (16), (26) and (6) we obtain

$$\begin{aligned} &f(z_n)(f'(x_n)-f'(y_n)) \\ &=[f'(\lambda)]^2[2c_2\varepsilon_n\tilde{d}_n+(3c_3-2c_2^2)\varepsilon_n^2\tilde{d}_n \\ &\quad +4(c_4-c_2c_3+c_2^3)\varepsilon_n^3\tilde{d}_n+\cdots] \end{aligned} \tag{28}$$

and

$$\begin{aligned} &f'(z_n)f(x_n) \\ &=[f'(\lambda)]^2[\varepsilon_n+c_2\varepsilon_n^2+c_3\varepsilon_n^3+c_4\varepsilon_n^4+c_5\varepsilon_n^5, \\ &\quad c_6\varepsilon_n^6+2c_2\varepsilon_n\tilde{d}_n+\cdots] \end{aligned} \tag{29}$$

therefore

$$\begin{aligned} &\frac{f(z_n)\big(f'(x_n)-f'(y_n)\big)}{2f(x_n)f'(z_n)} \\ &=c_2\tilde{d}_n+(1.5c_3-2c_2^2)\varepsilon_n\tilde{d}_n+(2c_4-4.5c_2c_3+4c_2^3)\varepsilon_n^2\tilde{d}_n+\cdots \end{aligned} \tag{30}$$

Finally, using (24), (27) and (30) we get the error relation:

$$\begin{aligned} &\varepsilon_{n+1} \\ &=\tilde{d}_n-[\tilde{d}_n-c_2\tilde{d}_n^2+\cdots][1+c_2\tilde{d}_n+(1.5c_3-2c_2^2)\varepsilon_n\tilde{d}_n \\ &\qquad +(2c_4-4.5c_2c_3+4c_2^3)\varepsilon_n^2\tilde{d}_n+\cdots] \\ &=(2c_2^2-1.5c_3)\varepsilon_n\tilde{d}_n^2+(4.5c_2c_3-2c_4-4c_2^3)\varepsilon_n^2\tilde{d}_n^2+\cdots \\ &=c_2^4(2c_2^2-1.5c_3)^3\varepsilon_n^{11}+O(\varepsilon_n^{12}) \end{aligned}$$

This means that the method defined by (23) is of the eleventh-order.

The method (5), requires 4 function evaluations per iteration, 2 of f and 2 of f', whereas the method (23), requires 6 function evaluations, 3 of f and 3 of f'. The methods (5) and (23) have the efficiency indexes $5^{1/4}=1.4953$ and $11^{1/6}=1.4913$, respectively, which are better than the efficiency index $2^{1/2}=1.4142$ of the Newton's method (1) and the double Newton's method (2) and the efficiency index $10^{1/6}=1.4678$ of the tenth-order (TO) method [12].

3. Numerical Examples and Conclusion

In this section, we employ the new methods defined by (5) and (23) to solve some nonlinear equations and compare them with Classical Newton's method (CN) (1) , double Newton's method (DN) (2) and the tenth-order (TO) method [12].

We use the following functions, [6]:

$$f_1(x)=x^3+4x^2-10,\ \lambda=1.3652300134140968879137 3,$$

$$f_2(x)=x^5+x^4+4x^2-20,\ \lambda=1.4662790738647226707058 7,$$

$$f_3(x)=e^{x^2+7x-30}-1,\ \lambda=3,$$

$$f_4(x)=(sinx)^2-x^2+1,\ \lambda=1.4044916482153411152467 0,$$

$$f_5(x)=e^x sinx+\ln(x^2+1),\ \lambda=0,$$

$$f_6(x)=x^3-sin^2x+3cosx+5,\ \lambda=-1.5826870457520698654008 1,$$

$$f_7(x)=x^3-e^{-x},\ \lambda=0.7728829591492101247496 29.$$

All numerical examples were performed in MatlabR2017b, using 200 digits floating point (digits: = 200), and variable precision arithmetic. We have computed the root of each test function for two different initial guesses x_0 for 7 real functions, listed above, while the iterative schemes were stopped when $|x_{n+1} - x_n| + |f(x_n)| < 10^{-200}$.

Table 1: The sequence of the approximation zeros of the function f_3 using Method (23) starting with $x_0 = 3.5$ under the stopping criterium $|x_{n+1} - x_n| + |f(x_n)| < 10^{-200}$

n	x_n
2	3.0000000000224591066252484811852536011949828439782793102939 2 6305524772982590027144516065588926636400611309778469349585291 5524513876430378167616940295342698380052510985697379029153578 129681631722907493
3	3.000000000000000000000000000000000000000000000000000000000000 00000000000000000000000000000000000000000000000010729445438011 74552422450923302147574650562879647923146061555972878557019644 172289755575487
4	3.0000000000000000000000000000000000000000000000000000000000000 000000000000000000000000000000000000000000000000000000000000000 000000000000000000000000000000000000000000000000000000000000000 0000000000000
5	3.0000000000000000000000000000000000000000000000000000000000000 000000000000000000000000000000000000000000000000000000000000000 000000000000000000000000000000000000000000000000000000000000000 0000000000000

Table 2: Numerical results for different methods with stopping criterium $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

		IT				
$f(x)$	x_0	CN (1)	DN (2)	TO [12]	Eq.(5)	Eq.(23)
f_1	1.9	9	5	3	4	3
	1	10	5	3	4	3
f_2	1.2	10	5	3	4	3
	2	11	6	3	4	3
f_3	3.5	17	9	5	7	5
	4	24	12	8	10	7
f_4	1.6	10	5	3	4	3
	2.5	11	6	3	5	3
f_5	3	11	5	4	4	3
	4.2	11	6	4	5	3
f_6	-1	10	5	3	4	3
	-3	11	6	3	4	3
f_7	0	11	6	3	4	3
	1.5	11	6	3	5	3

Table 3: Numerical results for different methods with stopping criterium
$|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$

$f(x)$	x_0	CN (1) (IT, ρ)	DN (2) (IT, ρ)	TO [12] (IT, ρ)	Eq.(5) (IT, ρ)	Eq.(23) (IT, ρ)
f_1	1.9	(9, 2.0037)	(5, 4.0221)	(3, 10.2305)	(4, 5.0674)	(3,11.3770)
	1	(10, 2.0018)	(5, 4.0213)	(3, 10.3010)	(4, 5.0676)	(3, 11.3879)
f_2	1.2	(10, 1.9999)	(5, 3.9992)	(3, 9.4298)	(4, 4.9916)	(3, 10.9442)
	2	(11, 2.0000)	(6, 3.9997)	(3, 9.5343)	(4, 4.9887)	(3, 10.9294)
f_3	3.5	(17,1.9957)	(9, 3.9742)	(5, 7.8713)	(7, 4.8251)	(5, 10.2332)
	4	(24, 1.9947)	(12, 3.9379)	(8, 9.3547)	(10, 4.7824)	(7, 10.2305)
f_4	1.6	(10, 1.8269)	(5, 4.0057)	(3, 9.9212)	(4, 5.0085)	(3, 11.0344)
	2.5	(11, 2.0006)	(6, 4.0033)	(3, 9.7953)	(5, 5.0043)	(3, 11.0908)
f_5	3	(10, 2.0002)	(5, 4.0024)	(3, 9.6135)	(4, 4.9953)	(3, 10.9560)
	4.2	(11, 2.0002)	(6, 4.0012)	(4, 9.9558)	(5, 4.9981)	(3, 10.9163)
f_6	-1	(10, 2.0029)	(5, 4.0353)	(3, 10.8454)	(4, 5.1260)	(3, 11.7674)
	-3	(11, 2.0022)	(6, 4.0134)	(3, 10.7783)	(4, 5.1786)	(3, 12.0135)
f_7	0	(11, 2.0002)	(6, 4.0010)	(3, 9.6018)	(4, 5.0050)	(3, 11.0227)
	1.5	(11, 2.0002)	(6, 4.0012)	(3, 9.6159)	(5, 5.0017)	(3, 11.0391)

Table 1 shows an example of the sequence of the approximation zeros of the function f_3 using the method (23) starting with $x_0 = 3.5$ under the stopping criterion $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$.
Displayed in Table 2 are the number of iterations (IT) required such that $|x_{n+1}-x_n|+|f(x_n)|<10^{-200}$. Table 3 shows the computational order ρ for all considered examples.
The computational results presented in Table 2 show that, the presented methods, (5) and (23) converge more rapidly than Classical Newton's method (1) and double Newton's method (2) and they require less number of iterations. Therefore, the new methods (5) and (23) have better convergence efficiency.
Table 3 shows the computational orders of 5 methods, CN (1), DN (2), TO[12], (5), and (23). It can be seen from the numerical results displayed in Tables 3 that the numerical results of the proposed methods support the theoretical results proved in Section 2.
Finally, we conclude that the new iterative methods (5) and (23), presented in this paper, can compete with other efficient equation solvers, such as the Classical Newton's method (1), and the double Newton's method (2), and the tenth-order method [12]. The results reflect the efficiency indexes 1.4953, 1.4913, 1.4142, 1.4142 and 1.4678 of the methods (5), (23), (1), (2), and TO [12], respectively.

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طريقتان تكراريتان من الرتبة الخامسة والرتبة الحادية عشرة لإيجاد جذور المعادلات غير الخطية

حسن محمد سعيد باوزير

ملخص

في هذا العمل ، تم بناء طريقتين تكراريتين ، بناءً على طريقة نيوتن ، للحصول على الحلول العددية للمعادلات غير الخطية. أثبتنا أن الطريقتين تتقارب من الرتبة الخامسة والرتبة الحادية عشرة. تم إجراء تحقيق تحليلي لتوضيح أن الطريقتين بها مؤشرات كفاية أعلى من بعض الطرائق الحديثة. تم تجربة أمثلة عددية للتحقق من أداء الطريقتين المقترحتين. علاوة على ذلك ، يتم التحقق من الرتب التقاربية في العمل التجريبي.

كلمات مفتاحية: معادلة غير خطية، طريقة تكرارية، طريقة نيوتن، رتبة التقارب.