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Characterizations of Hemi-Rings by their Bipolar-Valued Fuzzy h-Ideals

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Abstract: In this paper, we employed bipolar fuzzy set theory to hemi-rings and we popularized notation of bipolar valued fuzzy interior h-ideal, bipolar-valued fuzzy prime h-ideal, bipolar-valued fuzzy semiprime h-ideal, bipolar-valued fuzzy h-bi-ideal and bipolar-valued fuzzy h-quasi-ideal. We also introduced some basic properties and definition of bipolar-valued fuzzy h-ideals of hemi-ring. Then we introduced results of biplar-valued fuzzy h-ideal of h-hemi-regular and h-hemi-simple hemi-rings.

Keywords: Hemi-rings, bipolar valued fuzzy h-ideal, h-hemi-regular hemi-ring, h-hemi-simple hemi-rings

1 Introduction

Afterwards, Zadeh [12], popularized fuzzy set, there have been several generalizations of this essential concept. Fuzzy sets are extremely useful to deal many problems in applied mathematics, control engineering, information sciences, expert systems and theory of automata etc. Although there are many generalizations of fuzzy sets but non of these deal with the problems related to the contrary characteristics of the members having membership degree 0. Lee [7], handled this problem by introducing the concept of Bipolar-valued fuzzy (BVF) sets . The BVF sets aggregate a proper information conception structure to solving daily life problems. The sweet taste of foodstuffs is a bipolar valued fuzzy set. Assuming that sweet taste of foodstuff as a positive membership value then bitter taste of foodstuffs as a negative membership value. The remaining foodstuffs of taste like acidic, saline, chilly etc. are extraneous to the sweet and bitter foodstuffs. Thus these foodstuffs are accepted as zero membership values. There are two types of appearance in bipolar valued fuzzy sets so called approved display and diminished display.

With the broad concern, semirings presented by Vandiver [9], have been explored by many researchers [1, 2,3]. Ideals of hemi-rings, as a class of particular hemi-ring, play an essential role in the algebraic structure theories anyhow many properties of hemi-rings are described by ideals. On the other hand, generally ideals in hemi-rings do not correspond with the ideals in rings. Subsequently, Henriksen [4], defined k-ideals of hemi-rings. Iizuka [5], presented another more restricted ideals of hemi-rings called h-ideals. According to new concept of ideals, La Torre [6], analyzed exhaustively h-ideals and k-ideals of hemi-rings. In 2014 M. zhou et. al. [8], contemplated the applications of bipolar fuzzy theory to hemi-rings. In this paper, we introduced some basic definition, theorem and examples about bipolar-valued fuzzy h-ideals and we Characterized properties of h-hemi-regular and h-hemi-simple hemi-ring by using bipolar-valued fuzzy h-ideals, bipolar-valued h-bi-ideals and bipolar-valued h-quasi-ideals.

2 Preliminaries

In this section, for basic definitions of hemi-ring and basic concepts of BVF sets, we reffer to [3], and [7], respectively.

Let R be a universe. Expresses $\mho^{+} = \{\lambda^{+} \mid \lambda^{+} : R \longrightarrow [0,1]\}$, and $\mho^{-} = \{\lambda^{-} \mid \lambda^{-} : R \longrightarrow [-1,0]\}$. We symbolize $B = \{z, (\lambda^{+}(z), \lambda^{-}(z))\}$ a BVF set in R, where $\lambda^{+}(z) \in \mho^{+}$ denotes the satisfaction degree of $z \in R$ about some property, generally it is known as a positive membership degree and $\lambda^{-}(z) \in \mho^{-}$ denotes the satisfaction degree of $z \in R$ about some implicit counter-property, generally it is known as a negative membership degree. For the sake of

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simplicity, we shall use the symbol $B=(\lambda^+,\lambda^-)$ for BVF set $B=\{z,(\lambda^+(z),\lambda^-(z))\}$.

2.1 Definition

Let R be a universe and $M\subseteq R$. Then BVF characteristic function is given by $C_M=\left(C_M^+,C_M^-\right)$, where

$$C_M^+(z)=\begin{cases}1 & \text{if } z\in M\\ 0 & \text{if } z\notin M\end{cases},$$

$$C_M^-(z)=\begin{cases}-1 & \text{if } z\in M\\ 0 & \text{if } z\notin M\end{cases}.$$

2.2 Definition

Let R be a universe and $t\in(0,1]$. Then a BVF set $B=(\lambda^+,\lambda^-)$ in R of the form

$$\lambda^+(z)=\begin{cases}t^+ & if\ \ z=x\\ 0 & if\ \ z\neq x\end{cases},$$

$$\lambda^-(z)=\begin{cases}t^- & if\ \ z=x\\ 0 & if\ \ z\neq x\end{cases},$$

is called BVF point with value $t'=(t^+,t^-)\in(0,1]\times[-1,0)$ and support x. It is express as $x_{t'}=(x_t^+,x_{-t}^-)$. A BVF point $x_{t'}$ is said to belong to BVF subset B, written as $x_{t'}\in B$ if $B(x)\geq t'$ *i.e.*, $\lambda^+(x)\geq t^+$ and $\lambda^-(x)\leq t^-$.

Moreover, throught this paper R is hem-ring unless else particularized.

3 Main Results

In this section, we introduced some basic definitions of BVF h-ideals and some theorem regarding BVF h-ideals.

3.1 Definition

A BVF set $B=(\lambda^+,\lambda^-)$ is called BVF h-subhemi-ring of R if it holds

(1) $x_{t'}\in B,\, y_{r'}\in B\Longrightarrow(x+y)_{\min(t',r')}\in B$,

(2) $x_{t'}\in B,\, y_{r'}\in B\Longrightarrow(xy)_{\min(t',r')}\in B$,

(3) $x+a_1+z=a_2+z$, $(a_1)_{t'}\in B,\ (a_2)_{r'}\in B\Longrightarrow(x)_{\min(t',r')}\in B$, $\forall\, x,y,z,a_1,a_2\in R$ & $t'=(t^+,t^-)$, $r'=(r^+,r^-)\in(0,1]\times[-1,0)$.

3.2 Definition

A BVF set $B=(\lambda^+,\lambda^-)$ is called BVF left (resp. right) h-ideal of R if it holds (1), (3) and

(4) $x_{t'}\in B\Longrightarrow(yx)_{t'}\in B$ (resp. (5) $(xy)_{t'}\in B$), $\forall\ x, y\in R$ & $t'=(t^+,t^-)\in(0,1]\times[-1,0)$.

B is called a BVF h-ideal if it is both left and right BVF h-ideal of R.

3.3 Definition

A BVF set $B=(\lambda^+,\lambda^-)$ is called BVF interior h-ideal of R if it holds (1), (2), (3) and

(6) $y_{t'}\in B\Longrightarrow(xyz)_{t'}\in B, \forall x,y,z\in R$ & $t'=(t^+,t^-)\in(0,1]\times[-1,0)$.

3.4 Definition

A BVF set $B=(\lambda^+,\lambda^-)$ is called BVF h-bi-ideal of R if it holds (1), (2), (3) and

(7) $x_{t'}\in B,\, y_{r'}\in B\Longrightarrow(xzy)_{\min(t',r')}\in B, \forall x,\, y,\, z\in R$ & $t',r'\in(0,1]\times[-1,0)$.

3.5 Definition

Let $B=(\lambda^+,\lambda^-)$ be a BVF set of a commutative hemi-ring R with unity. Then B is called BVF prime h-ideal of R if it holds (1), (3), (4), (5) and

(8) $(xy)_{t'}\in B\Longrightarrow(x)_{t'}\in B$, or $y_{t'}\in B$, $\forall x,\, y\in R$ & $t'=(t^+,t^-)\in(0,1]\times[-1,0)$.

3.6 Definition

Let $B=(\lambda^+,\lambda^-)$ be a BVF subset of a commutative hemi-ring R with unity. Then B is called BVF semi-prime h-ideal of R if it holds $(1),(3),(4),(5)$ and

(9) $(x^2)_{t'}\in B\Longrightarrow x_{t'}\in B, \forall x\in R$ & $t'\in(0,1]\times[-1,0)$.

3.7 Remark

In rest of the paper, we denote set of BVF left h-ideals of R, BVF right h-ideals of R, BVF h-ideals of R, BVF interior h-ideals of R, BVF h-bi-ideals of R, BVF prime h-ideals of R and BVF semi-prime h-ideals of R by $BVFLhI(R)$, $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$, $BVFPhI(R)$ and $BVFShI(R)$ respectively.

3.8 Theorem

The conditions (1) to (9) are equivelent to $(1)'$ to $(9)'$ respectively, $\forall x_1, x_2, y, r_1, r_2$ where:

$(1)'$ $\lambda^+(x_1+x_2) \geq \min\{\lambda^+(x_1), \lambda^+(x_2)\}$,
$\lambda^-(x_1+x_2) \leq \max\{\lambda^-(x_1), \lambda^-(x_2)\}$,

$(2)'$ $\lambda^+(x_1x_2) \geq \min\{\lambda^+(x_1), \lambda^+(x_2)\}$,
$\lambda^-(x_1x_2) \leq \max\{\lambda^-(x_1), \lambda^-(x_2)\}$,

$(3)'$ $x_1+r_1+y = r_2+y$

$\implies \lambda^+(x_1) \geq \min\{\lambda^+(r_1), \lambda^+(r_2)\}$,
$\lambda^-(x_1) \leq \max\{\lambda^-(r_1), \lambda^-(r_2)\}$,

$(4)'$ $\lambda^+(x_1x_2) \geq \lambda^+(x_2)$,
$\lambda^-(x_1x_2) \leq \lambda^-(x_2)$,

$(5)'$ $\lambda^+(x_1x_2) \geq \lambda^+(x_1)$,
$\lambda^-(x_1x_2) \leq \lambda^-(x_1)$,

$(6)'$ $\lambda^+(x_1yx_2) \geq \lambda^+(y)$,
$\lambda^-(x_1yx_2) \leq \lambda^-(y)$,

$(7)'$ $\lambda^+(x_1zx_2) \geq \min\{\lambda^+(x_1), \lambda^+(x_2)\}$,
$\lambda^-(x_1zx_2) \leq \max\{\lambda^-(x_1), \lambda^-(x_2)\}$,

$(8)'$ $\lambda^+(x_1x_2) \geq \max\{\lambda^+(x_1), \lambda^+(x_2)\}$,
$\lambda^-(x_1x_2) \leq \min\{\lambda^-(x_1), \lambda^-(x_2)\}$,

$(9)'$ $\lambda^+(x_1^2) \geq \lambda^+(x_1)$,
$\lambda^-(x_1^2) \leq \lambda^-(x_1)$.

Proof. Straightforward.

3.9 Theorem

A BVF set $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$, $BVFPhI(R)$, $BVFShI(R)$) iff it holds following sets of conditions $\{(1)', (3)', (4)'\}$(resp. $\{(1)', (3)', (5)'\}$, $\{(1)', (3)', (4)', (5)'\}$, $\{(1)', (2)', (3)', (6)'\}$, $\{(1)', (2)', (3)', (7)'\}$, $\{(1)', (3)', (4)', (5)', (8)'\}$ and $\{(1)', (3)', (4)', (5)', (9)'\}$).

3.10 Example

Consider $R = \{0, 1, p, p^\star\}$ defined by

$+$	0	1	p	$p^\star$
0	0	1	p	$p^\star$
1	1	1	p	$p^\star$
p	p	p	p	$p^\star$
$p^\star$	$p^\star$	$p^\star$	$p^\star$	$p^\star$

,

.	0	1	p	$p^\star$
0	0	0	0	0
1	0	1	1	1
p	0	1	1	1
$p^\star$	0	1	1	1

.

We define BVF set B as follows

	0	1	p	$p^\star$
μ^+	0.52	0.52	0.32	0.32
μ^-	-0.73	-0.73	-0.23	-0.23

.

Clearly, $B \in BVFhI(R)$.

3.11 Example

Consider hemi-ring $\mathbb{N}_0$ with respect to the usual "+" and ".". Let $t_1', t_2' \in [0,1)$ be such that $t_1' \leq t_2'$ i.e, $(t_1, -t_1) \leq (t_2, -t_2)$. Define BVF set $B = (\lambda^+, \lambda^-)$ by

$$\lambda^+(x) = \begin{cases} t_1 & \text{if } x \in \langle 3 \rangle \\ t_2 & \text{if } x \notin \langle 3 \rangle \end{cases},$$

and

$$\lambda^-(x) = \begin{cases} -t_1 & \text{if } x \in \langle 3 \rangle \\ -t_2 & \text{if } x \notin \langle 3 \rangle \end{cases},$$

$\forall x \in \mathbb{N}_0$. Where $\langle 3 \rangle$ is set of generators of 3. Then $B = (\lambda^+, \lambda^-) \in BVFhbI(\mathbb{N}_0)$.

3.12 Remark

If $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ $(BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$, $BVFPhI(R)$, $BVFShI(R))$ then $\lambda^+(0) \geq \lambda^+(x)$ and $\lambda^-(0) \leq \lambda^-(x)$, $\forall x \in R$.

3.13 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_I \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, $BVFIhI(R)$, $BVFhbI(R)$) iff I is a left h-ideal (resp. right h-ideal, h-ideal, interior h-ideal, h-bi-ideal) of R.

Proof. Straightforward.

3.14 Theorem

Let $\emptyset \neq I \subseteq R$, where R is a commutative hemi-ring with unity. Then $C_I = \left(C_I^+, C_I^-\right) \in BVFPhI(R)$ (resp. $BVFShI(R)$) iff I is a prime h-ideal(resp. semi-prime h-ideals) of R respectively.

Proof. Straightforward.

3.15 Theorem

Every BVF h-ideal is a BVF interior h-ideal of R.

3.16 Remark

Generally, converse of Theorem 3.15, is not true.

3.17 Example

Consider $R = \{0, p, q, r\}$ defined by the following operations

$+$	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	q	p	0

,

$.$	0	p	q	r
0	0	0	0	0
p	0	q	0	q
q	0	0	0	0
r	0	q	0	q

.

Define B as follows

	0	p	q	r
μ^+	0.41	0.42	0.11	0.10
μ^-	-0.72	-0.71	-0.31	-0.33

.

Then $B = (\mu^+, \mu^-) \in BVFIhI(R)$ but $B = (\mu^+, \mu^-) \notin BVFhI(R)$.

As $B(pqr) = B(0) = (0.4, -0.7)$ and $B(q) = (0.1, -0.3)$, this shows $\mu^+(pqr) > \mu^+(q)$ and $\mu^-(pqr) < \mu^-(q)$. On the other hand $B(pp) = (0.1, -0.3)$ and $B(p) = (0.4, -0.7)$, this shows $\mu^+(pp) \ngeq \mu^+(p)$ and $\mu^-(pp) \nleq \mu^-(p)$.

3.18 Theorem

If $B = (\mu^+, \mu^-) \in BVFPhI(R)$ then $B = (\mu^+, \mu^-) \in BVFShI(R)$.

Proof. Suppose $B = (\lambda^+, \lambda^-) \in BVFPhI(R)$. Then by definition $\lambda^+(xy) \leq \max\{\lambda^+(x), \lambda^+(y)\}$ and $\lambda^-(xy) \geq \min\{\lambda^-(x), \lambda^-(y)\}$. For $y = x$ $\lambda^+(x) \geq \lambda^+(x^2)$ and $\lambda^-(x) \leq \lambda^-(x^2)$. Hence $B \in BVFShI(R)$. This complete the proof.

3.19 Remark

Generally, converse of Theorem 3.18, is not true.

3.20 Example

Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and p_1, p_2, p_3, ... be the distinct prime numbers in $\mathbb{N}_0$. If $K^0 = \mathbb{N}_0$ and $K^l = p_1 p_2 p_3 ... p_l \mathbb{N}_0$, where $l = 1, 2, 3, ...$ then $K^0 \supset K^1 \supset K^2 \supset .... K^n \supset K^{n+1} \supset ......$ As every non-zero elememt of $\mathbb{N}_0$ has unique prime factorization, for $l = 2, 3, ...$ K^l is a semi-prime h-ideal but not a prime h-ideal. Then by (3.14), $C_{K^l} = \left(C^+_{K^l}, C^-_{K^l}\right) \in BVFShI(R)$, but $C_{K^l} = \left(C^+_{K^l}, C^-_{K^l}\right) \notin BVFPhI(R)$.

3.21 Theorem

Let $B_i = \{(\lambda_i^+, \lambda_i^-) : i \in \Omega\}$ is a family BVF subsets of R and $B_i \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFhI(R)$, and $BVFIhI(R)$). Then $B = \underset{i\in\Omega}{\wedge} B_i \in BVFLhI(R)$ (resp. $BVFRhI(R)$, $BVFIhI(R)$, and $BVFIhI(R)$), where $B = (\lambda^+, \lambda^-)$ with $\lambda^+ = \underset{i\in\Omega}{\wedge} \lambda_i^+$ and $\lambda^- = \underset{i\in\Omega}{\vee} \lambda_i^-$ ($\lambda^+ \leq \lambda_i^+$, $\lambda^- \geq \lambda_i^-$, $\forall\, i \in \Omega$).

3.22 Definition

Let $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ be two BVF subset of R. The h-intrinsic product of $B_1 = (\lambda^+, \lambda^-)$ and $B_2 = (\mu^+, \mu^-)$ is denoted and described as $(B_1 \odot_h B_2)(x) = ((\lambda^+ \odot_h \mu^+)(x) \;, (\lambda^- \odot_h \mu^-)(x))$, $\forall x \in R$, if x can be signified as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z$, so that

$$(\lambda^+ \odot_h \mu^+)(x) = \bigvee_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}} \lambda^+(a_i)) \wedge (\overset{m}{\underset{i=1}{\wedge}} \mu^+(b_i)) \\ \wedge (\overset{n}{\underset{j=1}{\wedge}} \lambda^+(c_j)) \wedge (\overset{n}{\underset{j=1}{\wedge}} \mu^+(d_j)) \end{array} \right\},$$

$$(\lambda^- \odot_h \mu^-)(x) = \bigwedge_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}} \lambda^-(a_i)) \vee (\overset{m}{\underset{i=1}{\vee}} \mu^-(b_i)) \vee \\ (\overset{n}{\underset{j=1}{\vee}} \lambda^-(c_j)) \vee (\overset{n}{\underset{j=1}{\vee}} \mu^-(d_j)). \end{array} \right\}.$$

And $(B_1 \odot_h B_2)(x) = (0, 0)$ $\forall x \in R$, if x cannot be signified as $x + \Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z$.

3.23 Theorem

Let M_1, M_2 be h-ideals of R. Then we have

(i) $M_1 \subseteq M_2$, iff $C^+_{M_1} \leq C^+_{M_2}$, $C^-_{M_1} \geq C^-_{M_2}$,

(ii) $C^+_{M_1} \wedge C^+_{M_2} = C^+_{M_1 \cap M_2}$, $C^-_{M_1} \vee C^-_{M_2} = C^-_{M_1 \cup M_2}$,

(iii) $C^+_{M_1} \odot_h C^+_{M_2} = C^+_{\overline{M_1 M_2}}$, $C^-_{M_1} \odot_h C^-_{M_2} = C^-_{\overline{M_1 M_2}}$.

Proof. (i), (ii) Straightforwad.

(iii) Let $C_{\overline{M_1 M_2}} = \left(C^+_{\overline{M_1 M_2}}, C^-_{\overline{M_1 M_2}}\right)$. Suppose $x \in R$ and $x \in \overline{M_1 M_2}$ so $C^+_{\overline{M_1 M_2}} = 1$, $C^-_{\overline{M_1 M_2}} = -1$. Now, let $x + \Sigma_{i=1}^m p_i q_i + z = \Sigma_{j=1}^n r_j s_j + z$ for some p, $r \in A_1$ and q, $s \in M_2$. Then,

$$(C^+_{M_1} \odot_h C^+_{M_2})(x) = \bigvee_{x+\Sigma_{i=1}^m a_i b_i + z = \Sigma_{j=1}^n c_j d_j + z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}} C^+_{M_1}(a_i)) \wedge (\overset{m}{\underset{i=1}{\wedge}} C^+_{M_2}(b_i)) \\ \wedge (\overset{n}{\underset{j=1}{\wedge}} C^+_{M_1}(c_j)) \wedge (\overset{n}{\underset{j=1}{\wedge}} C^+_{M_2}(d_j)) \end{array} \right\}$$

$$\geq \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}} C^+_{M_1}(p_i)) \wedge (\overset{m}{\underset{i=1}{\wedge}} C^+_{M_2}(q_i)) \\ \wedge (\overset{n}{\underset{j=1}{\wedge}} C^+_{M_1}(r_j)) \wedge (\overset{n}{\underset{j=1}{\wedge}} C^+_{M_2}(s_j)) \end{array} \right\}$$

$= 1.$

$(C^-_{M_1} \odot_h C^-_{M_2})(x) =$

$$\bigwedge_{x+\Sigma_{i=1}^m a_ib_i+z=\Sigma_{j=1}^n c_jd_j+z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}} C^-_{M_1}(a_i)) \vee (\overset{m}{\underset{i=1}{\vee}} C^-_{M_2}(b_i)) \\ \vee(\overset{n}{\underset{j=1}{\vee}} C^-_{M_1}(c_j)) \vee (\overset{n}{\underset{j=1}{\vee}} C^+_{M_2}(d_j)) \end{array} \right\}$$

$$\leq \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}} C^-_{M_1}(p_i)) \vee (\overset{m}{\underset{i=1}{\vee}} C^-_{M_2}(q_i)) \\ \vee(\overset{n}{\underset{j=1}{\vee}} C^-_{M_1}(r_j)) \vee (\overset{n}{\underset{j=1}{\vee}} C^-_{M_2}(s_j)) \end{array} \right\}$$

$= -1.$

Hence (iii) is proved.

3.24 Theorem

A BVF subset $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$ (resp. $BVFRhI(R)$) iff it holds $(1)'$, $(3)'$ and $(C^+_R \odot_h \lambda^+)(x) \leq \lambda^+(x)$, $(C^-_R \odot_h \lambda^-)(x) \geq \lambda^-(x)$ (resp., $(\lambda^+ \odot_h C^+_R)(x) \leq \lambda^+(x)$, $(\lambda^- \odot_h C^-_R)(x) \geq \lambda^-(x)$).

3.25 Lemma

Let $B_1 = (\lambda^+, \lambda^-) \in BVFRhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVFLhI(R)$. Then $\lambda^+ \odot_h \mu^+ \leq \lambda^+ \wedge \mu^+$ and $\lambda^- \odot_h \mu^- \geq \lambda^- \vee \mu^-$.

3.26 Definition

A BVF set $B = (\lambda^+, \lambda^-)$ is a called BVF h-quasi-ideal of R iff it holds for $t, r, l, r_1, r_2 \in R$,

$(1)'$ $\lambda^+(t+r) \geq \min\{\lambda^+(t), \lambda^+(r)\}$,
$\lambda^-(t+r) \leq \max\{\lambda^-(t), \lambda^-(r)\}$.

$(3)'$ $t + r_1 + l = r_2 + l \Longrightarrow \lambda^+(t) \geq \min\{\lambda^+(r_1), \lambda^+(r_2)\}$,
$\lambda^-(t) \leq \max\{\lambda^-(r_1), \lambda^-(r_2)\}$.

$(10)'$ $(\lambda^+ \odot_h C^+_R) \cap (C^+_R \odot_h \lambda^+) \leq \lambda^+$, $(\lambda^- \odot_h C^-_R) \cup (C^-_R \odot_h \lambda^-) \geq \lambda^-$.

3.27 Remark

In rest of paper, set of BVF h-quasi-ideal of R is denoted by $BVFhqI(R)$.

3.28 Example

In Example 3.11, $B = (\lambda^+, \lambda^-) \in BVFhqI(\mathbb{N}_0)$.

3.29 Theorem

A BVF set $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$ iff all level subsets $U(B, t') \neq \emptyset$ are h-quasi-ideal of R.

3.30 Theorem

Let $\emptyset \neq I \subseteq R$. Then $C_I \in BVFhqI(R)$ iff I is a h-quasi-ideal of R.

3.31 Lemma

Let $B = (\lambda^+, \lambda^-) \in BVFRhI(R)$ and $B' = (\mu^+, \mu^-) \in BVFLhI(R)$. Then $B \cap B' \in BVFhqI(R)$.

Proof. Straightforward.

3.32 Lemma

If $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$ then $B = (\lambda^+, \lambda^-) \in BVFhbI(R)$.

Proof. Let $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$. It is sufficient to prove $\lambda^+(xyz) \geq \min\{\lambda^+(x), \lambda^+(z)\}$,
$\lambda^-(xyz) \leq \max\{\lambda^-(x), \lambda^-(z)\}$
and
$\lambda^+(xy) \geq \min\{\lambda^+(x), \lambda^+(y)\}$,
$\lambda^-(xy) \leq \max\{\lambda^-(x), \lambda^-(y)\}$ $\forall x, y, z \in R$. Now, we have

$$\lambda^+(xyz) \geq ((\lambda^+ \odot_h C^+_R) \cap (C^+_R \odot_h \lambda^+))(xyz)$$
$$= \min\{(\lambda^+ \odot_h C^+_R)(xyz), (C^+_R \odot_h \lambda^+)(xyz)\}$$
$$= \min \left\{ \begin{array}{c} \bigvee_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ (\overset{m}{\underset{i=1}{\wedge}} (\lambda^+)(a_i)) \wedge (\overset{n}{\underset{j=1}{\wedge}} (\lambda^+)(c_j)) \right\}, \\ \bigvee_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ (\overset{m}{\underset{i=1}{\wedge}} (\lambda^+)(b_i)) \wedge (\overset{n}{\underset{j=1}{\wedge}} (\lambda^+)(d_j)) \right\} \end{array} \right\}$$
$$\geq \min\{\min\{\lambda^+(0), \lambda^+(x)\}, \min\{\lambda^+(0), \lambda^+(z)\}\}$$
$$= \min\{\lambda^+(x), \lambda^+(z)\}.$$

Analogously, we have

$$\lambda^-(xyz) \leq ((\lambda^- \odot_h C^-_R) \cap (C^-_R \odot_h \lambda^-))(xyz)$$
$$= \max\{(\lambda^- \odot_h C^-_R)(xyz), (C^-_R \odot_h \lambda^-)(xyz)\}$$
$$= \max \left\{ \begin{array}{c} \bigwedge_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ (\overset{m}{\underset{i=1}{\vee}} (\lambda^-)(a_i)) \vee (\overset{n}{\underset{j=1}{\vee}} (\lambda^-)(c_j)) \right\}, \\ \bigwedge_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ (\overset{m}{\underset{i=1}{\vee}} (\lambda^-)(b_i)) \vee (\overset{n}{\underset{j=1}{\vee}} (\lambda^-)(d_j)) \right\} \end{array} \right\}$$

$\leq \max\{\max\{\lambda^-(0),\lambda^-(x)\},\max\{\lambda^-(0),\lambda^-(z)\}\}$
$= \max\{\lambda^-(x),\lambda^-(z)\}$.

Similarly, we can prove $\lambda^+(xy) \geq \min\{\lambda^+(x),\lambda^+(y)\}$, $\lambda^-(xy) \leq \max\{\lambda^-(x),\lambda^-(y)\}$. Therefore $B \in BVFhbI(R)$.

4 Characterization of hemi-rings by their bipolar-valued fuzzy *h*-ideals

In this section, we applied concepts of Y. Q. Yin et.al [10, 11] on BVF *h*-ideals of *R*.

4.1 Theorem

Let $B_1 = (\lambda^+,\lambda^-)$, $B_2 = (\mu^+,\mu^-)$ be two BVF subsets of R. Then we say $\lambda^+ \sim \mu^+$, $\mu^- \sim \lambda^-$ iff $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ and $\mu^+[\in]\lambda^+$, $\lambda^-[\in]\mu^-$.

4.2 Lemma

The relation "$\backsim$" is called equivalance relation on BVF subsets of R.

4.3 Lemma

Hemi-ring R is *h*-hemi-regular iff $\overline{MRM} = M$ for every *h*-quasi-ideal M of R.

4.4 Theorem

Let $B_1 = (\lambda^+,\lambda^-)$, $B_2 = (\mu^+,\mu^-) \in BVFhI(R)$. Then $\lambda^+[\in]\mu^+$, $\mu^-[\in]\lambda^-$ iff $\lambda^+ \leq \mu^+$, $\mu^- \leq \lambda^-$, $\forall x \in R$.

4.5 Theorem

If $B_1 = (\lambda^+,\lambda^-) \in BVFRhI(R)$ and $B_2 = (\mu^+,\mu^-) \in BVFLhI(R)$. Then $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$ iff R is *h*-hemi-regular.

Proof. Suppose R is *h*-hemi-regular. Let $B_1 \in BVFRhI(R)$ and $B_2 \in BVFLhI(R)$, then $\forall s \in R$ by the Lemma 3.25, $(\lambda^+ \odot_h \mu^+)(s) \leq (\lambda^+ \wedge \mu^+)(s)$, $(\lambda^- \odot_h \mu^-)(s) \geq (\lambda^- \vee \mu^-)(s)$ and so by the Theorem 4.4, $(\lambda^+ \odot_h \mu^+)(s)[\in](\lambda^+ \wedge \mu^+)(s)$, $(\lambda^- \vee \mu^-)(s)[\in](\lambda^- \odot_h \mu^-)(s)$. Now, since R is *h*-hemi-regular, so $\forall s \in R$, $\exists$ p, q, $z \in R$ such that $s + sps + z = sqs + z$. Thus $(\lambda^+ \odot_h \mu^+)(s) =$

$$\bigvee_{s+\Sigma_{i=1}^{n} p_i q_i + z = \Sigma_{j=1}^{m} r_j t_j + z} \left[(\bigwedge_{i=1}^{n} \lambda^+(p_i)) \wedge (\bigwedge_{i=1}^{n} \mu^+(q_i)) \wedge (\bigwedge_{j=1}^{m} \lambda^+(r_j)) \wedge (\bigwedge_{j=1}^{m} \mu^+(t_j)) \right]$$

$\geq \min\{\lambda^+(sp),\lambda^+(sq),\mu^+(s)\} = (\lambda^+ \wedge \mu^+)(s)$
$(\lambda^+ \odot_h \mu^+)(s) \geq (\lambda^+ \wedge \mu^+)(s)$
and
$(\lambda^- \odot_h \mu^-)(s) =$

$$\bigwedge_{s+\Sigma_{i=1}^{n} p_i q_i + z = \Sigma_{j=1}^{m} r_j t_j + z} \left[(\bigvee_{i=1}^{n} \lambda^-(p_i)) \vee (\bigvee_{i=1}^{n} \mu^-(q_i)) \vee (\bigvee_{j=1}^{m} \lambda^-(r_j)) \vee (\bigvee_{j=1}^{m} \mu^-(t_j)) \right]$$

$\leq \max\{\lambda^-(sp),\lambda^-(sq),\mu^-(s)\}$
$= (\lambda^- \vee \mu^-)(s)$
$(\lambda^+ \odot_h \mu^+)(s) \leq (\lambda^- \vee \mu^-)(s)$.

Hence $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$.

Conversely, let P be a right and Q be a left *h*-ideal of R. Then $C_P = (C_P^+, C_P^-) \in BVFRhI(R)$ and $C_Q = \left(C_Q^+, C_Q^-\right) \in BVFLhI(R)$. Now, by the Theorem 3.23, $C_{\overline{PQ}}^+ = C_P^+ \odot_h C_Q^+$, $C_{\overline{PQ}}^+ = C_P^+ \odot_h C_Q^+ \sim C_P^+ \wedge C_Q^+$ and $C_{\overline{PQ}}^+ = C_P^+ \odot_h C_Q^+ \sim C_P^+ \wedge C_Q^+ = C_{P \cap Q}^+ \Longrightarrow \overline{PQ} = P \cap Q$.

Hence R is *h*-hemi-regular.

4.6 Theorem

Let $B \in BVFhbI(R)$. Then $\lambda^+ \leq (\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)$, $\lambda^- \geq (\lambda^- \odot_h C_R^- \odot_h \lambda^-)$ iff R is *h*-hemi-regular.

*Proof.*Suppose $B \in BVFhbI(R)$.

$(\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)(x)$

$$= \bigvee_{xyz + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ \begin{array}{c} (\bigwedge_{i=1}^{m} (\lambda^+ \odot_h C_R^+)(a_i)) \wedge \\ (\bigwedge_{j=1}^{n} (\lambda^+ \odot_h C_R^+)(c_j)) \\ \wedge (\bigwedge_{i=1}^{m} (\lambda^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(d_j)) \end{array} \right\}$$

$$\geq \min \left\{ \begin{array}{c} ((\lambda^+ \odot_h C_R^+)(xa)), \\ ((\lambda^+ \odot_h C_R^+)(xc)), \lambda^+(x) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} \bigvee_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge_{i=1}^{m} (\lambda^+)(a_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(c_j)) \right\}, \\ \bigvee_{x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z_1} \left\{ (\bigwedge_{i=1}^{m} (\lambda^+)(b_i)) \wedge (\bigwedge_{j=1}^{n} (\lambda^+)(d_j)) \right\}, \\ \lambda^+(x) \end{array} \right\}$$

$\geq$
$\min\{\min\{\lambda^+(xax),\lambda^+(xcx)\},\min\{\lambda^+(xax),\lambda^+(xcx)\},$
$\lambda^+(x)\}$.

Since $xa+xaxa+za = xcxa+za$ and $xc+xaxc+zc = xcxc+zc$, we have that

$$(\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)(x) \geq \min\{\lambda^+(x), \lambda^+(x), \lambda^+(x)\} \geq \lambda^+(x).$$

Similarly,

$$(\lambda^- \odot_h C_R^- \odot_h \lambda^-)(x) = \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}}(\lambda^- \odot_h C_R^-)(a_i))\vee \\ (\overset{n}{\underset{j=1}{\vee}}(\lambda^- \odot_h C_R^-)(c_j)) \\ \vee(\overset{m}{\underset{i=1}{\vee}}(\lambda^-)(b_i))\vee(\overset{n}{\underset{j=1}{\vee}}(\lambda^-)(d_j)) \end{array} \right\}$$

$$\leq \max\left\{ \begin{array}{c} ((\lambda^- \odot_h C_R^-)(xa)), \\ ((\lambda^- \odot_h C_R^-)(xc)), \lambda^-(x) \end{array} \right\}$$

$$= \max\left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}}(\lambda^-)(a_i)) \\ \vee(\overset{n}{\underset{j=1}{\vee}}(\lambda^-)(c_j)) \end{array} \right\}, \\ \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}}(\lambda^-)(b_i)) \\ \vee(\overset{n}{\underset{j=1}{\vee}}(\lambda^-)(d_j)) \end{array} \right\}, \\ \lambda^-(x) \end{array} \right\}$$

$$\leq \max\{\max\{\lambda^-(xax), \lambda^-(xcx)\}, \max\{\lambda^-(xax), \lambda^-(xcx)\}, \lambda^-(x)\}.$$

Since $xa+xaxa+za = xcxa+za$ and $xc+xaxc+zc = xcxc+zc$, we have that

$$(\lambda^- \odot_h C_R^- \odot_h \lambda^-)(x) \leq \max\{\lambda^-(x), \lambda^-(x), \lambda^-(x)\} = \lambda^-(x).$$

This proves $\lambda^+ \leq (\lambda^+ \odot_h C_R^+ \odot_h \lambda^+)$, $\lambda^- \geq (\lambda^- \odot_h C_R^- \odot_h \lambda^-)$.

Conversely, let M be any h-bi-ideal of R. Then by the Theorem 3.13, $C_M \in BVFhbI(R)$. Since, $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+$, by the Theorem 3.23, $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+ = C^+_{\overline{MRM}}$ and $M \subseteq \overline{MRM}$. On the other hand, since M is h-bi-ideal of R so that $MRM \subseteq M$. This implies, $\overline{MRM} \subseteq \overline{M}$, and $\overline{MRM} \subseteq \overline{M} = M$, therefore $\overline{MRM} = M$. Hence R is h-hemi-regular.

4.7 Theorem

The following conditions for R are equivalent:

(i) R is h-hemi-regular hemi-ring,

(ii) $\min\{\lambda^+, \mu^+\} \leq \lambda^+ \odot_h \mu^+ \odot_h \lambda^+$, $\max\{\lambda^-, \mu^-\} \geq \lambda^- \odot_h \mu^- \odot_h \lambda^-$ for every $B = (\lambda^+, \lambda^-) \in BVFhbI(R)$ and for every $B' = (\mu^+, \mu^-) \in BVFhI(R)$,

(iii) $\min\{\lambda^+, \mu^+\} \leq \lambda^+ \odot_h \mu^+ \odot_h \lambda^+$, $\max\{\lambda^-, \mu^-\} \geq \lambda^- \odot_h \mu^- \odot_h \lambda^-$ for every $B = (\lambda^+, \lambda^-) \in BVFhqI(R)$ and for every $B' = (\mu^+, \mu^-) \in BVFhI(R)$.

Proof.$(i) \Longrightarrow (ii)$ Suppose (i) holds.

$$(\lambda^+ \odot_h \mu^+ \odot_h \lambda^+)(x) = \bigvee_{xyz+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}}(\lambda^+ \odot_h \mu^+)(a_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\lambda^+ \odot_h \mu^+)(c_j)) \\ \wedge(\overset{m}{\underset{i=1}{\wedge}}(\lambda^+)(b_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\lambda^+)(d_j)) \end{array} \right\}$$

$$\geq \min\left\{ \begin{array}{c} ((\lambda^+ \odot_h \mu^+)(xa)), \\ ((\lambda^+ \odot_h \mu^+)(xc)), \lambda^+(x) \end{array} \right\}$$

$$= \min\left\{ \begin{array}{c} \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}}(\lambda^+)(a_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\lambda^+)(c_j)) \\ \wedge(\overset{m}{\underset{i=1}{\wedge}}(\mu^+)(b_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\mu^+)(d_j)) \end{array} \right\}, \\ \bigvee_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z_1} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\wedge}}(\lambda^+)(a_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\lambda^+)(c_j)) \\ \wedge(\overset{m}{\underset{i=1}{\wedge}}(\mu^+)(b_i))\wedge \\ (\overset{n}{\underset{j=1}{\wedge}}(\mu^+)(d_j)) \end{array} \right\}, \\ \lambda^+(x) \end{array} \right\}$$

$$\geq \min\{\min\{\lambda^+(x), \mu^+(axa), \mu^+(cxa)\}, \min\{\lambda^+(x), \mu^+(axc), \lambda^+(cxc)\}, \lambda^+(x)\}.$$

Since $xa+xaxa+za = xcxa+za$ and $xc+xaxc+zc = xcxc+zc$, we have that

$$(\lambda^+ \odot_h \mu^+ \odot_h \lambda^+)(x)$$

$$\geq \min\{\min\{\lambda^+(x), \mu^+(x)\}, \min\{\lambda^+(x), \mu^+(x)\}\}$$

$$= \min\{\lambda^+(x), \mu^+(x)\}.$$

Similarly,

$$(\lambda^- \odot_h \mu^- \odot_h \lambda^-)(x)$$

$$= \bigwedge_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n c_jd_j+z} \left\{ \begin{array}{c} (\overset{m}{\underset{i=1}{\vee}}(\lambda^- \odot_h \mu^-)(a_i))\vee \\ (\overset{n}{\underset{j=1}{\vee}}(\lambda^- \odot_h \mu^-)(c_j)) \\ \vee(\overset{m}{\underset{i=1}{\vee}}(\lambda^-)(b_i))\vee \\ (\overset{n}{\underset{j=1}{\vee}}(\lambda^-)(d_j)) \end{array} \right\}$$

$$\leq \max\left\{ \begin{array}{c} ((\lambda^- \odot_h \mu^-)(xa)), \\ ((\lambda^- \odot_h \mu^-)(xc)), \lambda^-(x) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^-)(a_i)) \vee \\ (\bigvee_{j=1}^{n} (\lambda^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\mu^-)(b_i)) \vee \\ (\bigvee_{j=1}^{n} (\mu^-)(d_j)) \end{array} \right\}, \\ \bigwedge_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} c_j d_j + z} \left\{ \begin{array}{c} (\bigvee_{i=1}^{m} (\lambda^-)(a_i)) \vee \\ (\bigvee_{j=1}^{n} (\lambda^-)(c_j)) \\ \vee (\bigvee_{i=1}^{m} (\mu^-)(b_i)) \vee \\ (\bigvee_{j=1}^{n} (\mu^-)(d_j)) \end{array} \right\}, \\ \lambda^-(x) \end{array} \right\}$$

$\leq \max\{\max\{\lambda^-(x), \mu^-(axa), \mu^-(cxa)\}, \max\{\lambda^-(x), \lambda^-(axc), \mu^-(cxc)\}, \lambda^-(x)\}$

Since $xa + xaxa + za = xcxa + za$ and $xc + xaxc + zc = xcxc + zc$, we have that $(\lambda^- \odot_h \mu^- \odot_h \lambda^-)(x)$ $\leq \max\{\max\{\lambda^-(x), \mu^-(x)\}, \max\{\lambda^-(x), \mu^-(x)\}\}$ $= \max\{\lambda^-(x), \mu^-(x)\}$.

This proves (ii).

$(ii) \Longrightarrow (iii)$. By the Lemma 3.32. it is straightforward.

$(iii) \Longrightarrow (i)$. Assume that (iii) holds. Let M be any h-quasi-ideal of R. By the Theorem 3.13, $C_M \in BVFhqI(R)$. Since $C_R \in BVFhI(R)$. Now, from (iii) $C_M^+ \leq C_M^+ \odot_h C_R^+ \odot_h C_M^+$, by the Theorem 3.23, $C_M^+ \subseteq C_M^+ \odot_h C_R^+ \odot_h C_M^+ = C_{\overline{MRM}}^+$ and $M \subseteq \overline{MRM}$. On the other hand, since M is h-bi-ideal of R so that $MRM \subseteq M$. This implies, $\overline{MRM} \subseteq \overline{M}$ and $\overline{MRM} \subseteq \overline{M} = M$, therefore $\overline{MRM} = M$. Hence R is h-hemi-regular hemi-ring.

4.8 Theorem

Let R be a h-hemi-simple and $B = (\lambda^+, \lambda^-)$ be a BVF subset of R. Then $B \in BVFhI(R)$ iff $B \in BVFIhI(R)$.

Proof. By the Theorem 3.15, if $B = (\lambda^+, \lambda^-) \in BVFhI(R)$ then $B = (\lambda^+, \lambda^-) \in BVFIhI(R)$.

Conversely, assume $B = (\lambda^+, \lambda^-) \in BVFhI(R)$. Let $p, q \in R$, $\exists\, a_i, b_i, c_i, d_i, e_j, f_j, g_j, h_j \in R$ such that $p + \Sigma_{i=1}^{m} a_i p b_i c_i p d_i + z = \Sigma_{j=1}^{n} e_j p f_j g_j p h_j + z$. Which implies $pq + \Sigma_{i=1}^{m} a_i p b_i c_i p d_i q + zq = \Sigma_{j=1}^{n} e_j p f_j g_j p h_j q + zq$.

Thus $\lambda^+(pq) \geq \min\left\{\lambda^+\left(\Sigma_{i=1}^{m} a_i p b_i c_i p d_i q\right), \lambda^+\left(\Sigma_{j=1}^{n} e_j p f_j g_j p h_j q\right), 0.5\right\} \geq \lambda^+(p)$.

And $\lambda^-(pq) \leq \max\left\{\lambda^-\left(\Sigma_{i=1}^{m} a_i p b_i c_i p d_i q\right), \lambda^-\left(\Sigma_{j=1}^{n} e_j p f_j g_j p h_j q\right), -0.5\right\} \leq \lambda^-(p)$.

Thus $B \in BVFRhI(R)$. Similarly, we can show $B = (\lambda^+, \lambda^-) \in BVFLhI(R)$. Hence proved the theorem.

4.9 Theorem

If $B_1 = (\lambda^+, \lambda^-) \in BVFIhI(R)$ and $B_2 = (\mu^+, \mu^-) \in BVFIhI(R)$. Then $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$ iff R is h-semi-simple.

Proof. Suppose for any $B_1 = (\lambda^+, \lambda^-)$, $B_2 = (\mu^+, \mu^-) \in BVFLhI(R)$. By the Lemma (3.25) $(\lambda^+ \odot_h \mu^+) \leq (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \geq (\lambda^- \vee \mu^-)$. And by the Theorem (4.4) $(\lambda^+ \odot_h \mu^+)[\in](\lambda^+ \wedge \mu^+)$, $(\lambda^- \vee \mu^-)[\in](\lambda^- \odot_h \mu^-)$. Since, R is h-hemi-simple, so $\forall s \in R$, $\exists\, c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j \in R$ such that $s + \Sigma_{i=1}^{m} c_i s d_i e_i s f_i + z = \Sigma_{j=1}^{n} c'_j s d'_j e'_j s f'_j + z$. Thus

$(\lambda^+ \odot_h \mu^+)(s) =$

$$\bigvee_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigwedge_{i=1}^{m} \lambda^+(a_i)) \wedge (\bigwedge_{i=1}^{m} \mu^+(b_i)) \wedge (\bigwedge_{j=1}^{n} \lambda^+\left(a'_j\right)) \wedge (\bigwedge_{j=1}^{n} \mu^+\left(b'_j\right)) \right]$$

$\geq \min\{\lambda^+(c_i s d_i), \lambda^+\left(c'_j s d'_j\right), \mu^+(e_i s f_i), \mu^+\left(e'_j s f'_j\right)\}$
$\geq \min\{\lambda^+(s), \mu^+(s)\}$
$= (\lambda^+ \wedge \mu^+)(s)$
and
$(\lambda^- \odot_h \mu^-)(s) =$

$$\bigwedge_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a'_j b'_j + z} \left[(\bigvee_{i=1}^{m} \lambda^-(a_i)) \vee (\bigvee_{i=1}^{m} \mu^-(b_i)) \vee (\bigvee_{j=1}^{n} \lambda^-\left(a'_j\right)) \vee (\bigvee_{j=1}^{n} \mu^-\left(b'_j\right)) \right]$$

$\leq \max\{\lambda^-(c_i s d_i), \lambda^-\left(c'_j s d'_j\right), \mu^+(e_i s f_i), \mu^+\left(e'_j s f'_j\right)\}$
$\leq \max\{\lambda^-(s), \mu^-(s)\}$
$= (\lambda^- \vee \mu^-)(s)$

This proves $(\lambda^+ \odot_h \mu^+) \sim (\lambda^+ \wedge \mu^+)$, $(\lambda^- \odot_h \mu^-) \sim (\lambda^- \vee \mu^-)$.

Conversely, let I be a h-ideal of R. By the Theorem 3.15, I is an interior h-ideal of R. Now, by the Theorem 3.13, $C_I = \left(C_I^+, C_I^-\right) \in BVFIhI(R)$. We have that $C_I^+ = C_I^+ \wedge C_I^+$, and $C_I^+ = C_I^+ \odot_h C_I^+ \sim C_I^+ \wedge C_I^+$ and by the Theorem 3.23, $C_I^+ = C_{\overline{I^2}}^+ \Longrightarrow I = \overline{I^2}$. Therefore R is h-hemi-simple.

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