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A Mixture of Generalized Negative Binomial Distribution with Generalized Exponential Distribution

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ABSTRACT

The negative binomial distribution has become increasingly popular as a more flexible alternative to Poisson distribution, especially when it is questionable whether the strict requirements for Poisson distribution could be satisfied. But negative binomial distribution is better for overdispersed count data that are not necessarily heavy-tailed, for heavy tailed count data the traditional statistical distributions such as Poisson and negative binomial cannot be used efficiently. In this paper an attempt has been made to obtain a mixture of generalized negative binomial distribution with that of generalized exponential distribution, which is obtained by mixing the generalized negative binomial distribution with generalized exponential distribution. The new mixeddistribution so obtained generalizes several distributions that have been discussed in literature. Estimation of the parameters, factorial moment and ordinary (crude) moments of the new distribution has also been discussed. To justify the suitability, the distribution is fitted to a reported count data set. The resulting fit is found to be good in comparison to others.

Keywords: Generalized negative binomial distribution, generalized exponential distribution, mixture distribution and factorial moments.

1 Introduction

Traditional probability models such as Poisson, negative binomial distributions are regarded as the most suitable models to represent count data. Poisson distribution is a standard model for fitting count data when the number of occurrences of a phenomenon occurred at a constant rate with respect to time and an occurrence of the changes of any future occurrences. Equality of mean and variance is characteristic of the Poisson distribution but in a vast number of practical applications, the count data are either over-dispersed (variance is greater than mean) or under-dispersed (variance is less than mean) [12]. One frequent manifestation of over-dispersed data is that the incidence of zero counts is greater than expected for the Poisson distribution and this is of interests because zero counts frequently have special status, which is a violation of the Poisson restrictions that the variance of the observed random variable equals its mean [18]. Statistical procedures must eliminate for these sources of variability in count data which suggested a model in which the mean of Poisson distribution has a gamma distribution. Practically this leads to the negative binomial distribution. The negative binomial distribution has become increasingly popular as a more flexible alternative to Poisson distribution, especially when it is questionable whether the strict requirements for Poisson distribution could be satisfied [7] but negative binomial distribution is better for over-dispersed count data that are not necessarily heavy-tailed.Wang [17] showed that extremely heavy tail implies over-dispersion but the converse does not hold.Among the count data issues, which have a large number of zeros that lead to heavy tail. For such data sets, the number of sites where no crash is observed is so large that traditional statistical distributions or models, such as the Poisson and NB distributions, cannot be used efficiently. Lord and Geedipall [9] showed that Poisson distribution tends to underestimate the number of zeros given the mean of the data while the NB distributions may overestimate zeros, but underestimate observations with a count.

The mixing of probability distributions is an important way to construct new probability distributions that can be used as a more flexible alternative to traditional statistical distributions especially under overdispersion. Based on the mixing mechanism various authors explored new probability distributions for instance, Simon [16] obtained a negative binomial distribution by mixing the mean of the Poisson distribution with gamma distribution. Panger and Willmot [11] obtained a mixture of negative binomial distribution with exponential distribution and showed the flexibility of this particular mixture in fitting count data. Deniz, Sarabia and Ojeda [5] analyzed the univariate and multivariate versions of the mixture of negative binomial distribution with inverse Gaussian distribution [6]. The mixture of Poisson distribution with Lindley for modeling count data was introduced by Sankaran [15], later on Ghitany et.al [4] showed that in many ways Lindley is a better distribution compared to exponential and hence it is natural, one should expect a mixture of Poisson distribution with Lindley provides better fit compared to Poisson-exponential. Zamani and Ismail [19] introduced a new mixed negative binomial by mixing the negative binomial distribution with Lindley which has a thick tail and an alternative for modeling count data of insurance claims which has a thick tail and a large value at zero. Adil Rashid and Jan [13] obtained a compound of zero truncated generalized negative binomial distribution with generalized beta distribution by ascribing a generalized beta distribution to probability parameter. Aryuyuen and Bodhisuwan [1] obtained a mixture of negative binomial distribution with a generalized exponential distribution which includes two more free parameters. Most recently Adil Rashid and Jan [14] explored a new compound distribution by mixing Geeta distribution with generalized beta distribution which contains several distributions as special cases.

In this article, we introduce a new mixed generalized negative binomial distribution (m,θ,β) by mixing generalized negative binomial distribution with generalized exponential distribution (α,η) where reparametrization of $\theta=e^{-\lambda}$ is considered. This new mixed distributions has a thick tail and may be considered as alternative for modeling count data of insurance claims which has thick tail and large values of zeros.

2 Materials and methods

We begin by giving definitions and relations needed for mixing of probability distributions:

A discrete random variable is X said to have a generalized negative binomial distribution (GNBD) with parameters $\theta,\beta\,\&\,m$ if its probability mass function is given by

$$f_1(x;m,\theta,\beta)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}(1-\theta)^x\theta^{m+\beta x-x}, \qquad x=0,1,.. \qquad (1)$$

Where

$0 \le \theta < 1, \quad m > 0, \; \theta\beta < 1; \beta \ge 1,$ (i)

$0 \le \theta \le 1, \; m \in N, \; \beta = 0.$ (ii)

For $\beta = 1$ equation (1) reduces to the negative binomial distribution (NBD).If $m \in N$, *for* $\beta = 0$, one obtains from (1) binomial distribution (BD) and for $\beta = 1$, Pascal distribution see Johnson, Kotz and Kemp [7].

In GNBD the parameters θ, β *and* m are constants but here we have considered a problem in which the probability parameter $\theta = e^{-\lambda}$ where $\lambda > 0$ is itself a random variablefollowing generalizedexponential distribution (GED) with parameters α *and* η and has probability density function

$$f_2(\lambda;\alpha,\eta) = \alpha\eta(1-e^{-\eta x})^{\alpha-1} e^{-\eta x} \quad ; x > 0 \text{ for } \alpha,\eta > 0 \tag{2}$$

The GED was introduced by Gupta & Kundu [6]. The moment generating function (m.g.f) of (2) given by

$$M_X(t) = \int_0^\infty e^{tx} f_2(x;\alpha,\eta) = \frac{\Gamma(\alpha+1)\Gamma\left(1-\frac{t}{\eta}\right)}{\Gamma\left(\alpha-\frac{t}{\eta}+1\right)} \tag{3}$$

In order to obtain a mixture of GNBD with that of GED we first provide a general definition of this distribution which will subsequently expose its probability mass function.

3 Mixture of GNBD with GED

Let $X \mid \lambda$ be a random variable following GNBD ($m, \beta, \theta = e^{-\lambda}$) where λ is distributed as GED with positive parameters (α, η) i.e. $X \mid \lambda \sim GNBD(m, \beta, \theta = e^{-\lambda})$ & $\lambda \sim GED(\alpha, \eta)$, then determining the distribution of X that results from marginalizing over λ is known as a mixture of GNBD with GED.

Theorem3.1: The probability mass function of a mixture of generalized negative binomial distribution (GNBD) with generalized exponential distribution (GED) is given by the expression

$$f(x;m,\beta,\alpha,\eta)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{\Gamma(\alpha+1)\Gamma\left(1+\frac{j+m+\beta x-x}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x-x}{\eta}+1\right)} \quad (4)$$

where $x=0,1,\ldots;m,\alpha,\beta,\eta>0$.

Proof: If $X\mid\lambda \sim GNBD\,(m,\beta,\theta=e^{-\lambda})$ and $\lambda\sim GED\,(\alpha,\eta)$, then the probability mass function of X can be obtained by

$$f(x;m,\beta,\alpha,\eta)=\int_{0}^{\infty} f_1(X\mid\lambda)\, f_2(\lambda;\alpha,\eta)\,d\lambda \quad (5)$$

Where $f_1(X\mid\lambda)$ is defined by

$$f_1(X\mid\lambda)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}(1-e^{-\lambda})^{x}e^{-\lambda(m+\beta x-x)}$$

$$=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}e^{-\lambda j}\,e^{-\lambda(m+\beta x-x)}$$

$$=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\,e^{-\lambda(j+m+\beta x-x)} \quad (6)$$

Substituting (6) into (5) we obtain

$$f(x;m,\beta,\alpha,\eta)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\int_{0}^{\infty}e^{-\lambda(j+m+\beta x-x)}\,f_2(\lambda;\alpha,\eta)\,d\lambda$$

$$=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}M_{\lambda}\big(-(j+m+\beta x-x)\big)$$

Using the argument (3) we get

$$=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{\Gamma(\alpha+1)\Gamma\left(1+\frac{j+m+\beta x-x}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x-x}{\eta}+1\right)}$$

Equivalently it can be written in the form

$$f(x;m,\beta,\alpha,\eta)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{\alpha(j+m+\beta x-x)}{(\alpha\eta+(j+m+\beta x-x))}B\left(\alpha,\frac{j+m+\beta x-x}{\eta}\right)\right) \quad (7)$$

Where B(.) refers to the beta function

$$B(r,s)=\frac{\Gamma r\,\Gamma s}{\Gamma(r+s)};\ r,s>0$$

***Special Cases*:**

Case (i): For $\beta=1$ GNBD reduces to NBD, therefore the mixture of NBD with GED is followed from (7) by simply substituting $\beta=1$ in it.

$$f(x;m,\alpha,\eta)=\binom{m+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{\alpha(j+m)}{(\alpha\eta+(j+m))}B\left(\alpha,\frac{j+m}{\eta}\right)\right) \quad (8)$$

the mixture distribution displayed in (8) was introduced by Sirinapa Aryuyuen et.al [1].

Case (ii): For $\beta=0\ \&\ m\in N$ GNBD reduces to BD, therefore a mixture of BD with GED is followed from (8) by simply substituting $\beta=0$ in it.

$$f(x;m,\alpha,\eta)=\binom{m}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{\alpha(j+m-x)}{(\alpha\eta+(j+m-x))}B\left(\alpha,\frac{j+m-x}{\eta}\right)\right)$$

Case (iii): For $\alpha=1$, GED reduces to exponential distribution (ED) and a mixture of GNBD with ED is followed from (7) by simply substituting $\alpha=1$ in it.

$$f(x;m,\beta,\eta)=\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{(j+m+\beta x-x}{(\eta+(j+m+\beta x-x))}B\left(1,\frac{j+m+\beta x-x}{\eta}\right)\right)$$

Case (4): For $\beta=1, m>0$ and $\alpha=1$ in (7) we obtain mixture of NBD with ED and its pmf is given as

$$f(x;m,\eta)=\binom{m+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{(j+m)}{(\eta+(j+m))}B\left(1,\frac{j+m}{\eta}\right)\right)$$

$$=\binom{m+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(1-\frac{(-(m+j))}{\eta}\right)^{-1} \quad (9)$$

The mixture distribution displayed in (9) was introduced by Panger and Willmot [11].

Case (5): For $\beta=0, m\in N$ and $\alpha=1$ in (7) we obtain mixture of BD with ED, therefore, its pmf is given by the expression

$$f(x;m,\eta)=\binom{m}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\left(\frac{(j+m-x)}{(\eta+(j+m-x))}B\left(1,\frac{j+m-x}{\eta}\right)\right)$$

4 Factorial moments and ordinary (crude) moments of a mixture of negative binomial distribution with generalized exponential distribution

If $X\mid\lambda \sim$ NBD $(m,\theta=e^{-\lambda})$, where $\lambda \sim$ GED (α,η) then when one keeps in mind the so called factorial polynomial

$$x^{[l]}=x(x-1)(x-2)\ldots(x-l+1)$$

$$m_{[l]}(x)=E_{\lambda}\left[\mu_{l}(x\mid\lambda)\right] \tag{10}$$

is called the factorial moment of order l of a mixture of NBD with GED, where $\mu_l(x\mid\lambda)$ is the factorial moment of NBD.

Theorem 4.1: The factorial moment of order l of a mixture of negative binomial distribution with generalized exponential distribution is given by the expression

$$m_{[l]}(x)=\frac{\Gamma(m+l)}{\Gamma m}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}\frac{\left(\Gamma(\alpha+1)\Gamma\left(1-\frac{l-j}{\eta}\right)\right)}{\Gamma\left(\alpha-\frac{l-j}{\eta}+1\right)}$$

where $x=0,1,2\ldots,\ for\ m,\alpha\ and\ \eta>0$

Proof: The factorial moment of order l of NBD (m,θ) is

$$\mu_{[l]}(x)=\frac{\Gamma(m+l)(1-\theta)^{l}}{\Gamma(m)\theta^{l}}$$

Thus the factorial moment of order l of a mixture of NBD with GED is obtained from (10), if we let $\theta=e^{-\lambda}$

$$m_{[l]}(x)=E_{\lambda}\left(\frac{\Gamma(m+l)(1-e^{-\lambda})^{l}}{\Gamma(m)e^{-\lambda l}}\right)=\frac{\Gamma(m+l)}{\Gamma(m)}E_{\lambda}(e^{\lambda}-1)^{l}$$

Using the binomial expansion of $(e^{\lambda}-1)^{l}$ we have

$$=\frac{\Gamma(m+l)}{\Gamma(m)}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}E_{\lambda}\left(e^{\lambda(l-j)}\right)$$

$$m_{[l]}(x)=\frac{\Gamma(m+l)}{\Gamma(m)}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}M_{\lambda}\left(l-j\right) \quad (11)$$

From the moment generating function of GED (3) with $t=l-j$, we get

$$m_{[l]}(x)=\frac{\Gamma(m+l)}{\Gamma(m)}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}\frac{\left(\Gamma(\alpha+1)\Gamma\left(1-\frac{l-j}{\eta}\right)\right)}{\Gamma\left(\alpha-\frac{l-j}{\eta}+1\right)} \quad (12)$$

***Special case*:**

If $\alpha=1$ in (12) we get the factorial moment of order l of a mixture of NBD with ED

$$m_{[l]}(x)=\frac{\Gamma(m+l)}{\Gamma(m)}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}\frac{\left(\Gamma(2)\Gamma\left(1-\frac{l-j}{\eta}\right)\right)}{\Gamma\left(2-\frac{l-j}{\eta}\right)}$$

$$m_{[l]}(x)=\frac{\Gamma(m+l)}{\Gamma(m)}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j}\left(1-\frac{l-j}{\eta}\right)^{-1}$$

The ordinary (crude) moments of mixture distributions under considerations are obtained by using the formula

$$m_{l}=\sum_{J=0}^{l}S_{j}^{l}m_{[j]}$$

where S_{j}^{l} stands for the so called Sterling numbers of the second kind. Bohlmann [2] seems to be the first to give this formula; the tables for these numbers can be found, for instance, in Lukasiewicz and Warmus [10].

5 Parameter estimation

In this section the estimation of parameters of mixture of GNBD with GED will be discussed by using maximum likelihood estimation. The likelihood function ofa mixture of GNBD with GED (m,β,α,η) is given by

$$L(x,m,\beta,\alpha,\eta)=\prod_{i=1}^{n}\frac{m}{m+\beta x}\binom{m+\beta x}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^{j}\frac{\Gamma(\alpha+1)\Gamma\left(1+\frac{j+m+\beta x-x}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x-x}{\eta}+1\right)}$$

$$£(x,m,\beta,\alpha,\eta)=\log L(x,m,\beta,\alpha,\eta)=\sum_{i=1}^{n}\log\left(\frac{m}{m+\beta x_i}\right)+\sum_{i=1}^{n}\log\left(\frac{\Gamma(m+\beta x_i+1)}{\Gamma(x_i+1)\Gamma(m+\beta x_i-x_i+1)}\right)+$$

$$\sum_{i=1}^{n}\log\left(\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j}\frac{\Gamma(\alpha+1)\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}\right)$$

$$=\sum_{i=1}^{n}\left(\log m-\log(m+\beta x_i)\right)+\sum_{i=1}^{n}\left(\log\Gamma(m+\beta x_i+1)-\log\Gamma(x_i+1)-\log\Gamma(m+\beta x_i-x_i-1)\right)+$$

$$\sum_{i=1}^{n}\log\left(\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j}\frac{\Gamma(\alpha+1)\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}\right) \quad (13)$$

The first order conditions for finding the optimal values of the parameters obtained by differentiating (13) with respect $m,\beta,\alpha \; and \; \eta$ gives rise to the following differential equations

$$\frac{\partial £}{\partial m}=\sum_{i=1}^{n}\left(\frac{1}{m}-\frac{1}{m+\beta x_i}\right)+\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial m}\Gamma(m+\beta x_i+1)}{\Gamma(m+\beta x_i+1)}-\frac{\frac{\partial}{\partial m}\Gamma(m+\beta x_i-x_i+1)}{\Gamma(m+\beta x_i-x_i+1)}\right)+$$

$$\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j}\frac{\partial}{\partial m}\left(\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^{j}\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}}\right)$$

$$=\sum_{i=1}^{n}\left(\frac{\beta x_i}{m(m+\beta x_i)}\right)+\sum_{i=1}^{n}\Psi(m+\beta x_i+1)-\sum_{i=1}^{n}\Psi(m+\beta x_i-x_i+1)+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial m}\left(\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}}\right) \tag{14}$$

where $\Psi(x_i)=\frac{\Gamma'(x_i)}{\Gamma(x_i)}$ is a digamma function.

$$\frac{\partial £}{\partial\beta}=-\sum_{i=1}^{n}\left(\frac{x_i}{(m+\beta x_i)}\right)+\sum_{i=1}^{n}\left(\frac{\frac{\partial}{\partial\beta}\Gamma(m+\beta x_i+1)}{\Gamma(m+\beta x_i+1)}-\frac{\frac{\partial}{\partial\beta}\Gamma(m+\beta x_i-x_i+1)}{\Gamma(m+\beta x_i-x_i+1)}\right)+\sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\partial}{\partial\beta}\left(\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{\Gamma\left(1+\frac{j+m+\beta x_i-x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i-x_i}{\eta}+1\right)}}\right)$$

$$\frac{\partial \pounds}{\partial \beta} = -\sum_{i=1}^{n}\left(\frac{x_i}{(m+\beta x_i)}\right) + \sum_{i=1}^{n}\Psi(m+\beta x_i+1) - \sum_{i=1}^{n}\Psi(m+\beta x_i - x_i + 1) + \sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\partial}{\partial \beta}\left(\frac{\Gamma\left(1+\frac{j+m+\beta x_i - x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\Gamma\left(1+\frac{j+m+\beta x_i - x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}}\right) \quad (15)$$

$$\frac{\partial}{\partial \alpha} = \sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\partial}{\partial \alpha}\left(\frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}}\right) \quad (16)$$

$$\frac{\partial \pounds}{\partial \eta} = \sum_{i=1}^{n}\left(\frac{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\partial}{\partial \eta}\left(\frac{\Gamma\left(1+\frac{j+m+\beta x_i - x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}\right)}{\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j \frac{\Gamma\left(1+\frac{j+m+\beta x_i - x_i}{\eta}\right)}{\Gamma\left(\alpha+\frac{j+m+\beta x_i - x_i}{\eta}+1\right)}}\right) \quad (17)$$

The above equations have difficult and complicated algebraic solution therefore we use Newton-Raphson method which is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation [3]. Thus we estimated the parameters of distribution by equating (14-17) to zero, the MLE solution of

$\hat{m}, \hat{\beta}, \hat{\alpha}\, and\, \hat{\eta}$ can be obtained by solving the resulting equations simultaneously using a numerical procedure with Newton-Raphson method.

6 Results and Discussion

Here, we illustrate the applicability of the new proposed mixture distribution using a real data set, the data contains number of automobile liability policies in Switzerland for private cars taken form Klugman [8]. Normally these data sets are fitted by Poisson distribution, NB distribution and NB-GE distribution. The maximum likelihood method provides parameter estimation. By comparing these fitting distributions in table 1, based on the P value it is clear that a new mixture distribution of GNBD with GED provides better fit for the count data that have a large number of zeros.

Table:1 Observed and expected frequency of accident data

<table>
<tr><th rowspan="2">Number of accidents</th><th rowspan="2">Observed Frequency</th><th colspan="4">Fitted Distribution</th></tr>
<tr><th>Poisson</th><th>NBD</th><th>NBD-GED</th><th>GNBD-GED</th></tr>
<tr><td>0</td><td>103704</td><td>102633.7</td><td>103723.6</td><td>103708.8</td><td>103706.4</td></tr>
<tr><td>1</td><td>14075</td><td>15918.5</td><td>13989.9</td><td>14046.8</td><td>14068.6</td></tr>
<tr><td>2</td><td>1766</td><td>1234.5</td><td>1857.1</td><td>1797.8</td><td>1775.2</td></tr>
<tr><td>3</td><td>255</td><td rowspan="5">66.5</td><td>245.2</td><td>251.7</td><td>253.1</td></tr>
<tr><td>4</td><td>45</td><td rowspan="4">37.2</td><td>36</td><td>40.6</td></tr>
<tr><td>5</td><td>6</td><td rowspan="3">12</td><td rowspan="3">9.1</td></tr>
<tr><td>6</td><td>2</td></tr>
<tr><td>7</td><td>0</td></tr>
<tr><td>Total</td><td>119853</td><td>119853</td><td>119853</td><td>119853</td><td>119853</td></tr>
<tr><td colspan="2">Parameter Estimation</td><td>$\lambda = .1551$
$\theta = .1509$</td><td>$\hat{m} = 1.062$</td><td>$\hat{m} = 2.4132$
$\hat{\alpha} = 3.3010$
$\hat{\eta} = 30.9996$</td><td>$\hat{m} = 3.4193$
$\hat{\alpha} = 5.0139$
$\beta = 16.1333$
$\hat{\eta} = 33.0123$</td></tr>
<tr><td colspan="2">Chi-square</td><td>1332.3</td><td>12.12</td><td>4.26</td><td>0.675</td></tr>
<tr><td colspan="2">d.f.</td><td>2</td><td>2</td><td>2</td><td>2</td></tr>
<tr><td colspan="2">p-value</td><td><0.001</td><td>0.0023</td><td>0.1188</td><td>0.714</td></tr>
</table>

7 Conclusion

In this paper we have introduced a new mixture distribution by mixing the GNBD with GED known as a mixture of GNBD with GED,which contains several mixture distributions as its special cases. Further, the parameter estimation, factorial moments and ordinary (crude) moments of the new mixed distribution has also been obtained. Finally we described the usefulness of the new proposed mixed distribution to the count data sets characterized by large number of zeros.

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