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Generalization of Belief and Plausibility Functions to Fuzzy Sets

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Abstract: In order to process the fuzzy and imprecise information in the evidential reasoning, the scholars have made many attempts to generalize belief and plausibility functions based on the Dempster-Shafer(D-S) evidence theory to fuzzy sets for many decades. A new method for defining the fuzzy closeness degree is put forward in this paper. Based on the closeness degree, another generalization of belief and plausibility functions to fuzzy sets is proposed which discards the max and min operators in foregoing generalizations according to the measure of fuzzy inclusion. We then make the comparisons of the proposed extension with some methods available. The results of the numerical experiments show the effectiveness of the proposed generalization, especially for being able to catch more information about the change of fuzzy focal elements.

Keywords: Information fusion, D-S evidence theory, closeness degree, fuzzy sets.

1. Introduction

Uncertainty always exists in nature and real systems. It is known that probability has been used traditionally in modeling uncertainty. Since a belief function was proposed as another type of measuring uncertainty, D-S evidence theory [1–3] has been widely studied and applied in diverse areas [4–7]. Because of the advent of computer technology, the representation of human knowledge can be processed by a computer in complex systems.

As capably of disposing the uncertainty induced by ignorance, D-S evidence theory adopts belief function not the probability as the measurement, restraining the probability of some incidents to establish belief function without specifying the probability which in general is hard to obtain [8]. Since the fuzzy set concept was proposed by Zadeh [9], the analysis of fuzzy data becomes increasingly important. For decades, many scholars have made attempts to extend belief function and plausibility function to fuzzy sets. After Zadeh [10] proposed information granularity and extended belief function to fuzzy sets, some scholars [11–14] have made many attempts to extend the D-S evidence theory to fuzzy sets based on the inclusion degree of the fuzzy sets. These methods used the max or min operator to define the measure of fuzzy inclusion, its inclusion degree is decided by some critical points. Thus the belief function is not sensitive to the changes in focal element information. Afterwards, based on fuzzy decomposition theorem, Yen [15] and Yang et al. [16] proposed new extensions, which make certain improvements in the former methods but the problem of being not sensitive to the focal element. Our idea in this paper is to improve the efficiency of fuzzy closeness measure between fuzzy sets. To do this, our aim is further rational to gain value of the belief and plausibility functions.

Besides this introduction, this paper is organized as follows: the following section briefly reviews some existing extensions of belief and plausibility functions to fuzzy sets. In Section 3, we propose a new calculation method of fuzzy closeness degree and extend belief and plausibility functions to fuzzy sets based on fuzzy closeness degree. In Section 4, we make the comparisons with other existing extensions. Conclusions are then given in Section 5.

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2. Different generalizations of belief and plausibility functions

Up to date, several generalizations of belief and plausibility functions to fuzzy sets were proposed according to differently defined fuzzy inclusion $I(\tilde{A}\tilde{\subset}\tilde{B})$ with

$$Bel(\tilde{B}) = \sum_{\tilde{A}} I(\tilde{A}\tilde{\subset}\tilde{B})m(\tilde{A}) \tag{1}$$

These are Yager [12], Ishizuka et al. [13], Ogawa et al. [14], Yen [15] and Yang et al. [16] They extended the belief and plausibility functions with their own defined fuzzy inclusion as follows:

Yager [12]:

$$I(\tilde{A}\tilde{\subset}\tilde{B}) = \min_x\{\mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x)\} \tag{2}$$

Ishizuka et al. [13]:

$$I(\tilde{A}\tilde{\subset}\tilde{B}) = \frac{\min_x\{1, 1 + (\mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x))\}}{\max_x \mu_{\tilde{A}}(x)} \tag{3}$$

Ogawa et al. [14]:

$$I(\tilde{A}\tilde{\subset}\tilde{B}) = \frac{\sum_x \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}}{\sum_x \mu_{\tilde{B}}(x)} \tag{4}$$

Yen [15] indicated that: Yager, Ishizuka et al. and Ogawa et al. adopted the max and min operators in the definition of $I(\tilde{A}\tilde{\subset}\tilde{B})$. The fuzzy inclusion degree is decided by some critical points. Thus the belief function is not sensitive to the changes in focal element information. Therefore, he proposed a generalization with the construction of linear programming problems and decomposition theorem [17] for calculating the belief and plausibility functions of fuzzy sets. He defined the belief and plausibility functions of a fuzzy set $\tilde{B}$ as follows:

$$Bel(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \times \inf_{x\in A_{\alpha_i}} \mu_{\tilde{B}}(x) \tag{5}$$

$$Pls(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha_i} (\alpha_i - \alpha_{i-1}) \times \sup_{x\in A_{\alpha_i}} \mu_{\tilde{B}}(x) \tag{6}$$

where α_i is a cut set of fuzzy set $\tilde{A}$. $A_{\alpha_i} = \{x|\mu_{\tilde{A}}(x) \geq \alpha_i\}$, $\tilde{A} = \bigcup \alpha_i A_{\alpha_i}$, $\alpha_0 = 0$, $\alpha_n = 1$, $\alpha_{i-1} < \alpha_i$, $i = 1, 2, \ldots, n$. $m(A_{\alpha_i}) = (\alpha_i - \alpha_{i-1}) \times m(\tilde{A})$.

Yen's method is similar to those explained previously. Its fuzzy inclusion degree is equivalent to $I(\tilde{A}\tilde{\subset}\tilde{B}) = \sum_{\alpha_i}(\alpha_i - \alpha_{i-1}) \times \inf_{x\in A_{\alpha_i}} \mu_{\tilde{B}}(x)$, which makes certain improvements in the several methods mentioned before.

Yang et al. [16] proposed another extension based on the Yen's method:

$$Bel(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha} \frac{|\theta_\alpha|}{|\tilde{A}|} \times \inf_{x\in A_\alpha} \mu_{\tilde{B}}(x) \tag{7}$$

$$Pls(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha} \frac{|\theta_\alpha|}{|\tilde{A}|} \times \sup_{x\in A_\alpha} \mu_{\tilde{B}}(x) \tag{8}$$

where $\theta_\alpha = \{x|\mu_{\tilde{A}}(x) = \alpha\}$,$\alpha \in [0, 1]$, $|\tilde{A}| = \sum_x \mu_{\tilde{A}}(x)$, $|\theta_\alpha| = \sum_{x\in\theta_\alpha} \mu_{\tilde{A}}(x)$.

Lin [18] indicated that the methods of Yen and Yang et al. also have the problem of being not sensitive to the focal element changes. He, according to the similarity of the fuzzy sets, proposed the belief and plausibility functions as follows:

$$Bel(\tilde{B}) = \sum_j \left[1 - \frac{1}{|\tilde{A}|} \sum_i^{|\tilde{A}|} |\mu_{\tilde{B}}(x_i) - \mu_{\tilde{A}}(x_i)|\right] m(\tilde{A}_j) \tag{9}$$

$$Pls(\tilde{B}) = \sum_j \left[1 - \frac{1}{|\Theta|} \sum_i^{|\Theta|} |\mu_{\tilde{B}}(x_i) - \mu_{\tilde{A}}(x_i)|\right] m(\tilde{A}_j) \tag{10}$$

Hwang et al. [19] proposed the belief and plausibility functions based on Sugeno integration:

$$Bel(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{0\leq i\leq n} \frac{K_{\alpha_i} \int_{A_{\alpha_i}} \tilde{A}du}{\sum_{0\leq i\leq n} K_{\alpha_i} \int_{A_{\alpha_i}} \tilde{A}du} \times \inf_{x\in A_{\alpha_i}} \mu_{\tilde{B}}(x) \tag{11}$$

$$Pls(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A}) \sum_{0\leq i\leq n} \frac{K_{\alpha_i} \int_{A_{\alpha_i}} \tilde{A}du}{\sum_{0\leq i\leq n} K_{\alpha_i} \int_{A_{\alpha_i}} \tilde{A}du} \times \sup_{x\in A_{\alpha_i}} \mu_{\tilde{B}}(x) \tag{12}$$

Through the research on the fuzzy closeness degree, a new calculation method of closeness degree of fuzzy sets is proposed in this paper. Along with the approaches of Yen and Yang et al., based on improved fuzzy closeness degree, new belief and plausibility functions are proposed. This method is more sensitive to the focal element changes, and it is further rational to gain value of the belief and plausibility functions.

3. A new generalization of belief and plausibility functions to fuzzy sets

3.1. Fuzzy closeness degree

Definition 1. Let S be a mapping S: $\Re(X) \times \Re(X) \to [0, 1]$. If $S(\tilde{A}, \tilde{B})$ satisfies the following conditions [20]:

1) $0 \leq S(\tilde{A}, \tilde{B}) \leq 1$;

2) $S(\tilde{A}, \tilde{A}) = 1$;

3) $S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$;

4) If $\tilde{A}\tilde{\subseteq}\tilde{B}\tilde{\subseteq}\tilde{C}$ then $S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B}) \wedge S(\tilde{B}, \tilde{C})$.

Then $S(\tilde{A}, \tilde{B})$ is a closeness measure between fuzzy sets

$\tilde{A}$ and $\tilde{B}$. The degree $S(\tilde{A}, \tilde{B})$ is used to describe the degree of two fuzzy sets are close to each other. The distance can be utilized to define the closeness degree.

Definition 2. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $X = \{x_1, x_2, \ldots, x_n\}$. Let $\tilde{A} = (\mu_{\tilde{A}}(x_1)/x_1, \mu_{\tilde{A}}(x_2)/x_2, \ldots, \mu_{\tilde{A}}(x_n)/x_n)$, $\tilde{B} = (\mu_{\tilde{B}}(x_1)/x_1, \mu_{\tilde{B}}(x_2)/x_2, \ldots, \mu_{\tilde{B}}(x_n)/x_n)$. Functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are called the membership functions of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ respectively. The distance general form between the two fuzzy sets can be defined as:

$$d^p(\tilde{A}, \tilde{B}) = \left[\frac{1}{n}\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p\right]^{1/p} \quad (13)$$

where $p \geq 1$.

Generally, the smaller the distance between the two fuzzy sets, the closer the two fuzzy sets while the larger the distance, the smaller the similarity of the two fuzzy sets. Therefore, the closeness degree $S^p(\tilde{A}, \tilde{B})$ can be defined as:

$$S^p(\tilde{A}, \tilde{B}) = 1 - d^p(\tilde{A}, \tilde{B}) \quad (14)$$

Provided that $p = 1$, the Hamming closeness degree can be gained:

$$S_H^1(\tilde{A}, \tilde{B}) = 1 - \frac{1}{n}\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)| \quad (15)$$

Provided that $p = 2$, the Euclidean closeness degree can be gained:

$$S_E^2(\tilde{A}, \tilde{B}) = 1 - \left[\frac{1}{n}\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^2\right]^{\frac{1}{2}} \quad (16)$$

For $\mu_{\tilde{A}}(x_i)$ and $\mu_{\tilde{B}}(x_i)$ are the number between 0 and 1 in the D-S evidence theory. Therefore, $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|$ is also the number between 0 and 1. When the denominator n in Eq. (13) increases by 1, the increment of the numerator $\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p$ will be far smaller than 1, which makes distance value become exceptional sensitive to the changes of the denominator in Eq. (13), thus the closeness degree is also not sensitive to the subtle changes of the focal elements. More, for the value of $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|$ is small, and after the average value is extracted, the distance value $d^p(\tilde{A}, \tilde{B})$ will be smaller. The closeness value calculated with Eq. (14) is greater, thus the closeness value is not sensitive to the focal elements of the fuzzy set. For this, the improved fuzzy closeness degree in this paper will be defined as:

$$S'^p(\tilde{A}, \tilde{B}) = \frac{1 - d^p(\tilde{A}, \tilde{B})}{1 + d^p(\tilde{A}, \tilde{B})} \quad (17)$$

The improved closeness degree $S'^p(\tilde{A}, \tilde{B})$ obviously satisfies condition 1), 2) and 3) in the definition 1. Now, it is proved that $S'^p(\tilde{A}, \tilde{B})$ also satisfies condition 4).

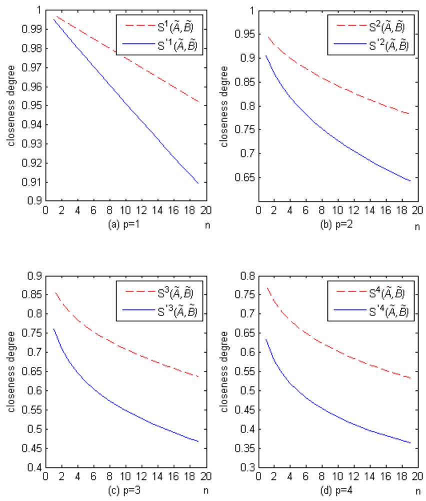

Figure 1 Comparison of two closeness degrees.

Proof. Since $\tilde{A}\tilde{\subseteq}\tilde{B}\tilde{\subseteq}\tilde{C}$, we can get $\mu_{\tilde{A}}(x_i) \leq \mu_{\tilde{B}}(x_i) \leq \mu_{\tilde{C}}(x_i)$, $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|^p \geq |\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p$, and $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{C}}(x_i)|^p \geq |\mu_{\tilde{B}}(x_i) - \mu_{\tilde{C}}(x_i)|^p$.

Thus, $d^p(\tilde{A}, \tilde{C}) \geq d^p(\tilde{A}, \tilde{B}), d^p(\tilde{A}, \tilde{C}) \geq d^p(\tilde{B}, \tilde{C})$.

For $f(x) = \dfrac{1-x}{1+x}$ is the strictly monotone decreasing function about x on [0, 1], the improved closeness degree is the strictly monotone decreasing function about the distance between fuzzy sets. Therefore, there will be:$S'^p(\tilde{A}, \tilde{C}) \leq S'^p(\tilde{A}, \tilde{B}), S'^p(\tilde{A}, \tilde{C}) \leq S'^p(\tilde{B}, \tilde{C})$, that is $S'^p(\tilde{A}, \tilde{C}) \leq S'^p(\tilde{A}, \tilde{B}) \wedge S'^p(\tilde{B}, \tilde{C})$.

Assuming $\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p$ increases in the form of $0.05^2, 0.10^2, 0.15^2, \ldots, 0.95^2$ and the corresponding n increases by 1, 2, 3,…, 19. The original closeness degree $S^p(\tilde{A}, \tilde{B})$ and the improved one $S'^p(\tilde{A}, \tilde{B})$ changes according to $\sum_{i=1}^{n}|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|^p$ and n value change curve, which is as shown by the Fig. 1. The dotted line in the Fig. 1 indicates the change curve of the original closeness degree while the solid line indicates the change curve of the improved closeness degree.

According to Fig. 1, the improved fuzzy closeness degree is more sensitive to changes of n and $|\mu_{\tilde{A}}(x_i) - \mu_{\tilde{B}}(x_i)|$ than the original closeness degree, more capable of catching the subtle changes of the focal elements and further properly describing the degree of the closeness of the two fuzzy sets.

3.2. Extension of belief and plausibility functions to fuzzy sets based on fuzzy closeness degree

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in $\Theta = \{x_1, x_2, \ldots, x_n\}$, and 2^Θ be fuzzy power set of Θ. Let $\tilde{A} = (\mu_{\tilde{A}}(x_1)/x_1, \mu_{\tilde{A}}(x_2)/x_2, \ldots, \mu_{\tilde{A}}(x_n)/x_n)$, $\tilde{B} = (\mu_{\tilde{B}}(x_1)/x_1, \mu_{\tilde{B}}(x_2)/x_2, \ldots, \mu_{\tilde{B}}(x_n)/x_n)$, $A_\alpha = \{x|\mu_{\tilde{A}}(x) \geq \alpha\}$, $\tilde{A} = \bigcup \alpha A_\alpha$, $K_\alpha = \{x|\mu_{\tilde{A}}(x) = \alpha\}$,$\alpha \in [0,1]$. In this paper, the distance $d^p(\tilde{B}, A_\alpha)$ and the closeness degree $S'^p(\tilde{B}, A_\alpha)$ between $\tilde{B}$ and A_α are defined as:

$$d^p(\tilde{B}, A_\alpha) = \left[\frac{1}{|K_\alpha|}\sum_{i=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_i) - \mu_{\tilde{A}}(x_i)|^p\right]^{1/p} \tag{18}$$

$$S'^p(\tilde{B}, A_\alpha) = \frac{1 - d^p(\tilde{B}, A_\alpha)}{1 + d^p(\tilde{B}, A_\alpha)} \tag{19}$$

where $x_i \in K_\alpha$, $|K_\alpha|$ is the number of elements of set K_α.

Utilizing Eq. (18) and (19) to define the belief function of a fuzzy set $\tilde{B}$ as:

$$m_*^p(\tilde{B} : A_\alpha) = m(\tilde{A}) \times \frac{\sum K_\alpha}{\sum \tilde{A}} \times S'^p(\tilde{B}, A_\alpha) \tag{20}$$

$$Bel^p(\tilde{B}) = \sum_{\tilde{A}} m_*^p(\tilde{B} : \tilde{A}) = \sum_{\tilde{A}}\sum_{\alpha} m_*^p(\tilde{B} : A_\alpha)$$
$$= \sum_{\tilde{A}} m(\tilde{A}) \sum_{\alpha}\left[\frac{\sum K_\alpha}{\sum \tilde{A}}\left[\frac{1 - d^p(\tilde{B}, A_\alpha)}{1 + d^p(\tilde{B}, A_\alpha)}\right]\right] \tag{21}$$

where $\sum K_\alpha = \sum_{x_j \in K_\alpha} \mu_{\tilde{A}}(x_j)$,$\sum \tilde{A} = \sum_{x_j \in \tilde{A}} \mu_{\tilde{A}}(x_j)$. The coefficient $\sum_\alpha \left[\frac{\sum K_\alpha}{\sum \tilde{A}}\left[\frac{1-d^p(\tilde{B},A_\alpha)}{1+d^p(\tilde{B},A_\alpha)}\right]\right]$ is called the contribution to $Bel(\tilde{B})$ from fuzzy set $\tilde{A}$.

In case of taking $p = 1$, there will be Eq.(22) in Eq.(21):

$$Bel^1(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A})$$
$$\sum_{\alpha}\left[\frac{\sum K_\alpha}{\sum \tilde{A}}\left[\frac{1 - \frac{1}{|K_\alpha|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|}{1 + \frac{1}{|K_\alpha|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|}\right]\right] \tag{22}$$

Converting the $|K_\alpha|$ in Eq. (18) into $|\tilde{A}|$ ($|\tilde{A}|$ refers to the number of elements of fuzzy set $\tilde{A}$), the plausibility function Pls of a fuzzy set $\tilde{B}$ is obtained:

$$Pls^p(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A})$$
$$\sum_{\alpha}\left[\frac{\sum K_\alpha}{\sum \tilde{A}}\left[\frac{1 - \left[\frac{1}{|\tilde{A}|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|^p\right]^{\frac{1}{p}}}{1 + \left[\frac{1}{|\tilde{A}|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|^p\right]^{\frac{1}{p}}}\right]\right] \tag{23}$$

In case of taking $p = 1$, there will be Eq.(24) in Eq.(23):

$$Pls^1(\tilde{B}) = \sum_{\tilde{A}} m(\tilde{A})$$
$$\sum_{\alpha}\left[\frac{\sum K_\alpha}{\sum \tilde{A}}\left[\frac{1 - \frac{1}{|\tilde{A}|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|}{1 + \frac{1}{|\tilde{A}|}\sum_{j=1}^{|K\alpha|}|\mu_{\tilde{B}}(x_j) - \mu_{\tilde{A}}(x_j)|}\right]\right] \tag{24}$$

4. Numerical examples

We use the following examples to compare the proposed method with the existing generalizations. Some data sets are from Yen [15].

Example 1. Let $\Theta = \{1,2,\ldots,10\}$. Let $\tilde{A}$ and $\tilde{C}$ be fuzzy sets in Θ with

$\tilde{A} = \{0.25/1, 0.5/2, 0.75/3, 1/4, 1/5, 0.75/6, 0.5/7, 0.25/8\}$

$\tilde{C} = \{0.5/5,\ 1/6,\ 0.8/7,\ 0.4/8\}$

where each member of the list is in the form of $\mu_{\tilde{A}}(x_i)/x_i$.

Let $\tilde{B}$ be a fuzzy set in 2^Θ with

$\tilde{B} = \{0.5/2,\ 1/3,\ 1/4,\ 1/5,\ 0.9/6,\ 0.6/7,\ 0.3/8\}$

The decomposition of the fuzzy focal $\tilde{A}$ consists of four nonfuzzy focal elements:

$\tilde{A} = \bigcup \alpha A_\alpha = 0.25A_{0.25} \bigcup 0.5A_{0.5} \bigcup 0.75A_{0.75} \bigcup 1.0A_{1.0}$

where $A_{0.25} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, $A_{0.5} = \{x_2, x_3, x_4, x_5, x_6, x_7\}$, $A_{0.75} = \{x_3, x_4, x_5, x_6\}$, $A_{1.0} = \{x_4, x_5\}$.

By definition $K_\alpha = \{x|\mu_{\tilde{A}}(x) = \alpha\}$, we can obtain

$K_{0.25} = \{x_1, x_8\}$, $K_{0.5} = \{x_2, x_7\}$, $K_{0.75} = \{x_3, x_6\}$, $K_{1.0} = \{x_4, x_5\}$

Then

$\sum K_{0.25}/\sum \tilde{A} = (0.25 + 0.25)/(0.25 + 0.5 + 0.75 + 1 + 1 + 0.75 + 0.5 + 0.25) = 0.1$

In case of taking $p = 1$, there will be from Eq. (18), (19) and (20):

$d^1(\tilde{B}, A_{0.25}) = \frac{1}{2} \times (|0 - 0.25| + |0.3 - 0.25|) = 0.15$

$m_*^1(\tilde{B} : A_{0.25}) = m(\tilde{A}) \times 0.1 \times \frac{1-0.15}{1+0.15} = 0.0739m(\tilde{A})$

Similarly,

$m_*^1(\tilde{B} : A_{0.5}) = 0.1810m(\tilde{A})$, $m_*^1(\tilde{B} : A_{0.75}) = 0.2m(\tilde{A})$, $m_*^1(\tilde{B} : A_{1.0}) = 0.4m(\tilde{A})$

Then

$Bel^1(\tilde{B} : \tilde{A}) = (0.0739 + 0.1810 + 0.2 + 0.4)m(\tilde{A}) = 0.8549m(\tilde{A})$

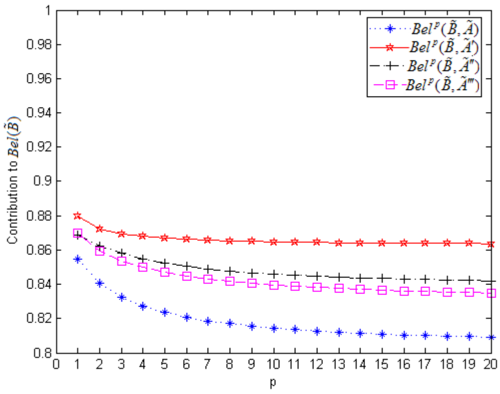

Figure 2 Changes to contribution of Bel caused by the changes of p values.

Similarly,

$Bel^1(\tilde{B}:\tilde{C}) = 0.6835m(\tilde{C})$

Thus, we obtain

$Bel^1(\tilde{B}) = 0.8549m(\tilde{A}) + 0.6835m(\tilde{C})$

Similarly, we have

$Pls^1(\tilde{B}) = 0.9593m(\tilde{A}) + 0.9053m(\tilde{C})$

We compare how these results are changed in response to a change of the membership function of the fuzzy focal element $\tilde{A}$ in three different ways denoted as

$\tilde{A}'$={0.166/1,0.5/2,0.833/3,1/4,1/5,0.75/6,0.5/7,0.25/8}

$\tilde{A}''$={0.25/1,0.75/2,1/3,1/4,1/5,0.75/6,0.5/7,0.25/8}

$\tilde{A}'''$={0/1,0.5/2,0.75/3,1/4,1/5,0.75/6,0.5/7,0.25/8}

Different methods adopted to calculate the contribution to belief function $Bel(\tilde{B})$ from the focal element $\tilde{A}$ and its variations. The results are as shown by Table 1 and Table 2. In Table 2, U, I and D denote unchanged, increased, and decreased, respectively.

According to the results of Table 2, it can be seen that in case of changes from $\tilde{A}$ to $\tilde{A}'$ and changes from $\tilde{A}$ to $\tilde{A}'''$, our generalization of the belief measure is identical to the methods of Yen, Yang et al., Lin and Hwang et al. In case of changes from $\tilde{A}$ to $\tilde{A}''$, for $S'^p(\tilde{B}, A_{1.0})$ keeps 1 and unchanged, but $\sum K_{1.0}/\sum \tilde{A}$=0.4 is changed into $\sum K_{1.0}/\sum \tilde{A}''$=0.5455, increasing too much, the closeness degree of $\tilde{B}$ and other A_α is small as the changes in $\sum K_\alpha$ and $\sum \tilde{A}$. Finally, in case of changes from $\tilde{A}$ to $\tilde{A}''$, $Bel(\tilde{B})$ enlarges (see Table 1).

Fig. 2 refers to the changes to contribution of the focal element $\tilde{A}$, $\tilde{A}'$, $\tilde{A}''$ and $\tilde{A}'''$ caused by the changes of p values.

According to Fig. 2, the contribution of the fuzzy focal element $\tilde{A}$ and its three changes to the belief function $Bel(\tilde{B})$ decreases according to the enlargement of p. However, this decreasing trend is not obvious. The contribution to the belief function $Bel(\tilde{B})$ basically remain unchanged when $p > 10$. The contribution to the belief function $Bel(\tilde{B})$ is the max when $p = 1$.

Now, compare how the results from Yen's [15], Yang's [16], Lin's [18], Hwang's [19], and our belief functions are changed in response to a change in a fuzzy focal element.

Example 2. As in *Example 1*,

$\tilde{B}$ = {0.5/2, 1/3, 1/4, 1/5, 0.9/6, 0.6/7, 0.3/8}

Let $\tilde{D}$ be a fuzzy set in Θ with

$\tilde{D} = \{0.75/2, 0.9/3, 1/4, 1/5, 0.5/6, 0.25/7, 0.1/8\}$

Let $\tilde{D}_1 \sim \tilde{D}_6$ be fuzzy sets constitution six changes in $\tilde{D}$ with

$\tilde{D}_1$ = {1/4, 1/5}

$\tilde{D}_2$ = {0.9/3, 1/4, 1/5}

$\tilde{D}_3$ = {0.75/2, 0.9/3, 1/4, 1/5}

$\tilde{D}_4$ = {0.75/2, 0.9/3, 1/4, 1/5, 0.5/6}

$\tilde{D}_5$ = {0.75/2, 0.9/3, 1/4, 1/5, 0.5/6, 0.25/7}

$\tilde{D}_5$ = {0.75/2,0.9/3,1/4,1/5,0.5/6,0.25/7,0.1/8} $= \tilde{D}$

Respectively adopting Yen's, Yang's, Lin's, Hwang's and our methods to calculate the contribution to belief function $Bel(\tilde{B})$ from $\tilde{D}_1 \sim \tilde{D}_6$. The results are as shown by Table 3.

According to the results of Table 3, it can be seen that Yen's belief function could not measure the changes in fuzzy focal element unless there are changes in the min value of the membership degree of the fuzzy focal element on critical point. For instance, as for fuzzy focal elements $\tilde{D}_3$, $\tilde{D}_4$ and $\tilde{D}_5$, the cut sets $D_{3,0.75}$, $D_{4,0.5}$, $D_{4,0.75}$ and $D_{5,0.25}$ have the same critical point with 2 ($\mu_{\tilde{B}}(2) = 0.5$), and $D_{3,0.9}$, $D_{3,1.0}$ and $D_{4,0.9}$ have the same critical points ($\mu_{\tilde{B}}(3) = \mu_{\tilde{B}}(4) = \mu_{\tilde{B}}(5) = 1$). Therefore, the contributions to $Bel(\tilde{B})$ from $\tilde{D}_3$, $\tilde{D}_4$ and $\tilde{D}_5$ are 0.625, changing until the critical point for $D_{6,0.1}$ changes to 8 ($\mu_{\tilde{B}}(8) = 0.3$). At this time, the contribution to $Bel(\tilde{B})$ changes to 0.605.

Though Yang's and Hwang's methods avoid the problems brought by the critical point of Yen's method, when the membership degree of elements in the fuzzy focal element $\tilde{D}_1$ and $\tilde{D}_2$ is not the same and that of those relative to the elements in the fuzzy focal element $\tilde{B}$ is equal, the contribution to $Bel(\tilde{B})$ of their methods is the same. From the inclusion degree, the degree of $\tilde{B}$ contained in $\tilde{D}_1$ is greater than that contained in $\tilde{D}_2$. Therefore, the methods of Yang and Hwang also could not catch the actual changes of fuzzy focal elements.

Lin's method could avoid the problems brought by the critical point but its result is reliant on the Haiming closeness degree. In the event that the membership degree of the element changes but its Hamming closeness degree does not change, the contribution gained does not change (such as the changes from $\tilde{A}$ to $\tilde{A}''$ in Table 2), which does not catch the actual change of the focal elements.

The fuzzy degree of the focal elements increases as the fuzzy focal element changes from $\tilde{D}_1$ to $\tilde{D}_6$ while the inclusion degree of the fuzzy focal element $\tilde{B}$ is more and

Table 1 Contribution to $Bel(\tilde{B})$ from the focal element $\tilde{A}$ and its variations.

Focal element	Yager [12]	Ishizuka [13]	Ogawa [14]	Yen [15]	Yang [16]	Lin [18]	Hwang [19]	Ours $p=1$	Ours $p=2$
$\tilde{A}$	0.5	0.750	0.8962	0.600	0.770	0.900	0.552	0.8549	0.8405
$\tilde{A}'$	0.5	0.834	0.9119	0.625	0.847	0.921	0.596	0.8799	0.8725
$\tilde{A}''$	0.5	0.750	0.9434	0.500	0.727	0.900	0.501	0.8688	0.8625
$\tilde{A}'''$	0.5	1	0.8962	0.675	0.826	0.931	0.691	0.8697	0.8592

Table 2 Changes to $Bel(\tilde{B})$ caused by changes in the focal element $\tilde{A}$.

Changes of focal element of $\tilde{A}$	Yager [12]	Ishizuka [13]	Ogawa [14]	Yen [15]	Yang [16]	Lin [18]	Hwang [19]	Ours $p=1$	Ours $p=2$
$\tilde{A} \to \tilde{A}'$	U	I	I	I	I	I	I	I	I
$\tilde{A} \to \tilde{A}''$	U	U	I	D	D	U	D	I	I
$\tilde{A} \to \tilde{A}'''$	U	I	U	I	I	I	I	I	I

Table 3 Contribution to $Bel(\tilde{B})$ from the focal element and its variations.

Focal element	Yen [15]	Yang [16]	Lin [18]	Hwang [19]	Ours $p=1$
$\tilde{D}_1$	1	1	1	1	1
$\tilde{D}_2$	1	1	0.9667	1	0.9436
$\tilde{D}_3$	0.625	0.8973	0.9125	0.852	0.8730
$\tilde{D}_4$	0.625	0.8494	0.8500	0.763	0.8194
$\tilde{D}_5$	0.625	0.8295	0.8167	0.700	0.8002
$\tilde{D}_6$	0.605	0.8178	0.8143	0.635	0.7973

more smaller and the contribution to $Bel(\tilde{B})$ is more and more smaller. The contribution to $Bel(\tilde{B})$ calculated by adopting our method decreases gradually. This just catches this change and is consistent with the actual. It becomes more sensitive to the focal element, can better gain information about changes of the focal element and useful to evidence combination than the methods of Yang and Lin.

5. Conclusion

According to differently defined fuzzy inclusion, some scholars extended the D-S evidence theory to fuzzy sets. However, these methods used the max or min operator to define the measure of fuzzy inclusion, its inclusion degree is decided by some critical points. Thus the belief function is not sensitive to the changes in focal element information.

In this paper, an improved fuzzy closeness degree calculation method was defined. The improved fuzzy closeness degree is more sensitive to the subtle changes of the focal element than the original closeness degree, more capable of catching the subtle changes of the focal elements and further properly describing the degree of the closeness of the two fuzzy sets. Based on the improved fuzzy closeness degree, the fuzzy belief function and plausibility function were proposed. Those calculation methods do not influenced by the critical point. The new fuzzy belief and plausibility functions can catch the actual focal element change information effectively and are sensitive to the subtle changes of the focal element. This is extraordinarily significant to expand the application range of evidence theory.

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