Measure of Noncompactness in Banach Algebra and Application to the Solvability of Integral Equations in BC(R+)

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Abstract: In this paper an attempt is made to prove a fixed point theorem for the product of two operators each of which satisfies a special conditions in Banach algebra, using the technique of measure of noncompactness. Also we show that how it can be used to investigate the solvability of integral equations.

Keywords: Measure of noncompactness, Fixed point.

1 Introduction

It is has been witnessed that the differential and integral equations that appear in many physical problems are generally nonlinear and fixed point theory presents a strong tool for obtaining the solutions of such equations which otherwise are hard to solve by other ordinary procedures (for example, see [1,2,3]). In this paper, we analyze solvability of a certain functional-integral equation which consist of many special cases of integral and functional-integral equations, which are applicable in various real world problems of engineering, economics, physics and similar fields (see [4,5]). Indeed, we are going to investigate the solvability of the integral equation

$$x(t) = \big((Tx)(t)\big)\big(f(t,x(t)) + \int_0^t g(t,s,x(s))ds\big), \quad (1)$$

where $t \in \mathbb{R}_+$ and T is an operator acting from the Banach algebra $BC(\mathbb{R}_+)$ consisting of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ which are continuous and bounded on $\mathbb{R}_+$ into itself and the functions f, g are continuous and satisfy certain conditions. Eq.1 includes many known integral equations as model cases. In the case $Tx \equiv 1$ the equation 1 turns into

$$x(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))ds,$$

which has been investigated in [6]. What we are going to achieve in this paper, will extend the findings already

obtained in [8,9,10,12,14]. The main tool used in our investigation is the technique associated with measure of noncompactness. For a discussion of existence of solution to the above-mentioned integral equation, the used measure of noncompactness must also satisfy an additional condition. Indeed, we will use a class of measures of noncompactness which satisfies a condition called (m). Such a condition will guarantee the solvability of operator equations in Banach algebra. It is worthwhile mentioning that the important measures of noncompactness in notable spaces satisfy condition (m) (see [8,9,10,12,18]). This condition had first been used in for Hausdorff measure of noncompactness in Banach algebra $C(I)$ consisting of real continuous functions defined on a closed and bounded interval I (see [15]).

2 Preliminaries

For this reason, suppose that E is a given Banach space which has the norm $\|.\|$ and zero element θ. If the closed ball in E is centered at x and has radius r, we show it by $B(x,r)$. In order to show $B(\theta,r)$, we write B_r. If X is a subset of E, in that case, we can show the closure and the closed convex hull of X with the symbols $\overline{X}$ and $ConvX$ respectively. Also $X + Y$ and λX $(\lambda \in \mathbb{R})$ are used to show the algebraic operation on sets. Moreover, by the

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symbol $\|X\|$ we will denote the norm of a bounded set X, i.e, $\|X\| = \sup\{\|x\| : \quad x \in X\}$.

Furthermore $\mathfrak{M}_E$ is used to denote the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ denote its subfamily includes all relatively compact sets.

Definition 1([11]). *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be measure of noncompactness in E if it is satisfies the following conditions*

(1) *The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.*
(2) $X \subseteq Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(X)$.
(4) $\mu(ConvX) = \mu(X)$.
(5) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$.
(6) *If (X_n) is a nested sequence of closed sets from $\mathfrak{M}_E$ such that $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^\infty X_n$ is nonempty.*

Observe that the intersection set X_∞ from axiom (6) is a member of the $\ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for any n, we have that $\mu(X_\infty) = 0$. This yields that $X_\infty \in \ker\mu$ (see [13]). Now suppose Banach space E has the structure of Banach algebra. For given subsets X and Y of a Banach algebra E, let

$$XY = \{xy : x \in X, y \in Y\}.$$

Definition 2. *We state that measure of noncompactness μ which has been defined in Banach algebra E satisfies the condition (m), if for arbitrary sets $X, Y \in \mathfrak{M}_E$, the following inequality is satisfied*

$$\mu(XY) \le \|X\|\mu(Y) + \|Y\|\mu(X).$$

Now we present an example of a measure of noncompactness in Banach algebra which satisfies condition (m). Let us consider the Banach space $BC(\mathbb{R}_+)$ consisting of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ which are continuous and bounded on $\mathbb{R}_+$. This space is endowed with the standard norm $\|x\| = \sup\{\|x(t)\| : \quad t \in \mathbb{R}_+\}$. Obviously $BC(\mathbb{R}_+)$ has also the structure of Banach algebra with the standard multiplication of functions. In addition, fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ and numbers $\varepsilon > 0$ and $L > 0$. For an arbitrary function $x \in X$ let us denote by $\omega^L(x, \varepsilon)$ the modulus of continuity of x on the interval $[0, L]$, i.e.

$$\omega^L(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, L], [t-s] \le \varepsilon\}.$$

In addition

$$\omega^L(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},$$
$$\omega_0^L(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon),$$
$$\omega_0^\infty(X) = \lim_{L \to \infty} \omega_0^L(X, \varepsilon).$$

Moreover, if $t \in \mathbb{R}_+$ is a fixed number, let us denote

$$X(t) = \{x(t) : x \in X\},$$
$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\},$$
$$c(X) = \limsup_{t\to\infty} diamX(t).$$

With help of the above mappings we denote the following measures of noncompactness in $BC(\mathbb{R}_+)$ (cf. [16,17])

$$\mu_c(X) = \omega_0^\infty(X) + c(X). \qquad (2)$$

Theorem 1. *The measure of noncompactness μ_c defined by 2 satisfies condition (m) on the family of all nonempty and bounded subsets X of Banach algebra $BC(\mathbb{R}_+)$ such that functions belonging to X are nonnegative on $\mathbb{R}_+$.*

Proof. If $x, y \in C[a, b]$, $\varepsilon > 0$ then for $t, s \in [a, b]$ such that $|t - s| \le \varepsilon$, we get

$$\begin{aligned}
|x(t)y(t) - x(s)y(s)| &\le |x(t)y(t) - x(t)y(s)| \\
&\quad + |x(t)y(s) - x(s)y(s)| \\
&\le |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\
&\le \|x\|\omega(y, \varepsilon) + \|y\|\omega(x, \varepsilon).
\end{aligned}$$

As a result

$$\omega(xy, \varepsilon) \le \|x\|\omega(y, \varepsilon) + \|y\|\omega(x, \varepsilon).$$

So $\omega_0^\infty(X)$ satisfies condition (m). Now, fix arbitrarily sets $X; Y \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Choose arbitrary functions $z_1, z_2 \in XY$. This means that there exist functions $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $z_1 = x_1y_1, z_2 = x_2y_2$. Next, for $t \in \mathbb{R}_+$ we get

$$\begin{aligned}
|z_1(t) - z_2(t)| &= |x_1(t)y_1(t) - x_2(t)y_2(t)| \\
&\le |x_1(t)y_1(t) - x_1(t)y_2(t)| \\
&\quad + |x_1(t)y_2(t) - x_2(t)y_2(t)| \\
&= |x_1(t)||y_1(t) - y_2(t)| + |y_2(t)||x_1(t) - x_2(t)| \\
&\le \|X\|diamY(t) + \|Y\|diamX(t).
\end{aligned}$$

Hence we obtain $diam(X(t)Y(t)) \le \|X\|diamY(t) + \|Y\|diamX(t)$ and consequently $c(XY) \le \|X\|c(Y) + \|Y\|c(X)$. So, that the measure of noncompactness μ_c satisfies condition (m).

In order to achieve the main purpose of this paper, the following theorem plays a crucial role

Theorem 2([7]). *Let Ω be a bounded, nonempty, convex and closed subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

Obviously the above formulated theorem constitutes the well know Schauder fixed point principle.

3 Main result

Now it is time to put forward the main theorem of this paper.

Theorem 3.*Assume that Ω is a nonempty, bounded, closed and convex subset of the Banach algebra E and the operators P and T continuously transform the set Ω into E such that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, we assume that the operator $S = P.T$ transform Ω into itself.*

If the operator P and T on the set Ω satisfy the following conditions

$$\begin{cases} \mu(P(X)) \leq \psi_1(\mu(X)), \\ \mu(T(X)) \leq \psi_2(\mu(X)), \end{cases}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of noncompactness satisfying condition (m) and $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions such that

$$\begin{cases} \lim_{n\to\infty} \psi_1^n(t) = 0, \\ \lim_{n\to\infty} \psi_2^n(t) = 0, \\ \lim_{n\to\infty} \left(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1 \right)^n(t) = 0, \end{cases}$$

for any $t \geq 0$, then S has at least fixed point in the set Ω.

*Proof.*Let us take an arbitrary nonempty subset X of the set Ω. Then in view of the assumption that μ satisfies condition (m) we obtain

$$\begin{aligned} \mu(S(X)) &\leq \mu(P(X).T(X)) \\ &\leq \|P(X)\|\mu(T(X)) + \|T(X)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\mu(T(X)) + \|T(\Omega)\|\mu(P(X)) \\ &\leq \|P(\Omega)\|\psi_2(\mu(X)) + \|T(\Omega)\|\psi_1(\mu(X)) \\ &= \left(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1 \right)(\mu(X)). \end{aligned} \quad (3)$$

Now, letting $\varphi(t) = \left(\|P(\Omega)\|\psi_2 + \|T(\Omega)\|\psi_1 \right)(t)$, then from **??** , we have $\mu(S(X)) \leq \varphi(\mu(X))$. Now, with regard to the fact that ψ_1 and ψ_2 are nondecreasing, we conclude φ is nondecreasing and in view of $\lim_{n\to\infty} \varphi^n(t) = 0$, we can apply main result in [6] , to get the desired result. But for the convenience of the reader, we add the scheme of proof of aforementioned theorem. We define sequence Ω_n as $\Omega_0 = \Omega$, $\Omega_n = ConvS\Omega_{n-1}$ for $n \geq 1$. Furthermore we assume $\mu(\Omega_n) > 0$ for all $n = 1, 2, \dots$. Keeping this condition in mind, we get

$$\begin{aligned} \mu(\Omega_{n-1}) &= \mu(ConvS\Omega_n) \\ &= \mu(S\Omega_n) \\ &\leq \varphi(\mu(\Omega_n)) \\ &\leq \varphi^2(\mu(\Omega_{n-1})) \\ &\leq \dots \\ &\leq \varphi^n(\mu(\Omega)). \end{aligned}$$

This showed that $\mu(\Omega_n) \to 0$ as $n \to \infty$. Now, we can use axiom 6 of definition of measure of noncompactness and

conclude that $\Omega_\infty = \bigcap_{n=1}^\infty \Omega_n$ is nonempty, convex and closed subset of the set Ω. Moreover it is noteworthy that Ω_∞ is compact. With regard to the above discussion Schauder fixed point principle guarantees the existence of a fixed point for the operator S .

*Remark.*By letting

$$\begin{cases} \psi_1(t) = k_1, & 0 \leq k_1 < 1 \\ \psi_2(t) = k_2, & 0 \leq k_2 < 1 \end{cases}$$

in Theorem 3, we obtain a special case of above theorem which has already been studied in([9, 10, 12, 18]), where the application of that special case in the existence of solutions of many integral equation has been investigated .

4 Application

In this section we use the main theorem of this paper to prove the solvability of integral equation

$$x(t) = (Tx)(t)(f(t,x(t)) + \int_0^t g(t,s,x(s))ds), \quad t \in \mathbb{R}_+,$$

we define

$$(Fx)(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))ds, \quad t \in \mathbb{R}_+$$

where the operator T, F are defined on the Banach algebra $BC(\mathbb{R}_+)$. Notice that F represented the so-called Volterra integral operator. Now, we formulate the assumptions under which the equation 1 will be investigated. We will assume the following hypotheses:

$(I)T$ is an operator acting continuously from Banach algebra $BC(\mathbb{R}_+)$ into itself which satisfies the following condition

$$\mu_c(T(X)) \leq \psi_1(\mu_c(X))$$

for any nonempty subset X of Ω in which Ω is a nonempty, bounded, closed and convex subset of the Banach algebra $BC(\mathbb{R}_+)$ and $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \psi_1^n(t) = 0$ for any $t \geq 0$.

(II)There exists a constant b such that

$$\|Tx\| \leq \psi_1(\|x\|) + b$$

$(III)f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Moreover, $t \to f(t,0)$ is a member of the space $BC(\mathbb{R}_+)$.

(IV)There exists an upper semicontinuous function $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function such that $\lim_{n\to\infty} \psi_2^n(t) = 0$ for any $t \geq 0$, we have that

$$|f(t,x) - f(t,y)| \leq \psi_2(|x-y|), \quad t \in \mathbb{R}_+, \quad x,y \in \mathbb{R}.$$

Moreover, we assume that ψ_2 is superadditive i.e., for each $t, s, \in \mathbb{R}_+, \psi_2(t) + \psi_2(s) \leqslant \psi_2(t+s)$.

(V) $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $c,d : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} c(t) \int_0^t d(s)ds = 0$ and $|g(t,s,x)| \leq c(t)d(s)$ for $t,s \in \mathbb{R}_+$ such that $s \leq t$, and for each $x \in \mathbb{R}$.

(VI) The inequality $(\psi_1(r) + b)(\psi_2(r) + q) \leq r$ has a positive solution r_0 in which q is constant and defined as

$$q = \sup\left\{ |f(t,0)| + c(t) \int_0^t d(s)ds \} : t \geq 0 \right\}.$$

Moreover, the number r_0 is such that $\big((\psi_2(r_0) + q)\psi_1 + (\psi_1(r_0) + b)\psi_2\big)(t) < t$ for $t \in \mathbb{R}_+$.

The following lemma is necessary to prove the theorem 4.

Lemma 1([6]). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following two conditions are equivalent*

$(1)\lim_{n\to\infty} \varphi^n(t) = 0$ *for each $t \geq 0$.*
$(2)\varphi(t) < t$ *for any $t > 0$.*

Theorem 4. *Under the assumptions (I) to (VI), the integral equation 1 has at least one solution in the space $BC(\mathbb{R}_+)$.*

Proof. We define the operator A as follows

$$(Ax)(t) = (Tx)(t)(Fx)(t).$$

With regard to the above assumptions, the functions Tx and Fx are continuous functions on $\mathbb{R}_+$ for any $x \in BC(\mathbb{R}_+)$. For an arbitrary fixed function $x \in BC(\mathbb{R}_+)$, we have

$$\begin{aligned}
|(Ax)(t)| &= |(Tx)(t)||(Fx)(t)| \\
&\leq (\psi_1(\|x\|) + b)(|f(t,x(t)) - f(t,0)| \\
&\quad + |f(t,0)| + |g(t,s,x(s))|ds) \\
&\leq (\psi_1(\|x\|) + b)(\psi_2(|x(t)|) \\
&\quad + |f(t,0)| + |g(t,s,x(s))|ds) \\
&\leq (\psi_1(\|x\|) + b)(\psi_2(|x(t)|) \\
&\quad + |f(t,0)| + c(t) \int_0^t d(s)ds) \\
&\leq (\psi_1(\|x\|) + b)(\psi_2(|x(t)|) + q).
\end{aligned}$$

So, we get

$$\|Ax\| \leq (\psi_1(\|x\|) + b)(\psi_2(\|x(t)\|) + q),$$

in which b and q are constant, defined in assumptions(II), (IV). So A maps the space $BC(\mathbb{R}_+)$ into itself. Moreover based of assumption (IV), we conclude that A maps the ball B_{r_0} into itself in which r_0 is a constant appearing in assumption (VI). Now we show that operator A is continuous on the ball B_{r_0}. To do this, let us first observe that the continuity of the operator T on the ball B_{r_0} is an easy consequence of the assumptions (I), (II), (VI).

Thus, it suffices to show that the operator F is continuous on B_{r_0}. Fix an arbitrary $\varepsilon > 0$ and $x,y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. So we can conclude

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq \psi_2(|x(t) - y(t)|) \\
&\quad + \int_0^t |g(t,s,x(s)) - g(t,s,y(s))|ds \\
&\leq \psi_2(|x(t) - y(t)|) \\
&\quad + \int_0^t |g(t,s,x(s))|ds \\
&\quad + \int_0^t |g(t,s,y(s))|ds \\
&\leq \psi_2(\varepsilon) + 2k(t), \quad (4)
\end{aligned}$$

where we denoted

$$k(t) = c(t) \int_0^t d(s)ds.$$

Further, in view of assumption (V), we deduce that there exists a number $L > 0$ such that

$$2k(t) = 2c(t) \int_0^t d(s)ds \leq \varepsilon, \quad (5)$$

for each $t \geq L$. Thus, taking into account Lemma 1 and linking 5 and 4, for an arbitrary $t \geq L$ we get

$$|(Fx)(t) - (Fy)(t)| \leq 2\varepsilon. \quad (6)$$

Now, we define the quantity $\omega^L(g,\varepsilon)$ as follows

$$\begin{aligned}
\omega^L(g,\varepsilon) = \sup\{|g(t,s,x) - g(t,s,y)| : \\
t,s \in [0,L], x,y \in [-r_0,r_0], \|x - y\| \leq \varepsilon\}.
\end{aligned}$$

Now with regard to the fact that the function $g(t,s,x)$ is uniformly continuous on the set $[0,L] \times [0,L] \times [-r_0,r_0]$, so

$$\lim_{\varepsilon\to 0} \omega^L(g,\varepsilon) = 0.$$

By considering 4 for an arbitrary fixed $t \in [0,L]$, we conclude that

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq \psi_2(\varepsilon) + \int_0^L \omega^L(g,\varepsilon)ds \\
&= \psi_2(\varepsilon) + L\omega^L(g,\varepsilon). \quad (7)
\end{aligned}$$

Combining 6 and 7, it is possible to conclude that the operator F is continuous on the ball B_{r_0}. Now, let X be an arbitrary nonempty subset of the ball B_{r_0}. Fix numbers $\varepsilon > 0$ and $L > 0$. Next, choose $t,s \in [0,L]$ such that $\|t - s\| \leq \varepsilon$. Without loss of generality, we assume that

$s < t$. Then, for $x \in X$ we conclude

$$|Fx)(t) - (Fx)(s)| \le |f(t,x(t)) - f(s,x(s))|$$
$$+ |\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau|$$
$$\le |f(t,x(t)) - f(s,x(t))| + |f(s,x(t)) - f(s,x(s))|$$
$$+ |\int_0^t g(t,\tau,x(\tau))d\tau - \int_0^t g(s,\tau,x(\tau))d\tau|$$
$$+ \int_0^t g(s,\tau,x(\tau))d\tau - \int_0^s g(s,\tau,x(\tau))d\tau|$$
$$\le \omega_1^L(f,\varepsilon) + \psi_2(|x(t) - x(s)|)$$
$$+ \int_0^t |g(t,\tau,x(\tau)) - g(s,\tau,x(\tau))|d\tau$$
$$+ \int_s^t |g(s,\tau,x(\tau))|d\tau$$
$$\le \omega_1^L(f,\varepsilon) + \psi_2(\omega^L(x,\varepsilon))$$
$$+ \int_0^t \omega_1^L(g,\varepsilon)d\tau + c(s)\int_s^t d(\tau)d\tau$$
$$\le \omega_1^L(f,\varepsilon) + \psi_2(\omega^L(x,\varepsilon))$$
$$+ L\omega_1^L(g,\varepsilon) + \varepsilon \sup\{c(s)d(t) : t,s \in [0,L]\} \quad (8)$$

where we denote

$$\omega_1^L(f,\varepsilon) = \sup\{|f(t,x) - f(s,x)| :$$
$$t,s \in [0,L], x \in [-r_0,r_0], |t-s| < \varepsilon\},$$
$$\omega_1^L(g,\varepsilon) = \sup\{|g(t,t,x) - g(s,t,x)| :$$
$$t,s,t \in [0,L], x \in [-r_0,r_0], |t-s| < \varepsilon\}.$$

Now with regard to the fact that f is uniformly continuous on the set $[0,L] \times [-r_0,r_0]$ and g is uniformly continuous on the set $[0,L] \times [0,L] \times [-r_0,r_0]$, we can conclude $\omega_1^L(f,\varepsilon) \to 0$ and $\omega_1^L(g,\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, since $c = c(t)$ and $d = d(t)$ are continuous on $\mathbb{R}_+$, the quantity $\sup\{c(s)d(t) : t,s \in [0,L]\}$ is finite. From 8, we conclude

$$\omega_0^L(FX) \le \lim_{\varepsilon \to 0} \psi_2(\omega^L(X,\varepsilon)).$$

Now with regard to the fact that ψ_2 is upper semicontinuous, so

$$\omega_0^L(FX) \le \psi_2(\omega_0^L(X)),$$

and so

$$\omega_0^\infty(FX) \le \psi_2(\omega_0^\infty(X)). \quad (9)$$

Now we choose two arbitrary functions $x, y \in X$. Then for $t \in \mathbb{R}$ we have

$$|(Fx)(t) - (Fy)(t)| \le |f(t,x(t)) - f(t,y(t))|$$
$$+ \int_0^t |g(t,s,x(s))|ds + \int_0^t |g(t,s,y(s))|ds$$
$$\le \psi_2(|x(t) - y(t)|)$$
$$+ 2c(t)\int_0^t d(s)ds$$
$$\le \psi_2(|x(t) - y(t)|) + 2k(t).$$

This estimate allows us to get the following one

$$diam(FX)(t) \le \psi_2(diamX(t)) + 2k(t).$$

Now with regard to the upper semicontinuity of the functions ψ_2 we obtain

$$c(FX) = \limsup_{t\to\infty} diam(FX)(t) \le \psi_2(\limsup_{t\to\infty} diamX(t)) = \psi_2(c(X)). \quad (10)$$

So, combining 9 and 10, we can conclude

$$\mu_c(FX) = \omega_0^\infty(FX) + c(FX)$$
$$= \omega_0^\infty(FX) + \limsup_{t\to\infty} diam(FX)(t)$$
$$\le \psi_2(\omega_0^\infty(X(t))) + \psi_2(\limsup_{t\to\infty} diam(X)(t))$$
$$\le \psi_2(\omega_0^\infty(X(t)) + \limsup_{t\to\infty} diam(X)(t))$$
$$\le \psi_2(\omega_0^\infty(X(t)) + c(X))$$

or, equivalently

$$\mu_c(FX) \le \psi_2(\mu_c(X)),$$

moreover, by considering assumption (I) we have

$$\mu_c(TX) \le \psi_1(\mu_c(X)),$$

in which μ_c is the defined measure of noncompactness on the space $BC(\mathbb{R}_+)$. Also, we get

$$\|TB_{r_0}\| \le \psi_1(r_0) + b \quad , \quad \|FB_{r_0}\| \le \psi_2(r_0) + q.$$

So, according to assumption (VI), we have

$$(\|FB_{r_0}\|\psi_1 + \|TB_{r_0}\|\psi_2)(t) < ((\psi_2(r_0) + q)\psi_1 + (\psi_1(r_0) + b)\psi_2)(t)$$
$$< t \quad \text{for all} \quad t \in \mathbb{R}_+ \quad (11)$$

Now, linking 11 and lemma 1 we get

$$\lim_{n\to\infty} (\|FB_{r_0}\|\psi_1 + \|TB_{r_0}\|\psi_2)^n(t) = 0.$$

Thus, all the conditions of Theorem 4 hold. Therefore Eq.1 has at least one solution in the space $BC(\mathbb{R}_+)$.

5 Example

*Example 1.*Consider the following functional integral equation

$$x(t) = \left(\frac{t^2}{1+t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{1+|\cos x(s)|}ds\right)$$
$$\times \left(\frac{t^2}{5+5t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{3+|\cos x(s)|}ds\right), \quad (12)$$

we define

$$(Tx)(t) = \frac{t^2}{1+t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{1+|\cos x(s)|}ds,$$

$$(Fx)(t) = \frac{t^2}{5+5t^4}\ln(1+|x(t)|) + \int_0^t \frac{se^{-t}\sin x(s)}{3+|\cos x(s)|}ds.$$

Now, we show that all the conditions of Theorem 4 are satisfied for the functional integral equation 12. To do so, first we checked out whether condition (IV) and (V) are satisfied. Similar fashion, by putting $\psi_1(t) = \frac{1}{4}\ln(1+t)$ condition (I) and (II) satisfied for the operator T, too. Moreover, we put
$f(t,x) = \frac{t^2}{1+t^6}\ln(1+|x|), g(t,s,x) = \frac{se^{-t}sinx}{1+|cosx|}$ and $\psi_2(t) = \frac{1}{5}\ln(1+t)$. obviously, ψ_2 is nondecreasing and concave on $\mathbb{R}_+$ and $\psi_2(t) < t$ for all $t > 0$. In addition, for arbitrarily fixed $x,y \in \mathbb{R}_+$ such that $|x| \geq |y|$ and for $t > 0$ we get

$$
\begin{aligned}
|f(t,x) - f(t,y)| &= \frac{1}{5}\frac{t^2}{2+2t^4}\ln\left(\frac{1+|x|}{1+|y|}\right) \\
&\leq \frac{1}{5}\ln\left(1 + \frac{|x|-|y|}{1+|y|}\right) \\
&< \frac{1}{5}\ln\left(1 + |x-y|\right) \\
&< \frac{1}{4}\ln\left(1 + 2|x-y|\right) \\
&= \psi_2\left(|x-y|\right).
\end{aligned}
$$

The case $|y| \geq |x|$ can be dealt with in the same way. Conditions (III) of Theorem 4 are clearly evident. In addition, pay close attention that the function g is continuous and maps the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$. Also, we have

$$|g(t,s,x)| \leq e^{-t}s$$

for $t,s \in \mathbb{R}$ and $x \in \mathbb{R}$. So, if we put $c(t) = e^{-t}$, and $d(s) = s$, then we can see that assumption (V) is satisfied. Indeed, we have

$$\lim_{t\to\infty} c(t)\int_0^t d(s)ds = 0.$$

Now, let us calculate the constant q which appears in assumption (VI). We obtain

$$
\begin{aligned}
q &= \sup\{|f(t,0)| + c(t)\int_0^t d(s)ds : t \geq 0\} \\
&= \sup\{2t^2 e^{-t/2} : t \geq 0\} = 2e^{-2}.
\end{aligned}
$$

Just like the above way checked out condition (II), we get $b = 2e^2$ (also see [6]). Furthermore, we can check that the inequality from assumption (VI) takes the form

$$\left(\frac{1}{4}\ln(1+r) + b\right)\left(\frac{1}{5}\ln(1+r) + q\right) < r.$$

It is obvious that this inequality has a positive solution r_0, say $r_0 = 1$. Moreover, we have
$\left((\psi_2(r_0) + q)\psi_1 + (\psi_1(r_0) + b)\psi_2\right)(t) < t$ for $t \in \mathbb{R}_+$. Consequently, all the conditions of Theorem 4 are satisfied. Therefore the functional integral equation 12 has at least one solution in the space $BC(\mathbb{R}_+)$.

References

[1] M. Kir and H.Kiziltunc, Fixed Point Theorems for Nonself Contraction Mappings in N-Normed Spaces, Inf. Sci. Lett. 3, No. 3, 111-115 (2014).

[2] A. Aghajani, M. Aliaskari, Generalization of Darbo's fixed point theorem and application. International Journal of Nonlinear Analysis and Applications, 5(2), (86-95), 2014.

[3] A. Aghajani, M. Aliaskari, Commuting mappings and generalization of Darbos fixed point theorem, Math. Sci. Lett. In press.

[4] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, New York, 1990.

[5] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[6] A. Aghajani, J. Banaś, N, Sabzali, Some generalizations of Darbo fixed point theorem and applictions, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 345-358.

[7] R. P. Agarwal, M. Meehan and D. O'Regan, Fixed Point Theory and Applications, Cambridge University Press, Cambridge, 2004.

[8] J. Caballero, A. B. Mingarelli, K. Sadarangani, Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer. Electronic Journal of Differential Equations, 2006. Chicago

[9] J. Banaś, S. Dudek, The Technique of Measures of Noncompactness in Banach Algebras and Its Applications to Integral Equations. In Abstract and Applied Analysis (Vol. 2013). Hindawi Publishing Corporation.

[10] J. Banaś, M. Lecko, Fixed points of the product of operators in Banach algebra, Panamerican Mathematical Journal, 12(2), (101109), 2002.

[11] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York, 1980.

[12] J. Banaś, L. Olszowy, On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations, Zeitschrift fur Analysis und ihre Anwendungen, 28(4), (475-498), 2009.

[13] J. Banas, On measures of noncompactness in Banach spaces, Comment. Math. Univ. Carolinae 21 (1980) 131-143.

[14] X. Yu, C. Zhu, J.Wang. On a weakly singular quadratic integral equations of Volterra type in Banach algebras. Advances in Difference Equations, 2014(1), 130.

[15] J. Banaś, M. Lecko, Fixed points of the product of operators in Banach algebra. Panamer. Math. J. 12 (2002), 101-109.

[16] J. Banaś,K. Goebel, Measures of Noncompactness in Banach Spaces. Lect. Notes Pure Appl. Math. 60. New York: Marcel Dekker 1980.

[17] J. Banaś, Measure of noncompactness in the space of continuous tempered functions. Demonstratio Math. 14 (1981), 127-133.

[18] H. K. Pathak, Study on existence of solutions for some nonlinear functional-integral equations with applications. Mathematical Communications, 18(1)(2013), 97-107.

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