Information Sciences Letters

Volume 3
Issue 2 *May 2014*

Article 2

2014

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Inf. Sci. Lett. **3**, No. 2, 45-53 (2014)

Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/030202

Dynamics of the Rational Difference Equation

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Received: 8 Dec. 2013, Revised: 17 Mar. 2014, Accepted: 18 Mar. 2014
Published online: 1 May 2014

Abstract: In this article, we study the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k} + hx_{n-l}}{dx_{n-k} + ex_{n-l}}, \qquad n = 0, 1, 2, \ldots..,$$

where the coefficients $A, B, C, D, b, d, e, h \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \ldots, x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$. We will prove that the equilibrium points of this equation are locally asymptotically stable, global attractor and hence they are global stability. This equation will have (or have not) prime period two solution under suitable conditions on these coefficients. The solutions of this equation will be proved to be bounded. Some numerical examples will be given to illustrate our results.

Keywords: Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability.

1 Introduction

The topic of difference equations is interesting and attractive to many mathematicians working in this field. It is a fertile research area. Many real life phenomena are modelling using these equations. Examples from economy, biology, etc. may be obtained in [11, 16, 17]. The study of some properties of these equations via the global attractivity, the boundedness and the periodicity of these equations is of great interest. For example, in the articles [2–10] [12–17] closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, (see [18–25], [27–29]) and the references cited therein.

The objective of this article is to investigate some qualitative behavior of the solutions of the nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k} + hx_{n-l}}{dx_{n-k} + ex_{n-l}}, \quad n = 0, 1, 2, \ldots.. \tag{1}$$

where the coefficients $A, B, C, D, b, d, e, h \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \ldots, x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$. Note that the special cases of Eq.(1) have been discussed in [1, 26, 30, 31].

Our interest now is to study the behavior of the solutions of Eq.(1) in its general form. For the related work, (see [32–37]). Let us now recall some well known results [11] which will be useful in the sequel.

Definition 1.*Consider a difference equation in the form*

$$x_{n+1} = F(x_n, x_{n-k}, x_{n-l}, x_{n-\sigma}), \quad n = 0, 1, 2, \ldots.. \tag{2}$$

where F is a continuous function, while k and l are positive integers such that $k < l < \sigma$. An equilibrium point $\tilde{x}$ of this equation is a point that satisfies the condition $\tilde{x} = F(\tilde{x}, \tilde{x}, \tilde{x}, \tilde{x})$. That is, the constant sequence $\{x_n\}$ with $x_n = \tilde{x}$ for all $n \geq -k \geq -l \geq \sigma$ is a solution of that equation.

Definition 2.*Let $\tilde{x} \in (0, \infty)$ be an equilibrium point of Eq.(2). Then we have*

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(i) An equilibrium point $\widetilde{x}$ of Eq.(2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-\sigma}, ..., x_{-l}, ..., x_{-k}, ..., x_{-1}, x_0 \in (0,\infty)$ with $|x_{-\sigma} - \widetilde{x}| + ... + |x_{-l} - \widetilde{x}| + ... + |x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \delta$, then $|x_n - \widetilde{x}| < \varepsilon$ for all $n \geq -k \geq -l$.

(ii) An equilibrium point $\widetilde{x}$ of Eq.(2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-\sigma}, ..., x_{-l}, ..., x_{-k}, ..., x_{-1}, x_0 \in (0,\infty)$ with $|x_{-\sigma} - \widetilde{x}| + ... + |x_{-l} - \widetilde{x}| + ... + |x_{-k} - \widetilde{x}| + ... + |x_{-1} - \widetilde{x}| + |x_0 - \widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of Eq.(2) is called a global attractor if for every $x_{-\sigma}, ..., x_{-l}, ..., x_{-k}, ..., x_{-1}, x_0 \in (0,\infty)$ we have

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of Eq.(2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of Eq.(2) is called unstable if it is not locally stable.

Definition 3.*A sequence $\{x_n\}_{n=-\sigma}^{\infty}$ is said to be periodic with period r if $x_{n+r} = x_n$ for all $n \geq -\sigma$. A sequence $\{x_n\}_{n=-\sigma}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.*

Definition 4.*Eq.(2) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-\sigma}, ..., x_{-l}, ..., x_{-k}, ..., x_{-1}, x_0 \in (0,\infty)$ there exists a positive integer N which depends on these initial conditions such that*

$$m \leq x_n \leq M \quad for\ all \quad n \geq N.$$

Definition 5.*The linearized equation of Eq.(2) about the equilibrium point $\widetilde{x}$ is defined by the equation*

$$z_{n+1} = \rho_0 z_n + \rho_1 z_{n-k} + \rho_2 z_{n-l} + \rho_3 z_{n-\sigma} = 0, \tag{3}$$

where

$$\rho_0 = \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial x_n}, \quad \rho_1 = \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial x_{n-k}},$$
$$\rho_2 = \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial x_{n-l}}, \quad \rho_3 = \frac{\partial F(\widetilde{x},\widetilde{x},\widetilde{x},\widetilde{x})}{\partial x_{n-\sigma}}.$$

The characteristic equation associated with Eq.(3) is

$$\rho(\lambda) = \lambda^{\sigma+1} - \rho_0 \lambda^{\sigma} - \rho_1 \lambda^{\sigma-k} - \rho_2 \lambda^{\sigma-l} - \rho_3 = 0. \tag{4}$$

Theorem 1.*[11]. Assume that F is a C^1- function and let $\widetilde{x}$ be an equilibrium point of Eq.(2). Then the following statements are true.*

(i) If all roots of Eq.(4) lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.

(ii) If at least one root of Eq.(4) has absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.

(iii) If all roots of Eq.(4) have absolute value greater than one, then the equilibrium point $\widetilde{x}$ is a source.

Theorem 2.*[17]. Assume that ρ_0, ρ_1, ρ_2 and $\rho_3 \in R$. Then*

$$|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| < 1, \tag{5}$$

is a sufficient condition for the asymptotic stability of Eq.(2).

Theorem 3.*[11]. Consider the difference equation (2). Let $\widetilde{x} \in I$ be an equilibrium point of Eq.(2). Suppose also that*

(i) F is a nondecreasing function in each of its arguments.

(ii) The function F satisfies the negative feedback property

$$[F(x,x,x,x) - x](x - \widetilde{x}) < 0 \quad for\ all \quad x \in I - \{\widetilde{x}\},$$

where I is an open interval of real numbers. Then $\widetilde{x}$ is global attractor for all solutions of Eq.(2).

2 The local stability of the solutions

The equilibrium point $\widetilde{x}$ of Eq.(1) is the positive solution of the equation

$$\widetilde{x} = (A+B+C+D)\widetilde{x} + \frac{(b+h)\ \widetilde{x}}{(d+e)\widetilde{x}}. \tag{6}$$

If $0 < A+B+C+D < 1$, then the only positive equilibrium point $\widetilde{x}$ of Eq.(1) is given by

$$\widetilde{x} = \frac{b\ +h}{(d+e)\left[1-(A+B+C+D)\right]}. \tag{7}$$

Let us now introduce a continuous function $F : (0,\infty)^4 \longrightarrow (0,\infty)$ which is defined by

$$F(u_0,u_1,u_2,u_3) = Au_0 + Bu_1 + Cu_2 + Du_3 + \frac{bu_1 + hu_2}{du_1 + eu_2}. \tag{8}$$

Consequently, we get

$$\begin{cases} \frac{\partial F(\tilde{x},\tilde{x},\tilde{x},\tilde{x})}{\partial u_0} = A = \rho_0, \\ \frac{\partial F(\tilde{x},\tilde{x},\tilde{x},\tilde{x})}{\partial u_1} = B + \frac{(be-dh)\,[1-(A+B+C+D)]}{(d+e)(b+h)} = \rho_1, \\ \frac{\partial F(\tilde{x},\tilde{x},\tilde{x},\tilde{x})}{\partial u_2} = C - \frac{(be-dh)[1-(A+B+C+D)]}{(d+e)(b+h)} = \rho_2, \\ \frac{\partial F(\tilde{x},\tilde{x},\tilde{x},\tilde{x})}{\partial u_3} = D = \rho_3. \end{cases} \tag{9}$$

Thus, the linearized equation of Eq.(1) about $\tilde{x}$ takes the form

$$z_{n+1} - \rho_0 z_n - \rho_1 z_{n-k} - \rho_2 z_{n-l} - \rho_3 z_{n-\sigma} = 0, \tag{10}$$

where ρ_0, ρ_1, ρ_2 and ρ_3 are given by (9).

Theorem 4.*(i) If $be-dh=0$, then the equilibrium point $\tilde{x}$ of Eq.(2) is locally asymptotically stable.*

(ii) If $be-dh \neq 0$, then equilibrium point $\tilde{x}$ of Eq.(2) is locally asymptotically stable if the following condition hold:

$$\begin{aligned} &|B(d+e)(b+h) + (be-dh)[1-(A+B+C+D)]| \\ &+ |C(d+e)(b+h) - (be-dh)[1-(A+B+C+D)]| \\ &< (d+e)(b+h)[1-(A+B)]. \end{aligned} \tag{11}$$

Proof.(i) If $be-dh=0$, we obtain

$$|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| = A+D+B+C < 1,$$

then we deduce from Theorem 2 that $\tilde{x}$ of Eq.(2) is locally asymptotically stable.

(ii) If $be-dh \neq 0$, we deduce from (9) and (11) that $|\rho_0| + |\rho_1| + |\rho_2| + |\rho_3| < 1$, and hence $\tilde{x}$ of Eq.(2) is locally asymptotically stable. Thus, the proof is now completed.□

3 Periodic solutions

In this section, we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that the equation (1) has periodic solutions of prime period two.

Theorem 5.*If k,l and σ are all even positive integers, then Eq.(1) has no prime period two solution.*

*Proof.*Assume that there exists distinct positive solutions

$$.......,P,Q,P,Q,........$$

of prime period two of Eq.(1). If k,l and σ are all even positive integers, then $x_n = x_{n-k} = x_{n-l} = x_{n-\sigma}$. It follows from Eq.(1) that

$$P = (A+B+C+D)Q + \frac{(b+h)}{(d+e)}, \tag{12}$$

and

$$Q = (A+B+C+D)P + \frac{(b+h)}{(d+e)}. \tag{13}$$

By subtracting (13) from (12), we get

$$(P-Q)[A+B+C+D+1] = 0.$$

Since $A+B+C+D+1 \neq 0$, then $P=Q$. This is a contradiction. Thus, the proof is now completed.□

Theorem 6.*If k,l and σ are all odd positive integers and $0 < A+B+C+D < 1$, then Eq.(1) has no prime period two solution.*

*Proof.*If k,l and σ are all odd positive integers, then $x_{n+1} = x_{n-k} = x_{n-l} = x_{n-\sigma}$. It follows from Eq.(1) that

$$P = AQ + (B+C+D)P + \frac{(b+h)}{(d+e)}, \tag{14}$$

and

$$Q = AP + (B+C+D)Q + \frac{(b+h)}{(d+e)}. \tag{15}$$

By subtracting (15) from (14), we get

$$(P-Q)[(A+1) - (B+C+D)] = 0.$$

Since $A+B+C+D < 1$, then $(A+1)-(B+C+D) > 2A > 0$, hence $P=Q$. This is a contradiction. Thus, the proof is now completed.□

Theorem 7.*If k,l are even and σ is odd positive integers and $0 < A+B+C+D < 1$,then Eq.(1) has no prime period two solution.*

*Proof.*If k,l are even and σ is odd positive integers, then $x_n = x_{n-k} = x_{n-l}$ and $x_{n+1} = x_{n-\sigma}$. It follows from Eq.(1) that

$$P = (A+B+C)Q + DP + \frac{(b+h)}{(d+e)}, \tag{16}$$

and

$$Q = (A+B+C)P + DQ + \frac{(b+h)}{(d+e)}. \tag{17}$$

By subtracting (17) from (16), we get

$$(P-Q)[(A+B+C+1) - D] = 0,$$

Since $A+B+C+D < 1$, then $(A+B+C+1) - D \neq 0$, hence $P=Q$. This is a contradiction. Thus, the proof is now completed.□

Theorem 8. *If σ is even and k,l are odd positive integers and $0<A+B+C+D<1$, then Eq.(1) has no prime period two solution.*

Proof. If σ is even and k,l are odd positive integers, then $x_n = x_{n-\sigma}$ and $x_{n+1} = x_{n-k} = x_{n-l}$. It follows from Eq.(1) that

$$P=(A+D)\,Q+(B+C)\,P+\frac{(b+h)}{(d+e)}, \tag{18}$$

and

$$Q=(A+D)\,P+(B+C)\,Q+\frac{(b+h)}{(d+e)}. \tag{19}$$

By subtracting (19) from (18), we get

$$(P-Q)\,[(A+D+1)-(B+C)]=0,$$

Since $A+B+C+D<1$, then $(A+D+1)-(B+C)\neq 0$, hence $P=Q$. This is a contradiction. Thus, the proof is now completed.□

Theorem 9. *If k is even and l,σ are odd positive integers, then Eq.(1) has prime period two solution if the condition*

$$(h-b)\,(d-e)\,\{(A+B)+[1-(C+D)]\} > 4\,\{hd\ (A+B)+be\ [1-(C+D)]\}, \tag{20}$$

is valid, provided $d>e$, $h>b$, $(C+D)<1$.

Proof. If k is even and l,σ are odd positive integers, then $x_n = x_{n-k}$ and $x_{n+1} = x_{n-l} = x_{n-\sigma}$. It follows from Eq.(1) that

$$P=(A+B)\,Q+(C+D)\,P+\frac{bQ+hP}{dQ+eP}, \tag{21}$$

and

$$Q=(A+B)\,P+(C+D)\,Q+\frac{bP+hQ}{dP+eQ}. \tag{22}$$

Consequently, we get

$$e\,P^2+dPQ=d\ (A+B)\,Q^2+e\ (A+B)\,PQ + (C+D)\,dPQ+e\,(C+D)\,P^2+bQ+hP, \tag{23}$$

and

$$e\,Q^2+dPQ=d\ (A+B)\,P^2+e\ (A+B)\,PQ + (C+D)\,dPQ+e\,(C+D)\,Q^2+bP+hQ. \tag{24}$$

By subtracting (24) from (23), we get

$$P+Q=\frac{h-b}{d\ (A+B)+e\ [1-(C+D)]}, \tag{25}$$

provided $h>b$, $(C+D)<1$. By adding (23) and (24), we obtain

$$PQ=\frac{(h-b)\,\{hd\,(A+B)+be\ [1-(C+D)]\}}{(d-e)\,[M_1+(A+B)]\,\{d\ (A+B)+e\,M_1\}^2}, \tag{26}$$

where $M_1=(1-(C+D))$, provided $d>e$, $h>b$, $(C+D)<1$. Assume that P and Q are two positive distinct real roots of the quadratic equation

$$t^2-(\,P+Q)\,t+PQ=0. \tag{27}$$

Thus, we deduce that

$$(P+Q)^2>4PQ. \tag{28}$$

Substituting (25) and (26) into (28), we get the condition (20). Thus, the proof is now completed.□

Theorem 10. *If l is even and k,σ are odd positive integers, then Eq.(1) has prime period two solution if the condition*

$$(b-h)\,(e-d)\,\{(A+C)+[1-(B+D)]\} > 4\,\{eb\ (A+C)+dh\ [1-(B+D)]\}, \tag{29}$$

is valid, provided $e>d$, $b>h$, $(B+D)<1$.

Proof. If l is even and k,σ are odd positive integers, then $x_n = x_{n-l}$ and $x_{n+1} = x_{n-k} = x_{n-\sigma}$. It follows from Eq.(1) that

$$P=(A+C)\,Q+(B+D)\,P+\frac{bP+hQ}{dP+eQ}, \tag{30}$$

and

$$Q=(A+C)\,P+(B+D)\,Q+\frac{bQ+hP}{dQ+eP}. \tag{31}$$

Consequently, we get

$$P+Q=\frac{b-h}{d\ [1-(B+D)]+e\ (A+C)}, \tag{32}$$

where $(B+D)<1, b>h$.

$$PQ=\frac{(b-h)\,\{e\,b\ (A+C)+dh\ [1-(B+D)]\}}{(e-d)\,\{(A+C)+M_2\}\,\{e\ (A+C)+d\,M_2\}^2}, \tag{33}$$

where $M_2=(1-(B+D))$, $(B+D)<1$. Substituting (32) and (33) into (28), we get the condition (29). Thus, the proof is now completed.□

Theorem 11. *If l,σ are even and k is odd positive integers, then Eq.(1) has prime period two solution if the condition*

$$(b-h)\,(e-d)\,\{(A+C+D)+(1-B)\} > 4\,\{eb\ (A+C+D)+dh\ (1-B)\}, \tag{34}$$

is valid, provided $e>d$, $b>h$, $B<1$.

Proof. If l,σ are even and k is odd positive integers, then $x_n = x_{n-l} = x_{n-\sigma}$ and $x_{n+1} = x_{n-k}$. It follows from Eq.(1) that

$$P=(A+C+D)\,Q+BP+\frac{bP+hQ}{dP+eQ}, \tag{35}$$

and

$$Q=(A+C+D)P+BQ+\frac{bQ+hP}{dQ+eP}. \quad (36)$$

Consequently, we get

$$P+Q=\frac{b-h}{d\ (1-B)+e\ (A+C+D)}, \quad (37)$$

where $b>h$, $B<1$,

$$PQ=\frac{(b-h)\{e\,b\ (A+C+D)+dh\ (1-B)\}}{(e-d)\{M_3+(1-B)\}\{e\,M_3+d\ (1-B)\}^2}, \quad (38)$$

where $M_3=(A+C+D)$, $e>d$, $b>h$, $B<1$. Substituting (37) and (38) into (28), we get the condition (34). Thus, the proof is now completed.□

Theorem 12.*If k,σ are even and l is odd positive integers, then Eq.(1) has prime period two solution if the condition*

$$(h-b)(d-e)\{(A+B+D)+(1-C)\}$$
$$>4\{eb\ (1-C)+dh\ (A+B+D)\}, \quad (39)$$

is valid, provided $d>e$, $h>b$, $C<1$.

*Proof.*If k,σ are even and l is odd positive integers, then $x_n=x_{n-k}=x_{n-\sigma}$ and $x_{n+1}=x_{n-l}$. It follows from Eq.(1) that

$$P=(A+B+D)Q+CP+\frac{bQ+hP}{dQ+eP}, \quad (40)$$

and

$$Q=(A+B+D)P+CQ+\frac{bP+hQ}{dP+eQ}. \quad (41)$$

Consequently, we get

$$P+Q=\frac{h-b}{e\ (1-C)+d\ (A+B+D)}, \quad (42)$$

where $h>b$, $C<1$,

$$PQ=\frac{(h-b)\{dh\ (A+B+D)+e\,b\ (1-C)\}}{(d-e)\{(1-C)+M_4\}\{d\,M_4+e\ (1-C)\}^2}, \quad (43)$$

where $M_4=(A+B+D)$, $d>e$, $h>b$, $C<1$. Substituting (42) and (43) into (28), we get the condition (39). Thus, the proof is now completed.□

4 Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of Eq.(1).

Theorem 13.*If $0<A+B+C+D<1$, then every solution of Eq.(1) is bounded from above by the quantity*

$$x_0+\left(1+\sum_{i=1}^{n-1}A^i\right)\left(\frac{b\,e+dh}{de}\right),$$

where $A<1$, and x_0 is an initial condition.

*Proof.*Let $\{x_n\}_{n=-\sigma}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$\begin{aligned}x_{n+1}&=Ax_n+Bx_{n-k}+Cx_{n-l}+Dx_{n-\sigma}+\frac{bx_{n-k}+hx_{n-l}}{dx_{n-k}+ex_{n-l}}\\&\le Ax_n+Bx_{n-k}+Cx_{n-l}+Dx_{n-\sigma}+\frac{b}{d}+\frac{h}{e}.\end{aligned}$$

Since the initial conditions satisfy the inequality $x_{-\sigma}<\ldots<x_{-l}<\ldots<x_{-k}<\ldots<x_{-1}<x_0$,then we get

$$x_1\le(A+B+C+D)x_0+\left(\frac{b}{d}+\frac{h}{e}\right),$$

and hence

$$x_1\le x_0+\left(\frac{be+hd}{de}\right). \quad (44)$$

With the aid of (44) we have

$$\begin{aligned}x_2&\le Ax_1+Bx_{1-k}+Cx_{1-l}+Dx_{1-\sigma}+\left(\frac{be+hd}{de}\right)\\&\le(A+B+C+D)x_0+(A+1)\left(\frac{be+hd}{de}\right),\end{aligned}$$

and hence

$$x_2\le x_0+(A+1)\left(\frac{be+hd}{de}\right). \quad (45)$$

With the aid of (45) we have

$$\begin{aligned}x_3&\le Ax_2+Bx_{2-k}+Cx_{2-l}+Dx_{2-\sigma}+\left(\frac{be+hd}{de}\right)\\&\le(A+B+C+D)x_0+(A+1)\left(\frac{be+hd}{de}\right),\end{aligned}$$

and hence

$$x_3\le x_0+\left[A^2+A+1\right]\left(\frac{be+hd}{de}\right). \quad (46)$$

By continuing this process, we deduce that

$$x_n\le x_0+\left(1+\sum_{i=1}^{n-1}A^i\right)\left(\frac{b\,e+dh}{de}\right), \qquad n=1,2,3,\ldots.. \quad (47)$$

where we define $\sum_{i=1}^{0}A^i=0$. Then every solution of Eq.(1) is bounded from above by the quantity

$$x_0+\left(1+\sum_{i=1}^{n-1}A^i\right)\left(\frac{b\,e+dh}{de}\right).$$

Thus, the proof is now completed.□

5 Global stability

In this section, we study the global asymptotic stability of the positive solutions of Eq.(1).

Theorem 14. *If $0 < A + B + C + D < 1$, then the equilibrium point $\widetilde{x}$ given by (7) of Eq.(1) is global attractor.*

Proof. We consider the following function

$$F(x,y,z,w) = Ax + By + Cz + Dw + \frac{by + hz}{dy + ez}. \tag{48}$$

It is easy to verify the condition (i) of Theorem 3. Let us now verify the condition (ii) of Theorem 3 as follows:

$$\begin{aligned}
[F(x,x,x,x) - x](x - \widetilde{x}) &= \left\{ (A+B+C+D)x + \frac{b+h}{d+e} - x \right\} \\
&\times \left\{ x - \frac{b+h}{(d+e)[1-(A+B+C+D)]} \right\} \\
&= -\left\{ [1-(A+B+C+D)]x - \frac{b+h}{d+e} \right\} \\
&\times \left\{ x - \frac{b+h}{(d+e)[1-(A+B+C+D)]} \right\} \\
&= -[1-(A+B+C+D)] \\
&\times \left\{ x - \frac{b+h}{(d+e)[1-(A+B+C+D)]} \right\}^2 .
\end{aligned} \tag{49}$$

Since $0 < A+B+C+D < 1$, then we deduce from (49) that

$$[F(x,x,x,x) - x](x - \widetilde{x}) < 0. \tag{50}$$

According to Theorem 3, $\widetilde{x}$ is global attractor. Thus, the proof is now completed.□

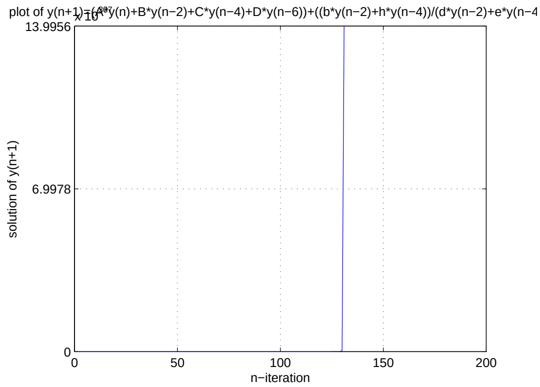
On combining the two Theorems 4 and 14, we have the following result:

Theorem 15. *The equilibrium point $\widetilde{x}$ given by (7) of Eq.(1) is globally asymptotically stable.*

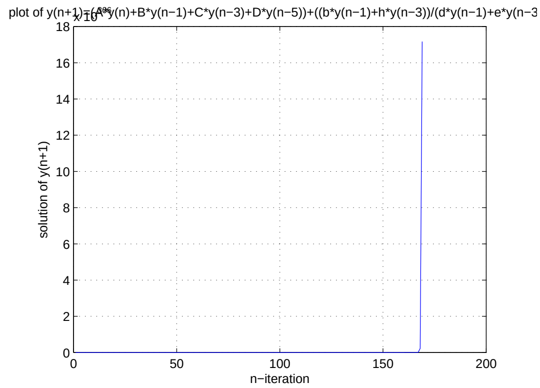
6 Numerical examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq.(1).

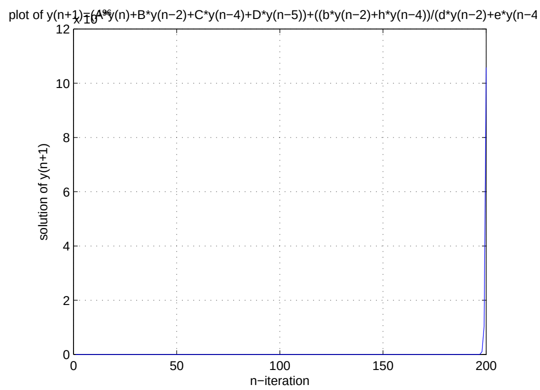
Example 1. Figure 1, shows that Eq.(1) has no prime period two solutions if $k = 2$, $l = 4$, $\sigma = 6$, $x_{-6} = 1$, $x_{-5} = 2$, $x_{-4} = 3$, $x_{-3} = 4$, $x_{-2} = 5$, $x_{-1} = 6$, $x_0 = 7$, $A = 300$, $B = 200$, $C = 100$, $D = 75$, $b = 50$, $h = 40$, $d = 30$, $e = 20$.



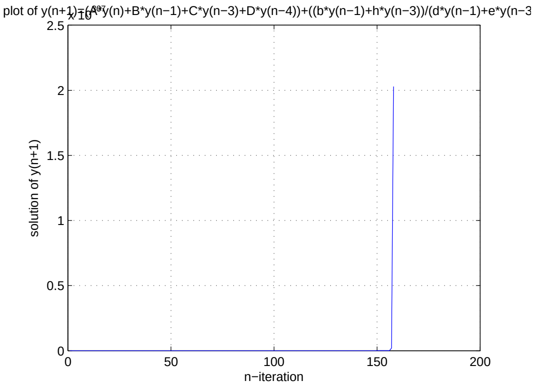
Example 2. Figure 2, shows that Eq.(1) has no prime period two solutions if $k = 1$, $l = 3$, $\sigma = 5$, $x_{-5} = 1$, $x_{-4} = 2$, $x_{-3} = 3$, $x_{-2} = 4$, $x_{-1} = 5$, $x_0 = 6$, $A = 75$, $B = 50$, $C = 25$, $D = 20$, $b = 5$, $h = 4$, $d = 3$, $e = 2$.



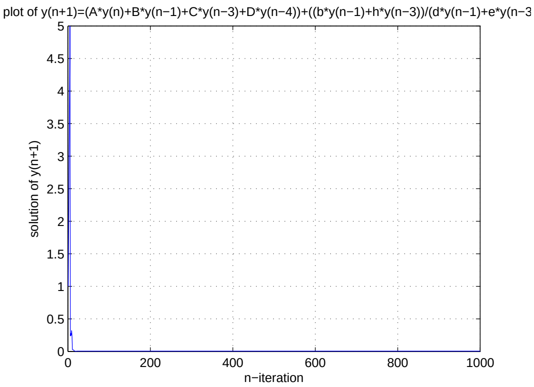
Example 3. Figure 3, shows that Eq.(1) has no prime period two solutions if $k = 2$, $l = 4$, $\sigma = 5$, $x_{-5} = 1$, $x_{-4} = 2$, $x_{-3} = 3$, $x_{-2} = 4$, $x_{-1} = 5$, $x_0 = 6$, $A = 10$, $B = 20$, $C = 30$, $D = 40$, $b = 5$, $h = 4$, $d = 3$, $e = 2$.



Example 4. Figure 4, shows that Eq.(1) has no prime period two solutions if $k=1,\ l=3,\ \sigma=4,\ x_{-4}=1,\ x_{-3}=2,\ x_{-2}=3,\ x_{-1}=4,\ x_0=5,\ A=100,\ B=50,\ C=25,\ D=200,\ b=5,\ h=4,\ d=3,\ e=2.$



Example 5. shows that Eq.(1) is globally asymptotically stable if $k=1,\ l=3,\ \sigma=4,\ x_{-4}=1,\ x_{-3}=2,\ x_{-2}=3,\ x_{-1}=4,\ x_0=5,\ A=0.01,\ B=0.02,\ C=0.03,\ D=0.04,\ b=5,\ h=4,\ d=3,\ e=2000.$



7 Conclusion

We have discussed some properties of the nonlinear rational difference equation (1), namely the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions. Our results in this article can be considered as a more generalization than the results obtained in Refs.[1,26,30,31]. Note that example 1 verifies Theorem 5 which shows that if k,l and σ are all even positive integers, then Eq.(1) has no prime period two solution and example 2 verifies Theorem 6 which shows that if k,l and σ are all odd positive integers, then Eq.(1) has no prime period two solution and example 3 verifies Theorem 7 which shows that if k,l are even and σ is odd positive integers, then Eq.(1) has no prime period two solution. But example 4 verifies Theorem 8 which shows that if σ is even and k,l are odd positive integers, then Eq.(1) has no prime period two solution, while example 5 verifies Theorem 15 which shows that Eq.(1) is globally asymptotically stable.

Acknowledgement

The authors wish to thank the referees for their suggestions and comments.

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