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Some Discussion on Behaviors of Markov Q-Process

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Abstract: Consider the limiting probability function of continuous-time Markov Branching Processes conditioned to be never extinct. Hereupon we receive a new stochastic population process as a continuous-time Markov chain called the Markov Q-Process. We study main properties of Markov Q-Process. The principal aim is to investigate asymptotic properties of Markov Q-Process. We investigate transition functions of this process and their convergence to stationary measures.

Keywords: Markov Branching Process; Markov Q-process; transition function; invariant measures; ergodic chain.
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1 Introduction

Considering a population of monotype individuals we will interested in its evolution. These individuals may be biological kinds, molecules in chemical reactions etc. Suppose the population size changes by random reproduction law as following. Each individual existing at epoch $t \in \mathscr{T} = [0; +\infty)$, independently of his history and of each other for a small time interval $(t; t+\varepsilon)$ transforms into $j \in \mathbb{N}_0 \backslash \{1\}$ individuals with probability $a_j\varepsilon + o(\varepsilon)$ and, with probability $1 + a_1\varepsilon + o(\varepsilon)$ each individual survives or makes evenly one descendant (as $\varepsilon \downarrow 0$); $\mathbb{N}_0 = \{0\} \cup \{\mathbb{N} = 1, 2, \ldots\}$. Here the numbers $\{a_j\}$ mean the evolution intensities of individuals that $a_j \geq 0$ for $j \in \mathbb{N}_0 \backslash \{1\}$ and $0 < a_0 < -a_1 = \sum_{j \in \mathbb{N}_0 \backslash \{1\}} a_j < \infty$. Appeared new individuals undergo transformations under same way as above. Letting $Z(t)$ be the population size at the moment t, we have the homogeneous continuous-time Markov Branching Process (MBP) which was first considered by Kolmogorov and Dmitriev [13].

The process $Z(t)$ is a Markov chain with the state space on $\mathbb{N}_0$ and transition functions

$$P_{ij}(t) := \mathsf{P}_i\{Z(t) = j\} = \mathsf{P}\{Z(t+\tau) = j \,|\, Z(\tau) = i\},$$

satisfying the branching property

$$P_{ij}(t) = \sum_{j_1 + \cdots + j_i = j} P_{1j_1}(t) \cdot P_{1j_2}(t) \cdots P_{1j_i}(t). \tag{1.1}$$

Probabilities P_{1j} in (1.1) are calculated using the local densities $\{a_j\}$ by relation

$$P_{1j}(\varepsilon) = \delta_{1j} + a_j\varepsilon + o(\varepsilon), \quad as \ \ \varepsilon \downarrow 0, \tag{1.2}$$

where δ_{ij} is the Kronecker's delta function. A Probability Generating Functions (PGF) version of the relation (1.2) is

$$F(\varepsilon; s) = s + f(s) \cdot \varepsilon + o(\varepsilon), \quad as \ \ \varepsilon \downarrow 0,$$

for all $0 \leq s < 1$, where

$$F(t; s) = \sum_{j \in \mathbb{N}_0} P_{1j}(t) s^j \qquad and \qquad f(s) = \sum_{j \in \mathbb{N}_0} a_j s^j.$$

Owing to Markovian property the PGF

$$F_i(t; s) := \sum_{j \in \mathbb{N}_0} P_{ij}(t) s^j = [F(t; s)]^i, \quad for\ all \ \ i \in \mathbb{N}. \tag{1.3}$$

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Assuming $a := f'(1)$ is finite and using the equation (1.4) we have $\mathsf{E}_i Z(t) = \sum_{j\in\mathbb{N}_0} jP_{ij}(t) = ie^{at}$. The last formula shows that long-term properties of MBP seem variously depending on parameter a. Hence the MBP is classified as critical if $a = 0$ and sub-critical or super-critical if $a < 0$ or $a > 0$ respectively. Monographs [2], [5], [19] are general references for mentioned and other classical facts on theory of MBP.

Throughout this paper we write $\mathsf{P}\{*\}$ and $\mathsf{E}[*]$ instead of $\mathsf{P}_1\{*\}$ and $\mathsf{E}_1[*]$ respectively.

Let random variable $\mathscr{H} := \inf\{t \in \mathscr{T} : Z(t) = 0\}$ be a hitting time of the zero state of MBP. By extinction theorem $\mathsf{P}_i\{\mathscr{H} < \infty\} = q^i$, where $q = \lim_{t\to\infty} P_{10}(t)$ is an extinction probability of MBP which is the least non-negative root of $f(s) = 0$. Moreover $\lim_{t\to\infty} F(t;s) = q$ uniformly by $0 \le s \le r < 1$. Let's consider the conditioned distribution function $\mathsf{P}_i^{\mathscr{H}(t)}\{*\} := \mathsf{P}_i\{*|t < \mathscr{H} < \infty\}$. It is known that if $a \le 0$ then $q = 1$. Therefore in this case $\mathsf{P}_i^{\mathscr{H}(t)}\{*\} = \mathsf{P}_i\{*|\mathscr{H} > t\}$ and

$$\mathsf{P}\{t < \mathscr{H} < \infty\} = \mathsf{P}\{\mathscr{H} > t\} \equiv \mathsf{P}\{Z(t) > 0\}.$$

On the other hand in this case $0 \le P_{1j}(t) \le \mathsf{P}\{\mathscr{H} > t\} \to 0$ as $t \to \infty$. But ratio $P_{1j}(t)/\mathsf{P}\{\mathscr{H} > t\}$ has a limiting finite law. So long-term properties of non-supercritical MBP are traditionally investigated on non-zero trajectories, that is under condition of event $\{\mathscr{H} > t\}$. Sevastyanov [18] proved that in the sub-critical case there is a limiting distribution law $\lim_{t\to\infty} \mathsf{P}^{\mathscr{H}(t)}\{Z(t) = j\}$ if and only if $\sum_{j\in\mathbb{N}} a_j j \ln j < \infty$. In the critical situation he also proved that if $2b := f''(1) < \infty$, then $Z(t)/bt$ has a limiting exponential law. In this case Chistyakov [3] proved that if $f^{(4)}(1) < \infty$ and j/bt is bounded, then $t \cdot \mathsf{P}^{\mathscr{H}(t)}\{Z(t) = j\} = 1/b + O\left(\sqrt{\ln t/t}\right)$ as $t \to \infty$. The author [6] improved this result being on the condition of $b < \infty$ only.

More interesting phenomenon arises if we observe the limit of conditioned distribution $\mathsf{P}_i^{\mathscr{H}(t+\tau)}\{*\}$ letting $\tau \to \infty$. In discrete-time situation this limit represents a distribution measure, which defines homogeneous Markov chain called the Q-process; see [2, pp. 56–60]. The Q-process was considered first by Lamperti and Ney [14]. Some properties of it were discussed by Pakes [15], [16], [17], Imomov [7], [9], [10], [11], Formanov and Imomov [4]. The considerable part of the paper of Klebaner, Rösler and Sagitov [12] is devoted to discussion of this process from the viewpoint of branching transformation called the Lamperti-Ney transformation. A closer look shows that in MBP case the limit $\lim_{\tau\to\infty} \mathsf{P}_i^{\mathscr{H}(t+\tau)}\{Z(t) = j\}$ has an honest probability measures $\mathbf{Q}(t) = \left\{\mathscr{Q}_{ij}(t)\right\}$ which defines the homogeneous continuous-time stochastic process as Markov chain with state space on $\mathbb{N}$. This process is called in [8] the Markov Q-Process. Let $W(t)$ to be the state size at the moment $t \in \mathscr{T}$ in Markov Q-Process. Then $W(0) \overset{d}{=} Z(0)$ and

$$\mathsf{P}_i\{W(t) = j\} = \mathscr{Q}_{ij}(t).$$

In the mentioned paper [8] some asymptotic properties of distribution of $W(t)$ are observed. Namely it was proved that if the corresponding MBP is critical, then $W(t)/\mathsf{E}W(t)$ has a limiting Erlang's law. In this case there is an invariant measure if second moment of PGF $f(s)$ is finite. In the non-critical situation under at some moment condition, there exists an invariant distribution for the process $W(t)$.

In Section 2 we define the Markov Q-Process and discuss properties concerning its construction and its transition function $\mathbf{Q}(t)$. In the Section 3 an ergodic property of $\mathbf{Q}(t)$ will be observed.

2 Construction of Markov Q-Process

In this section we will interested in the limiting interpretation of conditioned transition function $\mathsf{P}_i^{\mathscr{H}(t+\tau)}\{Z(t) = j\}$ letting $\tau \to \infty$ and for all fixed $t \in \mathscr{T}$. First by formula of full probability we write

$$\mathsf{P}_i\{t < \mathscr{H} < \infty,\ Z(t) = j\} = \mathsf{P}\{t < \mathscr{H} < \infty | Z(t) = j\} \cdot P_{ij}(t).$$

Since the probability of extinction of j particles is q^j then it follows that

$$\mathsf{P}_i\{t < \mathscr{H} < \infty,\ Z(t) = j\} = P_{ij}(t) \cdot q^j. \tag{2.1}$$

Using the formula (3.1) from last relation we receive that

$$\mathsf{P}_i\{t < \mathscr{H} < \infty\} = \sum_{j\in\mathbb{N}} \mathsf{P}_i\{Z(t) = j,\ t < \mathscr{H} < \infty\} = \sum_{j\in\mathbb{N}} P_{ij}(t) q^j. \tag{2.2}$$

Relation (2.1) implies

$$\mathsf{P}_i\{Z(t)=j,\ t+\tau<\mathscr{H}<\infty\} = P_{ij}(t)\cdot\sum_{k\in\mathbb{N}}\mathsf{P}_j\{\tau<\mathscr{H}<\infty,\ Z(\tau)=k\}$$
$$= P_{ij}(t)\cdot\sum_{k\in\mathbb{N}}P_{jk}(\tau)q^k.$$

Therefore considering identity (2.2) we have

$$\mathsf{P}_i^{\mathscr{H}(t+\tau)}\{Z(t)=j\} = P_{ij}(t)\cdot\frac{\sum_{k\in\mathbb{N}}\frac{P_{jk}(\tau)}{P_{11}(\tau)}q^k}{\sum_{j\in\mathbb{N}}\frac{P_{ij}(t+\tau)}{P_{11}(t+\tau)}q^j}\cdot\frac{P_{11}(\tau)}{P_{11}(t+\tau)}.$$

Using the ratio limit property [6, Lemma 7] and after short calculation it follows that

$$\lim_{\tau\to\infty}\mathsf{P}_i^{\mathscr{H}(t+\tau)}\{Z(t)=j\} = \frac{jq^{j-i}}{i\beta^t}P_{ij}(t) =: \mathscr{Q}_{ij}(t),$$

where as before $\beta=\exp\{f'(q)\}$. It is easy to be convinced that $0<\beta\le 1$ decidedly. To wit $\beta=1$ if $a=0$ and $\beta<1$ otherwise. Since $F'(t;q)=\beta^t$

$$\sum_{j\in\mathbb{N}}\mathscr{Q}_{ij}(t) = \sum_{j\in\mathbb{N}}\frac{jq^{j-i}}{i\beta^t}P_{ij}(t) = \frac{F_i'(t;q)}{iq^{i-1}\beta^t} = 1,$$

so we have an honest probability measure $\mathbf{Q}(t)=\left\{\mathscr{Q}_{ij}(t)\right\}$. This measure defines a new stochastic process $W(t)$, $t\in\mathscr{T}$, called Markov Q-Process (MQP) to be the homogeneous continuous-time Markov chain with the state space $\mathscr{E}\subseteq\mathbb{N}$; see [8]. In consequence of the Markovian nature of this process the transition functions $\mathscr{Q}_{ij}(t)$ satisfy the Kolmogorov-Chapman equations:

$$\mathscr{Q}_{ij}(t+\varepsilon) = \sum_{k\in\mathscr{E}}\mathscr{Q}_{ik}(\varepsilon)\mathscr{Q}_{kj}(t). \tag{2.3}$$

Thus the random function $W(t)$ denotes the state size at the moment $t\in\mathscr{T}$ in MQP, so

$$\mathscr{Q}_{ij}(t) = \mathsf{P}_i\{W(t)=j\} = \frac{jq^{j-i}}{i\beta^t}P_{ij}(t). \tag{2.4}$$

Considering together equalities (1.2) and (2.4) entail the following important representation for transition functions $\mathscr{Q}_{1j}(\varepsilon)$:

$$\mathscr{Q}_{1j}(\varepsilon) = \delta_{1j}+p_j\varepsilon+o(\varepsilon), \quad \textit{as}\ \ \varepsilon\downarrow 0, \tag{2.5}$$

with probability densities

$$p_0=0,\quad p_1=a_1-\ln\beta,\quad \textit{and}\quad p_j=jq^{j-1}a_j\ge 0\quad \textit{for}\ \ j\in\mathscr{E}\backslash\{1\},$$

where $\left\{a_j\right\}$ are evolution intensities of MBP $Z(t)$. It follows from (2.5) that PGF of intensities $\left\{p_j\right\}$ has the form of

$$g(s) := \sum_{j\in\mathscr{E}}p_js^j = s\left[f'(qs)-f'(q)\right]. \tag{2.6}$$

We see that $g(1)=0$, so the infinitesimal PGF $g(s)$ completely defines the process $W(t)$, where $\left\{p_j\right\}$ are intensities of process evolution that $p_j>0$ for $j\in\mathscr{E}\backslash\{1\}$ and

$$0<-p_1=\sum_{j\in\mathscr{E}\backslash\{1\}}p_j<\infty.$$

In the following theorem we discuss basic properties of transition matrix $\mathbf{Q}(t)=\left\{\mathscr{Q}_{ij}(t)\right\}$. Herewith we will follow methods and facts from monograph of Anderson [1].

Theorem 1. *The transition matrix $\mathbf{Q}(t)$ of the MQP is standard and honest. Its components $\mathscr{Q}_{ij}(t)$ are positive and uniformly continuous functions of $t\in\mathscr{T}$ for all $i,j\in\mathscr{E}$.*

Proof. According to the branching property (1.1) for chain $Z(t)$, we see

$$P_{ij}(\varepsilon) = \delta_{ij}+ia_{j-i+1}\varepsilon+o(\varepsilon), \quad \textit{as}\ \ \varepsilon\downarrow 0.$$

Hence seeing representation (2.4)

$$\begin{cases} \mathcal{Q}_{ii}(\varepsilon) = 1 + (ia_1 - \ln\beta)\,\varepsilon + o(\varepsilon), \\ \mathcal{Q}_{ij}(\varepsilon) = jq^{j-i}a_{j-i+1}\varepsilon + o(\varepsilon), \end{cases} \quad as \ \ \varepsilon \downarrow 0, \tag{2.7}$$

for all $i, j \in \mathscr{E}$. It follows from (2.7) that

$$\begin{aligned} \sum_{j\in\mathscr{E}} \left|\mathcal{Q}_{ij}(\varepsilon) - \delta_{ij}\right| &= \sum_{j\in\mathscr{E}\setminus\{i\}} \mathcal{Q}_{ij}(\varepsilon) + |\mathcal{Q}_{ii}(\varepsilon) - 1| \\ &= \sum_{j\in\mathscr{E}\setminus\{i\}} \mathcal{Q}_{ij}(\varepsilon) + 1 - \mathcal{Q}_{ii}(\varepsilon) \\ &\le 2\,|1 - \mathcal{Q}_{ii}(\varepsilon)| \to 0, \quad as \ \ \varepsilon \downarrow 0. \end{aligned}$$

So $\mathcal{Q}_{ij}(t)$ is standard. Easily to be convinced that a PGF version of (2.4) is

$$G_i(t;s) := \mathsf{E}_i s^{W(t)} = \sum_{j\in\mathscr{E}} \mathcal{Q}_{ij}(t)s^j = \frac{qs}{i\beta^t}\left[\frac{\partial}{\partial x}\left(\frac{F(t;x)}{q}\right)^i\right]_{x=qs},$$

or more obviously that

$$G_i(t;s) = \left[\frac{F(t;qs)}{q}\right]^{i-1} G(t;s), \tag{2.8}$$

where

$$G(t;s) := G_1(t;s) = \frac{s}{\beta^t}\left.\frac{\partial F(t;x)}{\partial x}\right|_{x=qs}.$$

It is known that $F(t;q) = q$ and $F'(t;q) = \beta^t$; see [19, pp. 52–53]. In our presupposition the MBP is honest. Therefore it follows from (2.8) that $\sum_{j\in\mathscr{E}} \mathcal{Q}_{ij}(t) = G_i(t;1) = 1$.

A positiveness of functions $\mathcal{Q}_{ij}(t)$ is obvious owing to (2.7). Supposing $\varepsilon > 0$ it follows from equation (2.3) that

$$\begin{aligned} \mathcal{Q}_{ij}(t+\varepsilon) - \mathcal{Q}_{ij}(t) &= \sum_{k\in\mathscr{E}} \mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t) \\ &= \sum_{k\in\mathscr{E}\setminus\{i\}} \mathcal{Q}_{ik}(\varepsilon)\mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t)\cdot[1 - \mathcal{Q}_{ii}(\varepsilon)]. \end{aligned}$$

The last relation gives

$$\begin{aligned} -[1 - \mathcal{Q}_{ii}(\varepsilon)] &\le -\mathcal{Q}_{ij}(t)\cdot[1 - \mathcal{Q}_{ii}(\varepsilon)] \\ &\le \mathcal{Q}_{ij}(t+\varepsilon) - \mathcal{Q}_{ij}(t) \le \sum_{k\in\mathscr{E}\setminus\{i\}} \mathcal{Q}_{ik}(t)\mathcal{Q}_{kj}(\varepsilon) \\ &\le \sum_{k\in\mathscr{E}\setminus\{i\}} \mathcal{Q}_{kj}(\varepsilon) = 1 - \mathcal{Q}_{ii}(\varepsilon), \end{aligned}$$

so $\left|\mathcal{Q}_{ij}(t+\varepsilon) - \mathcal{Q}_{ij}(t)\right| \le 1 - \mathcal{Q}_{ii}(\varepsilon)$. Similarly

$$\begin{aligned} \left|\mathcal{Q}_{ij}(t-\varepsilon) - \mathcal{Q}_{ij}(t)\right| &= \left|\mathcal{Q}_{ij}(t) - \mathcal{Q}_{ij}(t-\varepsilon)\right| \\ &\le 1 - \mathcal{Q}_{ii}(t - (t-\varepsilon)) = 1 - \mathcal{Q}_{ii}(\varepsilon). \end{aligned}$$

Therefore we obtain $\left|\mathcal{Q}_{ij}(t+\varepsilon) - \mathcal{Q}_{ij}(t)\right| \le 1 - \mathcal{Q}_{ii}(|\varepsilon|)$ for any $\varepsilon \ne 0$ and for all $i, j \in \mathscr{E}$. The obtained relation implies that $\mathcal{Q}_{ij}(t)$ is uniformly continuous function of $t \in \mathscr{T}$ because $\lim_{\varepsilon\downarrow 0} \mathcal{Q}_{ii}(\varepsilon) = 1$ for all $i \in \mathscr{E}$. □

It can easily be seen that a PGF version of the relation (2.5) is

$$G(\varepsilon;s) = s + g(s)\cdot\varepsilon + o(\varepsilon), \quad \begin{array}{l} as\ \varepsilon \downarrow 0\ and \\ for\ all\ 0 \le s < 1. \end{array} \tag{2.9}$$

By the way according to formulas (1.3) and (2.8) one can see that the PGF $G(t;s)$ satisfies the following functional equation:

$$G(t+\tau;s) = \frac{G\left(t;\widehat{F}(\tau;s)\right)}{G\left(0;\widehat{F}(\tau;s)\right)} G(\tau;s), \tag{2.10}$$

where $\widehat{F}(t;s) = F(t;qs)/q$ is the PGF of sub-critical MBP. Using formularizations (2.9) and (2.10) for the difference $\Delta_\varepsilon G(t;s) = G(t-\varepsilon;s) - G(t+\varepsilon;s)$ yields that

$$\Delta_\varepsilon G(t;s) = [some\ function]\cdot\varepsilon + o(\varepsilon), \quad as\ \ \varepsilon\downarrow 0,$$

for any $t\in\mathscr{T}$ and all $0\le s<1$, which implies that $G(t;s)$ is differentiable. It has been shown in [8] that

$$G(t;s) = s\exp\left\{\int_0^t h\left(\widehat{F}(\tau;s)\right)d\tau\right\}, \tag{2.11}$$

where $h(s) = g(s)/s$.

3 Classification and Ergodic behavior of transition functions

Note that evolution of MQP is ruled in essence by the positive parameter β. Afterwards we will convinced that two types of processes will be subdivided depending on value of this parameter. Putting together (2.8) and (2.11) we write

$$G_i(t;s) = s\left[\widehat{F}(t;s)\right]^{i-1}\exp\left\{\int_0^t h\left(\widehat{F}(\tau;s)\right)d\tau\right\}. \tag{3.1}$$

Let $\alpha := g'(1)$ is finite. Direct differentiating in point $s=1$, it follows from (3.1) that

$$\mathsf{E}_i W(t) = (i-1)\beta^t + \mathsf{E}W(t)$$

and

$$\mathsf{E}W(t) = \begin{cases} 1+\gamma(1-\beta^t), & when\ \ \beta<1,\\ \alpha t+1 \qquad\quad , & when\ \ \beta=1. \end{cases} \tag{3.2}$$

Moreover we obtain the variance structure

$$\mathsf{Var}_i W(t) = \begin{cases} [\gamma+(i-1)(1+\gamma)\beta^t](1-\beta^t), & when\ \ \beta<1,\\ \alpha it \qquad\qquad\qquad\qquad\qquad , & when\ \ \beta=1. \end{cases} \tag{3.3}$$

Where $\gamma=\alpha/|\ln\beta|$ and $\mathsf{Var}_i W(t) = \mathsf{Var}[W(t)|W(0)=i]$ in (3.3).

The formula (3.2) implies that when $\beta=1$

$$\mathsf{E}_i W(t) \sim \alpha t, \quad as\ \ t\to\infty,$$

and if $0<\beta<1$

$$\mathsf{E}_i W(t) \to 1+\gamma, \quad as\ \ t\to\infty.$$

So in the case of $\beta=1$ the MQP has transience property.

We classify the MQP as *restrictive* if $\beta<1$ and *explosive* if $\beta=1$.

Theorem 2. *The MQP is*

(i) *positive if it is restrictive and $\alpha := g'(1)$ is finite;*

(ii) *null if it is explosive.*

Proof. To prove the assertion (i) from (2.11) we get

$$\ln\mathscr{Q}_{11}(t) = \int_0^t h\left(\widehat{F}(\tau;0)\right)d\tau = \int_0^{\widehat{F}(t;0)}\frac{h(x)}{\widehat{f}(x)}dx \to \int_0^1\frac{h(x)}{\widehat{f}(x)}dx,$$

since $\widehat{F}(t;0)\uparrow 1$ as $t\to\infty$, where $\widehat{f}(s) = f(qs)/q$. Herein we used the fact that $\lim_{s\downarrow 0}[G(t;s)/s] = \mathscr{Q}_{11}(t)$. The condition $\alpha<\infty$ implies that integral in right-hand side converges. Hence $\lim_{t\to\infty}\mathscr{Q}_{11}(t)>0$. For part (ii) we recall that in this case $q=1$ and $h(s)=f'(s)$ if $\beta=1$. Similarly

$$\ln\mathscr{Q}_{11}(t) = \int_0^t h(F(\tau;0))d\tau = \int_0^{F(t;0)}\frac{h(x)}{f(x)}dx \to \int_0^1\frac{f'(x)}{f(x)}dx = -\infty.$$

So that $\lim_{t\to\infty}\mathscr{Q}_{11}(t)=0$. □

Now let's recall the following assertion.

Lemma [6]. *The following assertions are valid.*

–*Let $a \neq 0$. Then*

$$\frac{\partial F(t;s)}{\partial s} = \frac{|f'(q)|}{f(s)} \mathscr{A}(s) \cdot \beta^t (1 + o(1)), \quad as \ t \to \infty, \tag{3.4}$$

where

$$\mathscr{A}(s) = (q - s) \exp\left\{ \int_s^q \left[\frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\}. \tag{3.5}$$

–*Let $a = 0$. If the second moment $f''(1) =: 2b$ is finite, then*

$$\frac{\partial F(t;s)}{\partial s} = \frac{b(1-s)^2}{f(s)\left[bt(1-s)+1\right]^2} (1 + o(1)), \quad as \ t \to \infty. \tag{3.6}$$

Putting together (3.1) and (3.4)–(3.6) and considering that $\lim_{t\to\infty} \widehat{F}(t;s) = 1$ uniformly for all $0 \le s \le r < 1$, we obtain following theorem.

Theorem 3. *Let $\alpha := g'(1)$ is finite.*

(i) *If MQP is restrictive, then*

$$G_i(t;s) = s\frac{|f'(q)|}{f(qs)} \mathscr{A}(qs) (1 + o(1)), \quad as \ t \to \infty, \tag{3.7}$$

where the function $\mathscr{A}(s)$ has the form of (3.5).

(ii) *If MQP is explosive, then*

$$G_i(t;s) = s\frac{2\alpha}{f(s)} \left[\frac{(1-s)}{(1-s)\alpha t + 2} \right]^2 (1 + o(1)), \quad as \ t \to \infty. \tag{3.8}$$

Since $\mathscr{Q}_{11}(t) = \lim_{s\downarrow 0} \left[G(t;s)/s \right]$, it follows from (3.7) and (3.8) the following local limit theorem.

Theorem 4. *Let $\alpha := g'(1)$ is finite.*

(i) *If MQP is restrictive, then*

$$\mathscr{Q}_{11}(t) = \frac{|\ln \beta|}{a_0} \mathscr{A}(0) (1 + o(1)), \quad as \ t \to \infty,$$

(ii) *If MQP is explosive, then*

$$t^2 \mathscr{Q}_{11}(t) = \frac{2}{a_0 \alpha} \left(1 + O\left(\frac{1}{t} \right) \right), \quad as \ t \to \infty.$$

Further we observe limit properties of $\left\{ \mathscr{Q}_{ij}(t) \right\}$ for all $i, j \in \mathscr{E}$. For the general MQP the following ratio limit property holds.

Theorem 5. *The limits*

$$\lim_{t\to\infty} \frac{\mathscr{Q}_{ij}(t)}{\mathscr{Q}_{11}(t)} = \omega_j \tag{3.9}$$

exist for all $i, j \in \mathscr{E}$, and these determined by the PGF

$$\mathscr{U}(s) = \sum_{j\in\mathscr{E}} \omega_j s^j = s \exp\left\{ \int_0^s \frac{|h(x)|}{m(x)} dx \right\}, \tag{3.10}$$

where $h(s) = g(s)/s$ and $m(s) = s \ln \beta + \int_0^s h(x) dx$. The limiting PGF $\mathscr{U}(s)$ converges for all $0 \le s < 1$.

Proof. Let's consider the PGF

$$\mathscr{U}_i(t;s) = \sum_{j\in\mathscr{E}} \frac{\mathscr{Q}_{ij}(t)}{\mathscr{Q}_{11}(t)} s^j = \frac{1}{\mathscr{Q}_{11}(t)} G_i(t;s) = \left[\widehat{F}(t;s) \right]^{i-1} \mathscr{U}(t;s), \tag{3.11}$$

where

$$\mathscr{U}(t;s) = \sum_{j\in\mathscr{E}} \frac{\mathscr{Q}_{1j}(t)}{\mathscr{Q}_{11}(t)} s^j.$$

It follows from (3.11) that it suffice to consider the case $i=1$ because $\widehat{F}(t;s)\uparrow 1$ as $t\to\infty$ uniformly for all $0\le s\le r<1$. So write

$$\mathscr{U}(t;s)=s\exp\left\{\int_0^t\left[h\left(\widehat{F}(u;s)\right)-h\left(\widehat{F}(u;0)\right)\right]du\right\}.$$

One can choose $\tau\in\mathscr{T}$ for any $0\le s<1$ so that $s=\widehat{F}(\tau;0)$. On the other hand we know that $\widehat{F}\left(t;\widehat{F}(\tau;0)\right)=\widehat{F}(t+\tau;0)$; [19, p. 24]. Therefore we obtain equalities

$$\begin{aligned}\mathscr{U}(t;s)&=s\exp\left\{\int_\tau^{t+\tau}h\left(\widehat{F}(u;0)\right)du-\int_0^t h\left(\widehat{F}(u;0)\right)du\right\}\\&=s\exp\left\{\int_0^\tau\left[h\left(\widehat{F}(t;\widehat{F}(u;0))\right)-h\left(\widehat{F}(u;0)\right)\right]du\right\}\\&=s\exp\left\{\int_0^s\frac{h\left(\widehat{F}(t;x)\right)-h(x)}{\widehat{f}(x)}dx\right\},\end{aligned}$$

where $\widehat{f}(s):=f(qs)/q$. In the last step we have used the Kolmogorov backward equation

$$\frac{\partial F(t;s)}{\partial t}=f(F(t;s)),\quad \text{for all } 0<s<1;$$

see [19, pp. 27–30]. We see that $\widehat{f}(s)$ is equal to $m(s)$. To get to (3.10) it suffice to take limit as $t\to\infty$ in obtained relation for $\mathscr{U}(t;s)$ being that $\widehat{F}(t;s)\to 1$ and $h(1)=0$. Assertion (3.9) follows now from continuity theorem for PGF. Lastly it is easily to be convinced that $\mathscr{U}(s)<\infty$ for all $0\le s<1$. □

Aggregating Theorems 4 and 5, yields the following

Theorem 6. *Let $\alpha:=g'(1)$ is finite.*

(i) *If MQP is restrictive, then*

$$\mathscr{Q}_{ij}(t)=\omega_j\frac{|\ln\beta|}{a_0}\mathscr{A}(0)(1+o(1)),\quad as\ t\to\infty.$$

(ii) *If MQP is explosive, then*

$$t^2\mathscr{Q}_{ij}(t)=\omega_j\frac{2}{a_0\alpha}\left(1+O\left(\frac{1}{t}\right)\right),\quad as\ t\to\infty.$$

Now using the Kolmogorov-Chapmen equation (2.3) we obtain that

$$\frac{\mathscr{Q}_{ij}(t+\tau)}{\mathscr{Q}_{11}(t+\tau)}\cdot\frac{\mathscr{Q}_{11}(t+\tau)}{\mathscr{Q}_{11}(t)}=\sum_{k\in\mathscr{E}}\frac{\mathscr{Q}_{ik}(t)}{\mathscr{Q}_{11}(t)}\mathscr{Q}_{kj}(\tau).$$

On the other hand setting $s=0$ in (2.10) we can see that $\mathscr{Q}_{11}(t+\tau)/\mathscr{Q}_{11}(t)\to 1$ as $t\to\infty$. Hence we get the following invariance equation for $\{\omega_j\}$:

$$\omega_j=\sum_{k\in\mathscr{E}}\omega_k\mathscr{Q}_{kj}(t),\quad for\ all\ t\in\mathscr{T}. \tag{3.12}$$

The PGF version of (3.12) is

$$\mathscr{U}\left(\widehat{F}(t;s)\right)=\frac{\widehat{F}(t;s)}{G(t;s)}\mathscr{U}(s),\quad for\ 0\le s<1,$$

the functional equation of generalized Schroeder form. So the set $\{\omega_j\}$ to be the ergodic invariant measure for MQP.

We complete the paper stating the following limit theorem.

Theorem 7. *Let $\alpha:=g'(1)$ is finite.*

(i) *If MQP is restrictive, then the variable $W(t)$ tends in mean square and with probability one to the random variable W having the finite mean and variance:*

$$\mathsf{E}W=1+\gamma\qquad and\qquad \mathsf{Var}W=\gamma.$$

(ii) *If MQP is explosive, then for any $x>0$*

$$\lim_{t\to\infty}\mathsf{P}\left\{\frac{W(t)}{\mathsf{E}W(t)}\le x\right\}=1-e^{-2x}-2xe^{-2x}.$$

For *Proof* see [8].

4 Conclusion remark

The paper is devoted to research of the population process which is defined as the long-living continuous-time Markov branching process. This is the homogenous Markov chain and called the Markov Q-process (MQP). In a discrete-time situation a same process was defined in [2]. In our case the process was considered first by author [8]. We see that the structural parameter $\beta = f'(q)$ enters a role of the regulating one. In fact the long-time behaviors of MQP depend on this parameter and unlike the branching process this is classified only two types. In research of transition functions $\mathcal{Q}_{ij}(t)$ we essentially use asymptotic properties of the first derivative of PGF of Markov Branching process. Ratio limit property (Theorem 5) for transition functions states an existence of invariant measure for MQP without any moment assumptions. The Theorem 7 shows the limit properties of states of process. In our subsequent researches the considered model will be spread to the age-depended Bellman-Harris process case.

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