Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.12785/jsap/040206

Bayesian Inference for The Inverse Exponential Distribution Based on Pooled Type-II Censored Samples

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Received: 26 Feb. 2015, Revised: 2 Jun. 2015, Accepted: 3 Jun. 2015
Published online: 1 Jul. 2015

Abstract: In this paper, the maximum likelihood and Bayesian estimation are developed based on pooled sample of two independent Type-II censored samples from the inverse exponential distribution. The Bayesian estimation is discussed using different loss functions. The problem of predicting the failure times from a future sample from the sample population is also discussed from a Bayesian viewpoint. A Monte Carlo simulation study is conducted to compare the maximum likelihood estimator with the Bayesian estimators. Finally, illustrative example is presented to illustrate the different inference methods discussed here.

Keywords: Bayesian estimation; Bayesian prediction; Pooled Type-II censored samples; Inverse exponential distribution; Maximum likelihood estimation.

1 Introduction

In reliability analysis, experiments often get terminated before all units on test fail based on cost and time considerations. In such cases, failure information is available only on part of the sample, and only partial information on all units that had not failed. Such data are called censored data. There are several forms of censored data. One of the most common forms of censoring is Type-II right censoring which can be described as follows: Consider n identical units under observation in a life-testing experiment and suppose only the first $r \leq n$ failure times $X_{1:n},\ldots,X_{r:n}$ are observed and the rest of the data are only known to be larger than $X_{r:n}$.

In Type-II censoring scheme, if r is small and n is relatively large compared to r, the precision of the estimates of parameters obtained from such a censored data will be very low. In such a situation, if it will be possible and convenient to take an additional Type-II right censored data from another independent sample (possibly of small size s), it might be possible to use the combined ordered sample from these two Type-II right censored samples in order to increase the precision of the estimation. There are a variety of scenarios wherein one can obtain combined ordered sample from two independent Type-II censored samples arising from a common parent distribution. One possible situation is when the number of items placed on a life test per run are limited, so that several independent runs need to be done. Another scenario is in the context of a meta-analysis when similar life-testing experiments from different facilities need to be pooled together.

Balakrishnan et al in [1] considered the situation in which two independent Type-II right censored samples are pooled, and demonstrated the advantage of pooling samples and expressed the joint distribution of order statistics from the pooled sample as a mixture of progressively Type-II censored samples. Using these mixture forms, they then derived nonparametric prediction intervals for order statistics from a future sample. Recently, Mohie El-Din et al. [2] considered the pooled sample of two independent Type-II censored samples from the left truncated exponential distribution and derived the maximum likelihood (ML) and Bayesian estimators for the unknown parameters, and then they discussed the problem of predicting the failure times from a future sample from a Bayesian viewpoint. In this paper, we discuss the

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same problem when the observed sample is a pooled sample from two independent Type-II right censored samples from the inverse exponential distribution.

The role of inverse exponential distributions is indispensable in many applications of reliability theory, for its memoryless property and its constant failure rate, see; [3], [4]. In the life distribution, if the random variable X has an exponential distribution then the random variable $T=(1/X)$ has an inverse exponential distribution. The later inverse exponential distribution has been considered by Killer and Kamath in [5], and Duran and Lewis in [6] among many others. The probability density (PDF) and cumulative (CDF) functions of the inverse exponential distribution can be given by

$$f(x;\theta)=\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right) \text{ and } F(x;\theta)=\exp\left(-\frac{\theta}{x}\right),\ x\geq 0, \tag{1}$$

respectively.

For the Bayesian estimation in this context, we consider here three types of loss functions. The first is the squared error (SE) loss function which is a symmetric function that gives equal importance to overestimation and underestimation in the parameter estimation. The second is the linear-exponential (LINEX) loss function, introduced by Varian in [7], which is asymmetric and gives differing weights to overestimation and underestimation. This function rises approximately exponentially on one side of zero and approximately linearly on the other side. These loss functions have been used by many authors; see, for example, [8], [9], [10], [11], [12], [13], and [14]. The third loss function is the generalization of the entropy (GE) loss used by several authors (see, for example, [15]). This more general version allows for different shapes of the loss function.

The rest of this paper is organized as follows. In Section 2, the description of the model of the pooled sample from two independent Type-II censored samples is presented. The ML estimator and the Bayesian estimators of the unknown parameters under SE, LINEX, and GE loss functions are derived in Section 3. The problem of predicting the order statistics from a future sample then is discussed in Section 4. Finally, in Section 5, some computational results are presented for illustrating all the inferential methods developed here.

2 The model description

Let $X_{1:n},\ldots,X_{r:n}$ and $Y_{1:m},\ldots,Y_{s:m}$ be independent right Type-II censored samples from two independent random samples $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_m$, respectively, drawn from a population with distribution function F. In the following, the pooled sample from $X_{1:n},\ldots,X_{r:n};Y_{1:m},\ldots,Y_{s:m}$ will be denoted by $\mathbf{Z}=(Z_{(1)},\ldots,Z_{(r+s)})$ where $Z_{(1)}\leq\ldots\leq Z_{(r+s)}$.

Balakrishnan et al.in [1] derived the joint density function of $\mathbf{Z}=(Z_{(1)},\ldots,Z_{(r+s)})$ as a mixture of progressively Type-II censored samples given by

$$f_{\mathbf{Z}}(\mathbf{z})=\sum_{i=0}^{r-1}\beta_i f_{\mathbf{T}_i}(\mathbf{z})+\sum_{j=0}^{s-1}\beta_j^* f_{\mathbf{T}_j^*}(\mathbf{z}), \tag{2}$$

where $\mathbf{z}=(z_1,\ldots,z_{r+s})$ is a vector of realizations, $\mathbf{T}_i=(T_{1:r+s:n+m}^{\mathscr{R}_i},\ldots,T_{r+s:r+s:n+m}^{\mathscr{R}_i})$ for $i=0,\ldots,r-1$, and $\mathbf{T}_j^*=(T_{1:r+s:n+m}^{\mathscr{R}_j^*},\ldots,T_{r+s:r+s:n+m}^{\mathscr{R}_j^*})$ for $j=0,\ldots,s-1$, are progressively Type-II censored samples from the same population based on the progressive censoring schemes

$$\mathscr{R}_i=(\underbrace{0,\ldots,0,m-s}_{s+i},\underbrace{0,\ldots,0,n-r}_{r-i}),$$

$$\mathscr{R}_j^*=(\underbrace{0,\ldots,0,n-r}_{r+j},\underbrace{0,\ldots,0,m-s}_{s-j}),$$

respectively, and the constants β_i and β_j^* are given by

$$\beta_i=\frac{\binom{s+i-1}{s-1}\binom{n+m-s-i}{n-i}}{\binom{n+m}{n}} \text{ for } i=0,1,\ldots,r-1,$$

$$\beta_j^*=\frac{\binom{r+j-1}{r-1}\binom{n+m-r-j}{m-j}}{\binom{n+m}{m}} \text{ for } j=0,1,\ldots,s-1.$$

By using the joint density function of the progressively Type-II censored sample [see Balakrishnan and Aggarwala (2000) and Balakrishnan (2007)], the joint density function in (2) becomes

$$f_Z(z)=\sum_{i=0}^{r-1}A_i\left[1-F(z_{s+i})\right]^{m-s}\left[1-F(z_{r+s})\right]^{n-r}\prod_{q=1}^{r+s}f(z_q)+\sum_{j=0}^{s-1}A_j^*\left[1-F(z_{r+j})\right]^{n-r}\left[1-F(z_{r+s})\right]^{m-s}\prod_{q=1}^{r+s}f(z_q), \tag{3}$$

where

$$A_i = \frac{(n+m)!(n-i)!}{(n+m-s-i)!(n-r)!}\beta_i\ , \quad \text{for } \ i = 0,\ldots,r-1,$$

and

$$A_j^* = \frac{(n+m)!(m-j)!}{(n+m-r-j)!(m-s)!}\beta_j^*, \quad \text{for } \ j = 0,\ldots,s-1.$$

Using (1) and (3), we obtain the likelihood function of θ based on the combined ordered sample $\mathbf{Z} = (Z_{(1)},\ldots,Z_{(r+s)})$ as

$$L(\theta \mid \mathbf{z}) = \frac{1}{\prod_{q=1}^{r+s} z_q^2}\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_i C_{h_1,h_2}\theta^{r+s}\exp\left[-\theta\left(u_i\right)\right] + \sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_j^* C_{h_1,h_2}\theta^{r+s}\exp\left[-\theta\left(u_i\right)\right]\right\}, \tag{4}$$

where

$$u_i = \frac{h_1}{z_{s+i}} + \frac{h_2}{z_{r+s}} + \sum_{q=1}^{r+s}\frac{1}{z_q}, \text{ for } i = 1,\ldots,r-1,$$

$$u_j^* = \frac{h_2}{z_{r+j}} + \frac{h_1}{z_{r+s}} + \sum_{q=1}^{r+s}\frac{1}{z_q}, \text{ for } j = 1,\ldots,s-1,$$

and

$$C_{h_1,h_2} = (-1)^{h_1+h_2}\,\frac{(m-s)!\,(n-r)!}{(m-s-h_1)!\,(n-r-h_2)!h_1!h_2!} \text{ for } h_1 = 0,\ldots,m-s,\ h_2 = 0,\ldots,n-r.$$

3 ML and Bayesian estimation

In this section, we derive the ML estimator and the Bayesian estimators for the unknown parameter θ . when the observed sample is the ordered pooled sample $\mathbf{Z} = (Z_{(1)},\ldots,Z_{(r+s)})$.From (4), the log-likelihood function of θ is given by

$$\log L(\theta \mid \mathbf{z}) = \log\left\{\frac{1}{\prod_{q=1}^{r+s} z_q^2}\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_i C_{h_1,h_2}\theta^{r+s}\exp\left(-\theta u_i\right) + \sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_j^* C_{h_1,h_2}\theta^{r+s}\exp\left(-\theta u_j^*\right)\right\}\right\}, \tag{5}$$

and so the ML estimator $\hat{\theta}_{ML}$ of θ is readily obtained by solving the following equation

$$\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_i C_{h_1,h_2}\left(r+s-\theta u_i\right)\exp\left(-\theta u_i\right) + \sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_j^* C_{h_1,h_2}\left(r+s-\theta u_j^*\right)\exp\left(-\theta u_j^*\right) = 0. \tag{6}$$

In the Bayesian approach, θ is viewed as realization of a random variable distributed according to $\pi(\theta)$ on Θ, which is the prior distribution. For this purpose, we consider here the conjugate gamma

$$\pi(\theta; a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp\left(-b\ \theta\right),\ \theta > 0, \tag{7}$$

where a and b are positive hyperparameters that could be chosen, for example, from a prior knowledge of the mean and variance of θ, and $\Gamma(.)$ denotes the complete gamma function.

Upon combining (4) and (7), the posterior density function of θ, given $\mathbf{Z} = \mathbf{z}$, is obtained as

$$\pi^*(\theta|\mathbf{z}) = I^{-1}\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_i C_{h_1,h_2}\theta^G\exp\left(-\theta H_i\right) + \sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_j^* C_{h_1,h_2}\theta^G\exp\left(-\theta H_j^*\right)\right\}, \tag{8}$$

where $G = r+s+a-1$, $H_i = u_i + b$, $H_j^* = u_j^* + b$, and I is the normalizing constant given by

$$I = \Gamma(G+1)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_i C_{h_1,h_2}\left[H_i\right]^{-(G+1)} + \sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r} A_j^* C_{h_1,h_2}\left[H_j^*\right]^{-(G+1)}\right\}. \tag{9}$$

Hence, the Bayesian estimator of θ under the SE loss function is given by

$$\begin{aligned}\hat{\theta}_{BS} &= E[\theta] \\ &= I^{-1}\Gamma(G+1)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_iC_{h_1,h_2}[H_i]^{-(G+2)}+\sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_j^*C_{h_1,h_2}\left[H_j^*\right]^{-(G+2)}\right\}.\end{aligned} \tag{10}$$

The LINEX loss function can be expressed as

$$L_{BL}\left(\hat{\theta},\theta\right) = \exp\left[-\nu\left(\hat{\theta}-\theta\right)\right]-\nu\left(\hat{\theta}-\theta\right)-1, \tag{11}$$

where $\nu \neq 0$. The sign and magnitude of the shape parameter ν represent the direction and degree of asymmetry, respectively. The problem of choosing the value of the parameter ν has been discussed by Calabria and Pulcini in [18]. The Bayesian estimator of θ under the LINEX loss function is given by

$$\begin{aligned}\hat{\theta}_{BL} &= \frac{-1}{\nu}\log\{E[\exp(-\nu\theta)]\} \\ &= \frac{-1}{\nu}\log\left\{I^{-1}\Gamma(G+1)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_iC_{h_1,h_2}[H_i+\nu]^{-(G+1)}+\sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_j^*C_{h_1,h_2}\left[H_j^*+\nu\right]^{-(G+1)}\right\}\right\}.\end{aligned} \tag{12}$$

The GE loss function, is given by

$$L_{BE}\left(\hat{\theta},\theta\right) \propto \left(\frac{\hat{\theta}}{\theta}\right)^d - d\ln\left(\frac{\hat{\theta}}{\theta}\right)-1. \tag{13}$$

It may be noted that when $d > 0$, a positive error is regarded as more serious than a negative error; on the other hand, when $d < 0$, a negative error is regarded as more serious than a positive error. The Bayesian estimator of θ under the GE loss function is given by

$$\begin{aligned}\hat{\theta}_{BE} &= \left[E\left(\theta^{-d}\right)\right]^{\frac{-1}{d}} \\ &= \left\{I^{-1}\Gamma(G+1)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_iC_{h_1,h_2}[H_i+\nu]^{-(G+1)}+\sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}A_j^*C_{h_1,h_2}\left[H_j^*+\nu\right]^{-(G+1)}\right\}\right\}^{\frac{-1}{d}}.\end{aligned} \tag{14}$$

4 Bayesian prediction of order statistics from a future sample

Let $W_{1:\rho},\ldots,W_{\rho:\rho}$ be the order statistics from a future random sample of size ρ from the same population. We discuss here the Bayesian prediction of $W_{q:\rho}$, for $q = 1,\ldots,\rho$, based on the observed pooled sample $Z = \left(Z_{(1)},\ldots,Z_{(r+s)}\right)$. We derive the Bayesian predictive distribution for $W_{q:\rho}$ and then find the Bayesian point predictor and prediction interval for $W_{q:\rho}$.

It is well known that the marginal density function of the q-th order statistic from a sample of size ρ from a continuous distribution with cdf $F(x)$ and pdf $f(x)$ is given by

$$f_{W_{q:\rho}}(w|\theta) = \frac{\rho!}{(q-1)!(\rho-q)!}[F(w)]^{q-1}[1-F(w)]^{\rho-q}f(w),\quad w \geq 0, \tag{15}$$

for $1 \leq q \leq \rho$; see [19].

Upon substituting (1) in (15), the marginal density function of the $W_{q:\rho}$ becomes

$$f_{W_{q:\rho}}(w|\theta) = \sum_{h_3=0}^{\rho-q}K_{h_3}\frac{\theta}{w^2}\exp\left(\frac{-\theta}{w}(q-h_3)\right), 1 \leq q \leq \rho, \tag{16}$$

where $K_{h_3} = (-1)^{h_3}\frac{\rho!(\rho-q)!}{(q-1)!(\rho-q)!(\rho-q-h_3)!h_3!}$ for $h_3 = 0,\ldots,\rho-q$.

Upon combining (8) and (16), the Bayesian predictive density function of $W_{q:\rho}$, given $\mathbf{Z}=\mathbf{z}$, is then

$$f^*_{W_{q:\rho}}(w|\mathbf{z}) = \int_0^\infty \pi^*(\theta|\mathbf{z}) f_{W_{q:\rho}}(w|\theta)\, d\theta$$
$$= I^{-1}\Gamma(G+2)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}\sum_{h_3=0}^{\rho-q}\frac{A_i C_{h_1,h_2}K_{h_3}}{w^2}\left[H_i+\frac{(q+h_3)}{w}\right]^{-(G+2)}\right.$$
$$\left.+\sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}\sum_{h_3=0}^{\rho-q}\frac{A^*_j C_{h_1,h_2}K_{h_3}}{w^2}\left[H^*_j+\frac{(q+h_3)}{w}\right]^{-(G+2)}\right\}. \tag{17}$$

From (17), we simply obtain the predictive survival function of $W_{q:\rho}$, given $\mathbf{Z}=\mathbf{z}$, as

$$\bar{F}^*_{W_{q:\rho}}(t|\mathbf{z}) = \int_t^\infty f^*_{W_{q:\rho}}(w|\mathbf{z})\, dw$$
$$= I^{-1}\Gamma(G+1)\left\{\sum_{i=0}^{r-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}\sum_{h_3=0}^{\rho-q}\frac{A_i C_{h_1,h_2}K_{h_3}}{(q+h_3)}\left\{(H_i)^{-(G+1)}-\left[H_i+\frac{(q+h_3)}{t}\right]^{-(G+1)}\right\}\right.$$
$$\left.+\sum_{j=0}^{s-1}\sum_{h_1=0}^{m-s}\sum_{h_2=0}^{n-r}\sum_{h_3=0}^{\rho-q}\frac{A^*_j C_{h_1,h_2}K_{h_3}}{(q+h_3)}\left\{(H^*_j)^{-(G+1)}-\left[H^*_j+\frac{(q+h_3)}{t}\right]^{-(G+1)}\right\}\right\}. \tag{18}$$

The Bayesian point predictor of $W_{q:\rho}$, under SEL is obtained as the mean of the predictive density, given by (17). We have no closed-form expression for the point predictor but it is not difficult to carry out a numerical integration for this propose. The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $W_{q:\rho}, 1\le q\le\rho$ can be obtained by solving the following two equations:

$$\bar{F}^*_{W_{q:\rho}}(L\mid z) = 1-\frac{\gamma}{2} \text{ and } \bar{F}^*_{W_{q:\rho}}(U\mid z) = \frac{\gamma}{2},$$

where $\bar{F}^*_{W_{q:\rho}}(t\mid z)$ is as in (18), and L and U denote the lower and upper bounds, respectively.

For the highest posterior density (HPD) method, we need to solve the following two equations

$$\bar{F}^*_{W_{q:\rho}}\left(L_{W_{q:\rho}}\mid z\right) - \bar{F}^*_{W_{q:\rho}}\left(U_{W_{q:\rho}}\mid z\right) = 1-\gamma$$

and

$$f^*_{W_{q:\rho}}\left(L_{W_{q:\rho}}\mid z\right) - f^*_{W_{q:\rho}}\left(U_{W_{q:\rho}}\mid z\right) = 0.$$

where $f^*_{W_{q:\rho}}(w\mid z)$ is as in (17), and $L_{W_{q:\rho}}$ and $U_{W_{q:\rho}}$ denote the HPD lower and upper bounds,respectively.

5 Numerical results and an illustrative example

In this section, the ML and Bayesian estimates based on the SE, LINEX and GE loss functions are all compared by means of a Monte Carlo simulation study. A numerical example is finally presented to illustrate all the inferential results established in the preceding sections.

5.1 Monte Carlo simulation

A simulation study is carried out for evaluating the performance of the ML estimate and all the Bayesian estimates discussed in Section 4. We chose the parameter θ to be 0.1, 0.5 and 1 and the two sample sizes $(m,n)=(10,10)$ with different choices of r and s. For these cases, we computed the ML estimate and the Bayesian estimates of θ under the SE, LINEX (with $\nu=0.5$) and GE (with $d=0.5$) loss functions using informative priors (IP) and non-informative prior (NIP). We repeated this process 1000 times and computed, for each estimate, the estimated bias (EB) and the estimated risk (ER) by using the root mean square error. The EB and ER of all the estimates of θ are summarized in Tables 1.

From Table 1, we observe that, for the different choices of θ, the estimated bias and risk of the Bayesian estimates based on the SE, LINEX and GE loss functions are smaller than those of the ML estimates. We also observe that the estimated bias and risk of all the estimates decrease with increasing r and s. Moreover, a comparison of the results for the informative priors with the corresponding ones for non-informative priors reveals that the former produce more precise results, as we would expect.

Table 1: Values of EB and ER of the ML and Bayes estimators for θ with different choices of r and s.

			$\hat{\theta}_{ML}$			$\hat{\theta}_{BS}$		$\hat{\theta}_{BL}$		$\hat{\theta}_{BE}$	
θ	r	s	EB	ER		EB	ER	EB	ER	EB	ER
0.5	4	4	0.4242	0.7866	IP	0.0025	0.0230	0.0023	0.0227	0.0017	0.0221
			–	–	NIP	0.0080	0.0268	0.0080	0.0272	0.0039	0.0250
	6	4	0.3691	0.5845	IP	0.0024	0.0225	0.0023	0.0224	0.0016	0.0216
			–	–	NIP	0.0079	0.0261	0.0078	0.0260	0.0037	0.0244
	6	6	0.1262	0.5236	IP	0.0024	0.0224	0.0022	0.0224	0.0015	0.0215
			–	–	NIP	0.0078	0.0261	0.0076	0.0259	0.0037	0.0243
	8	6	0.1028	0.4936	IP	0.0023	0.0223	0.0021	0.0223	0.0015	0.0214
			–	–	NIP	0.0077	0.0259	0.0076	0.0257	0.0035	0.0240
	8	8	0.0146	0.1763	IP	0.0022	0.0219	0.0017	0.0222	0.0014	0.0210
			–	–	NIP	0.0076	0.0256	0.0074	0.0257	0.0032	0.0238
1	4	4	0.4096	1.4118	IP	0.0387	0.1575	0.0292	0.1531	0.0119	0.1513
			–	–	NIP	0.0796	0.2682	0.0639	0.2566	0.0402	0.2471
	6	4	0.3722	1.3250	IP	0.0378	0.1565	0.0288	0.1520	0.0115	0.1486
			–	–	NIP	0.0789	0.2613	0.0628	0.2504	0.0368	0.2431
	6	6	0.3637	1.3199	IP	0.0378	0.1552	0.0287	0.1507	0.0112	0.1474
			–	–	NIP	0.0776	0.2606	0.0624	0.2494	0.0368	0.2422
	8	6	0.3111	1.1482	IP	0.0369	0.1541	0.0278	0.1496	0.0108	0.1463
			–	–	NIP	0.0772	0.2586	0.0622	0.2476	0.0351	0.2404
	8	8	0.0222	0.4132	IP	0.0364	0.1509	0.0270	0.1462	0.0105	0.1425
			–	–	NIP	0.0758	0.2565	0.0606	0.2450	0.0344	0.2390
5	4	4	1.9896	5.5699	IP	0.3466	1.2985	0.0192	1.0861	0.1378	1.2080
			–	–	NIP	0.3978	1.3406	0.0371	1.1115	0.1952	1.2395
	6	4	1.6327	4.6798	IP	0.3440	1.2642	0.0133	1.0650	0.1378	1.1810
			–	–	NIP	0.3942	1.3067	0.0347	1.0895	0.1838	1.2156
	6	6	0.4020	4.2442	IP	0.3385	1.2596	0.0093	1.0590	0.1367	1.1760
			–	–	NIP	0.3886	1.3031	0.0340	1.0838	0.1774	1.2079
	8	6	0.3182	2.4480	IP	0.3370	1.2503	0.0068	1.0530	0.1304	1.1678
			–	–	NIP	0.3859	1.2928	0.0277	1.0770	0.1756	1.2020
	8	8	0.2037	2.0131	IP	0.3297	1.2381	0.0037	1.0341	0.1284	1.1503
			–	–	NIP	0.3789	1.2823	0.0233	1.0583	0.1661	1.1904

5.2 Illustrative example

In order to illustrate all the inferential results established in the preceding sections, we simulated two samples with sizes $(m,n) = (10,10)$ from the inverse exponential distribution with $\theta = 1$, and then applied The right Type-II censoring scheme with $r = 6$ and $s = 4$. The two right Type-II censored samples are as follows:

Group X	0.2633	0.6081	1.0516	1.0786	1.1286	1.1769	*	*	*	*
Group Y	0.4465	0.6425	0.9016	1.0807	*	*	*	*	*	*

These two samples are now assumed to have come from the inverse exponential distribution, with parameter θ being unknown. Based on Type-II pooled sample $Z = (0.2633, 0.4465, 0.6081, 0.6425, 0.9016, 1.0516, 1.0786, 1.0807, 1.1286, 1.1769)$ from these two samples, we computed the ML estimate and the Bayesian estimates of θ based on the SE, LINEX (with $\nu = 0.5$) and GE (with $d = 0.5$) loss functions using informative prior with $(a,b) = (10,10)$ and non-informative prior with $(a,b) = (0,0)$. Also, we computed the point predictors as well as the bounds of the equi-tailed prediction intervals for the order statistics $W_{q:\rho}$, $q = 1,\ldots,10$ from a future sample with size $\rho = 10$ from the same population. All these results are summarized in Tables 2 and 3.

5.3 Conclusion and discussion

In this paper, the Bayesian estimation based on the SE, LINEX and GE loss functions for the unknown parameter of inverse exponential distributions has been discussed based on Pooled Type-II Censored Samples. Both Bayesian point and interval predictions of the future failures have been developed based on the observed Pooled Type-II Censored data.

Table 2: The ML and Bayes estimates for θ.

$\widehat{\theta}_{ML}$		$\widehat{\theta}_{BS}$	$\widehat{\theta}_{BL}$	$\widehat{\theta}_{BE}$
1.0726	IP	1.0474	1.0381	1.0206
–	NIP	1.0746	1.0598	1.0328

Table 3: Bayesian prediction of $W_{q:10}$ for $q = 1, \ldots, 10$.

	Point predictor		Equi-tailed interval		HPD interval	
q	IP	NIP	IP	NIP	IP	NIP
1	0.3545	0.3154	(0.154, 1.690)	(0.146, 1.801)	(0.150, 1.510)	(0.155, 1.803)
2	0.6586	0.6245	(0.280, 2.355)	(0.298, 2.424)	(0.123, 1.890)	(0.127, 2.163)
3	1.0694	0.9001	(0.397, 3.035)	(0.309, 3.256)	(0.269, 2.566)	(0.273, 2.879)
4	1.2224	1.1124	(0.581, 3.777)	(0.570, 4.600)	(0.389, 3.318)	(0.410, 3.651)
5	1.2021	1.1985	(0.795, 4.912)	(0.719, 5.001)	(0.565, 4.210)	(0.580, 4.582)
6	1.6541	1.5325	(1.151, 6.101)	(1.003, 6.322)	(0.789, 5.227)	(0.795, 5.757)
7	1.9154	1.7254	(1.415, 7.566)	(1.312, 8.114)	(1.014, 6.773)	(1.064, 7.358)
8	2.9120	3.0654	(1.424, 7.241)	(1.254, 8.012)	(1.058, 7.562)	(1.074, 7.954)
9	5.4005	5.3258	(1.845, 9.154)	(1.754, 10.562)	(1.413, 8.254)	(1.437, 11.124)
10	5.9245	5.8457	(2.745, 15.246)	(2.621, 17.321)	(1.820, 12.669)	(1.965, 15.268)

The ML and Bayesian estimates have then been compared through a Monte Carlo simulation study and a numerical example has also been presented to illustrate all the inferential results established here.

The computational results show that the Bayesian estimation based on the SE, LINEX and GE loss functions is more precise than the ML estimation. Also, the ERs of all the estimates decrease with increasing r and s even when the sample sizes m and n are small. Moreover, a comparison of the results for the informative priors with the corresponding ones for non-informative priors reveals that the former produce more precise results, as we would expect. Finally, the HPD prediction intervals seem to be more precise than the equi-tailed prediction intervals.

Acknowledgement

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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