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On Convergence of Intermediate Order Statistics under Power Normalization

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Abstract: We discuss the convergence of the moments of intermediate order statistics under power normalization. The moments convergence is established for four p-max-stable laws according to conditions imposed on the considered distribution and on the rank sequence.

Keywords: Intermediate order statistics, moment convergence, power normalization

1 Introduction

Let $X_1, X_2, ..., X_n$ be independent random variables (rv's) with the same distribution function (df) $F(x)$ and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the corresponding order statistics. We call $X_{k_n:n}$ and $X_{r_n:n}$ the upper and lower intermediate order statistics, respectively, if $k_n = n - r_n + 1, \frac{r_n}{n} \to 0$, as $n \to \infty$. A sequence $\{r_n\}$ is said to satisfy Chibisov's condition, if

$$\lim_{n\to\infty}\left(\sqrt{r_{n+z_n}} - \sqrt{r_n}\right) = \frac{\alpha l \nu}{2},\ l > 0, \tag{1}$$

for any sequence $\{z_n\}$ of integer-values, where $\frac{z_n}{n^{1-\frac{\alpha}{2}}} \to \nu$, as $n \to \infty$ ($0 < \alpha < 1$ and ν is any arbitrary real number). As Chibisov in [4] himself noted, the condition (1) implies that $\frac{r_n}{n^\alpha} \to \ell^2$, as $n \to \infty$. It is noteworthy to mention that the latter condition implies Chibisov's condition (see, [1] and [3]), which means that the class of intermediate rank sequences which satisfy the Chibisov condition is a very wide class. Chibisov [4] showed that, whenever $\{r_n\}$ satisfies (1), the possible nondegenerate types of the limiting distribution of the lower intermediate term $X_{r_n:n}$, under linear normalization are $G_{1,\beta}(x) = \mathcal{N}(v_1(x;\beta)) = \mathcal{N}(\beta \log x),\ x > 0,\ G_{2,\beta}(x) = \mathcal{N}(v_2(x;\beta)) = \mathcal{N}(-\beta \log |x|),\ x \leq 0,$ and $G_3(x) = \mathcal{N}(v_3(x)) = \mathcal{N}(x)$, where $\mathcal{N}(.)$ stands for the standard normal distribution. The corresponding possible nondegenerate limiting distributions for the upper intermediate term $X_{k_n:n}$ are $\Psi_{i;\beta}(x) = 1 - \mathcal{N}(v_i(-x,\beta)),\ i = 1,2,3$ (note that $\Psi_{3;\beta}(x) = 1 - \mathcal{N}(v_3(-x))$). Clearly, $\Psi_{1;\beta} = G_{2;\beta},\ \Psi_{2;\beta} = G_{1;\beta}$ and $\Psi_{3;\beta} = G_{3;\beta}$. Therefore, we have $\{G_{i;\beta},\ i = 1,2,3\} \equiv \{\Psi_{i;\beta},\ i = 1,2,3\}$. The intermediate order statistics have many applications. For example, intermediate order statistics can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are extremes relative to the available sample size. Pickands [12] has shown that intermediate order statistics can be used in constructing consistent estimators for the shape parameter of the limiting extremal distribution in the parametric form. Many authors, e.g. [13] and [5], have also found estimators that are based, in part, on intermediate order statistics.

During the last two decades E. Pancheva and her collaborators (e,g., see, [6]-[10]) developed the extreme value theory under nonlinear monotone increasing normalizing mappings in order to get a wider class of limit laws, which can be used in solving approximation problems. Barakat and Omar [2] (see also [3]) showed that the possible nondegenerate types of the limit df of the lower intermediate order statistics $X_{r_n:n}$ under the power normalization $T_n(x) = a_n \mid x \mid^{b_n} \text{sign}(x), a_n, b_n > 0$, are

$$L_{1,\beta}(x) = \mathcal{N}(\beta \log\log x),\ x > 1; \quad L_{2,\beta}(x) = \mathcal{N}(-\beta \log(-\log x)),\ 0 < x \leq 1;$$

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$$L_{3,\beta}(x)=\mathscr{N}(\beta\log(-\log|x|)),\ -1<x\le 0;\quad L_{4,\beta}(x)=\mathscr{N}(-\beta\log\log|x|),\ x\le -1;$$

$$L_{5;\beta}(x)=L_5(x)=\mathscr{N}(\log x),\ x>0;\quad L_{6;\beta}(x)=L_6(x)=\mathscr{N}(-\log|x|),\ x\le 0.$$

The corresponding types of the upper intermediate order statistics are $H_{1;\beta}(x)=1-\mathscr{N}(\beta\log((\log\mid x\mid))),\ x\le -1;$ $H_{2;\beta}(x)=\mathscr{N}(\beta\log(-\log\mid x\mid)),\ -1<x\le 0;$ $H_{3;\beta}(x)=1-\mathscr{N}(\beta\log(-\log x)),\ 0<x\le 1;$ $H_{4;\beta}(x)=\mathscr{N}(\beta\log(\log x)),\ x>1;$ $H_{5;\beta}(x)=H_5(x)=\mathscr{N}(-\log\mid x\mid),\ x\le 0;$ and $H_{6;\beta}(x)=H_6(x)=\mathscr{N}(\log x),\ x>0.$ Although, in general, we have $H_{i;\beta}\ne L_{i;\beta},\ i=1,2,...,6,$ we note that the two classes of possible limit laws of lower and upper intermediate order statistics under power normalization shows that they coincide, i.e., $\{H_{i;\beta},\ i=1,2,...,6,\}\equiv\{L_{i;\beta},\ i=1,2,...,6,\}.$

2 Moment convergence of the intermediate order statistics

Recently, the moment convergence of the extremes under power normalization have been studied by [11]. In this section, we study the moments convergence of the intermediate order statistics under power normalization, i.e., for some suitable normalizing constants $a_n,b_n>0$, and for some positive integer k,

$$\lim_{n\to\infty}\mathrm{E}\left(T_n^{-1}(X_{r_n:n})\right)^k=\lim_{n\to\infty}\mathrm{E}\left(\mid\frac{X_{r_n:n}}{a_n}\mid^{\frac{1}{b_n}}\mathrm{sign}(X_{r_n:n})\right)^k=\int_{\ell(L_{i,\beta})}^{r(L_{i,\beta})}x^k dL_{i,\beta}(x), i\in\{1,2,...,6\},$$

where $\ell(F)=\inf\{x:F(x)>0\}$ and $r(F)=\sup\{x:F(x)<1\}$ are the left and right end-points for the df F, respectively. Obviously, for every integer $k>0$, $\int_{\ell(L_{i,\beta})}^{r(L_{i,\beta})}x^k d(L_{i,\beta}(x)), i\in\{2,3,5,6\}$, converges (for being $\ell(L_{i,\beta})$ and $r(L_{i,\beta}), i=2,3,$ are finite, while $L_{5;\beta}(x)=L_5(x)$ and $L_{6;\beta}(x)=L_6(x)$ are a log-normal and a negative log-normal df's, respectively. Since $L_{1;\beta}(x)$ is a log-log-normal df, then, $\int_{\ell(L_{1,\beta})}^{r(L_{1,\beta})}x^k d(L_{1,\beta}(x))=\int_0^\infty x^k d(L_{1,\beta}(x))=\mathrm{E}\left(e^{k\eta}\right)$ where η has a log-normal distribution. On the other hand, the expected value $\mathrm{E}\left(e^{t\eta}\right)$ is not defined for any positive value of the argument t as the defining integral diverges. Therefore, $\int_{\ell(L_{1,\beta})}^{r(L_{1,\beta})}x^k d(L_{1,\beta}(x))$ is divergent. Moreover, since $L_{4;\beta}(x)$ is a negative log-log-normal df, i.e., $L_{4;\beta}(x)=1-L_{1;\beta}(-x)$, we deduce that $\int_{\ell(L_{4,\beta})}^{r(L_{4,\beta})}x^k d(L_{4,\beta}(x))$ is also divergent for every positive k.

Barakat and Omar [1] found the domains of attraction of all possible limit laws of the df of the power normalized lower intermediate order statistic $T_n^{-1}(X_{r_n:n})$. In Theorem 2.1 we present the results of [1] only for the four remaining cases (i.e., for $L_{i,\beta}(x)), i=2,3,5,6)$ because these results are essential in the study of moment convergence. Throughout this theorem, we write $F\in D_p(L)$ to indicate that F belongs to the domain of attraction of the law L, under power normalization. Also, for any nondecreasing function F we write $F^-(y)=\inf\{x:F(x)>y\}$.

Theorem 2.1.

1.A df $F\in D_p(L_{2,\beta})$ if and only if $\ell(F)=0$ and for any $\tau>0$,

$$\lim_{x\to-\infty}\frac{F(\exp(\tau x))-F(\exp(x))}{[F(\exp(x))]^{\frac{2-\alpha}{2-2\alpha}}}=-\ell^{\frac{-1}{1-\alpha}}\beta\log\tau.$$

We may set $a_n=1$ and $b_n=-\log(F^-(\frac{r_n}{n}))\to\infty$.

2.A df $F\in D_p(L_{3,\beta})$ if and only if $\exists x_0$ such that $F(-e^{-x_0})=0$. Moreover, $F(-e^{-x_0}+\varepsilon)>0$, $\forall\varepsilon>0$ (i.e., $-\infty<\ell(F)=-e^{-x_0}<0$). Moreover, for any $\tau>0$,

$$\lim_{x\downarrow 0}\frac{F(-\exp(-(x_0+x\tau)))-F(-\exp(-(x_0+x)))}{[F(-\exp(-(x_0+x)))]^{\frac{2-\alpha}{2-2\alpha}}}=\ell^{\frac{-1}{1-\alpha}}\beta\log\tau.$$

In this case we may set $a_n=e^{-x_0}$ and $b_n=-(\log(-F^-(\frac{r_n}{n}))+x_0)\to\infty$.

3.A df $F\in D_p(L_5)$ if and only if $\ell(F)\ge 0$ and the sequence $\{\beta_n\}$ defined as the smallest numbers for which $F(e^{\beta_n})\le\frac{k}{n}\le F(e^{\beta_n}+0)$ (i.e., $\beta_n=\log F^-(\frac{k}{n})\to\log\ell(F)$), satisfies the condition

$$\lim_{n\to\infty}\frac{\beta_{n+z_n(\nu)}-\beta_n}{\beta_{n+z_n(\mu)}-\beta_n}=\frac{\nu}{\mu},\qquad (2)$$

for all sequences $\{z_n(t)\}, t \in \mathbb{R}$, satisfying $\frac{z_n(t)}{n^{1-\frac{\alpha}{2}}} \to t$, as $n \to \infty$.

We may set in this case $a_n = e^{\beta_n} = F^-(\frac{r_n}{n})$ and $b_n = \log \frac{F^-(\frac{r_n+\sqrt{r_n}}{n})}{F^-(\frac{r_n}{n})}$.

4. A df $F \in D_p(L_6)$ if and only if $\ell(F) < 0$ and the sequence $\{\beta_n\}$, defined as the smallest numbers for which $F(-e^{-\beta_n}) \leq \frac{r_n}{n} \leq F(-e^{-\beta_n}+0)$, satisfies the condition (2).
In this case we may set $a_n = e^{-\beta_n}$ and $b_n = \frac{1}{\sqrt{\xi}} \log \frac{F^-(\frac{r_n}{n})}{F^-(\frac{r_n+\sqrt{r_n\xi}}{n})}$.

Theorem 2.2 (the main result).

(a) Under the conditions of Theorem 2.1, Part (1) (or Part (2)), we have

$$\lim_{n\to\infty} \mathrm{E}\left(\mid \frac{X_{r_n:n}}{a_n} \mid^{\frac{1}{b_n}} \mathrm{sign}(X_{r_n:n}) \right)^k = \int_{\ell(L_{i,\beta})}^{r(L_{i,\beta})} x^k dL_{i,\beta}(x),\ i \in \{2,3\}, \tag{3}$$

if $r_n \sim \ell^2 n^\alpha$ and F are such that $\frac{n}{b_n\sqrt{r_n}} \to 0$, as $n \to \infty$, and

$$\int_1^\infty y^{\varepsilon-1}(1-F(y))dy < \infty,\ \text{for some}\ \varepsilon > 0.$$

(b) Under the conditions of Theorem 2.1, Part (3) (or Part (4)), we have

$$\lim_{n\to\infty} \mathrm{E}\left(\mid \frac{X_{r_n:n}}{a_n} \mid^{\frac{1}{b_n}} \mathrm{sign}(X_{r_n:n}) \right)^k = \int_{\ell(L_{i,\beta})}^{r(L_{i,\beta})} x^k dL_{i,\beta}(x),\ i \in \{5,6\},$$

if $r(F) = \sup\{x : F(x) < 1\} < \infty$.

Proof. Let $F_{r_n:n}(T_n(x)) = P(T_n^{-1}(X_{r_n:n}) \leq x) = P(X_{r_n:n} \leq a_n \mid x \mid^{b_n} \mathrm{sign}(x)) = \mathrm{I}_{F((T_n(x)))}(r_n, n-r_n+1)$, where $\mathrm{I}_F(a,b) = \frac{(a+b-1)!}{(a-1)!(b-1)!}\int_0^F t^{a-1}(1-t)^{b-1}dt,\ a,b \geq 1$, is the incomplete beta function. Clearly, for any $M > 0$ and $i \in \{2,3,5,6\}$, we get

$$\begin{aligned}
&\left| \mathrm{E}\left(a_n \mid X_{r_n:n} \mid^{b_n} \mathrm{sign}(X_{r_n:n})\right)^k - \int_{-\infty}^{\infty} x^k dL_{i,\beta}(x) \right| \\
&\leq \int_{-\infty}^{-M} |x|^k dF_{r_n:n}(-a_n \mid x \mid^{b_n}) + \left| \int_{-\infty}^{-M} x^k dL_{i,\beta}(x) \right| \\
&+ \left| \int_{-M}^{M} x^k dF_{r_n:n}(a_n \mid x \mid^{b_n} \mathrm{sign}(x)) - \int_{-M}^{M} x^k dL_{i,\beta}(x) \right| \\
&+ \int_M^\infty x^k dF_{r_n:n}(a_n x^{b_n}) + \int_M^\infty x^k dL_{i,\beta}(x).
\end{aligned}$$

Moreover, in view of the conditions of Theorem 2.1, we get, as $n \to \infty$,

$$\int_{-M}^{M} x^k dF_{r_n:n}(a_n \mid x \mid^{b_n} \mathrm{sign}(x)) \to \int_{-M}^{M} x^k dL_{i,\beta}(x), i = 2,3.5,6.$$

Therefore, in order to prove Theorem 2.2 for $i = 2,3,5,6$, we have to prove

$$\lim_{M\to\infty}\lim_{n\to\infty} \int_{-\infty}^{-M} |x|^k dF_{r_n:n}(-a_n \mid x \mid^{b_n}) = 0 \tag{4}$$

and

$$\lim_{M\to\infty}\lim_{n\to\infty} \int_M^\infty x^k dF_{r_n:n}(a_n x^{b_n}) = 0. \tag{5}$$

Clearly, (4) holds, for all $i = 2,3,5$ and $i = 6$, since $\ell(F) = \ell(F_{r_n:n})$ and $\ell(F) = 0, \ell(F) > -\infty, \ell(F) \geq 0$ and $0 > \ell(F) > -\infty$ for $i = 2,3,5$ and $i = 6$, respectively. On the other hand, by using Fubini's theorem, we get

$$\int_M^\infty x^k dF_{r_n:n}(a_n x^{b_n}) = \int_M^\infty \int_0^x k y^{k-1} dy dF_{r_n:n}(a_n x^{b_n})$$

$$= \int_M^\infty \int_0^M ky^{k-1} dy dF_{r_n:n}(a_n x^{b_n}) + \int_M^\infty \int_M^x ky^{k-1} dy dF_{r_n:n}(a_n x^{b_n})$$

$$= M^k(1-F_{r_n:n}(a_n M^{b_n})) + \int_M^\infty ky^{k-1}(1-F_{r_n:n}(a_n y^{b_n}))dy = A_n(M)+B_n(M). \quad (6)$$

Now, under the conditions of Theorem 2.1, we get $A_n(M) = M^k(1-F_{r_n:n}(a_n M^{b_n})) \to M^k(1-L_{i,\beta}(M)),\ i \in \{2,3,5,6\}$. Since the kth moments $k = 1,2,...,$ of the limit laws $L_{i,\beta}(x),\ i \in \{2,3,5,6\}$ exist, we get

$$\lim_{M\to\infty}\lim_{n\to\infty} A_n(M) = \lim_{M\to\infty} M^k(1-L_{i,\beta}(M)) = 0. \quad (7)$$

Thus, in view of (6) and (7), the proof of theorem follows if we prove $\lim_{M\to\infty}\lim_{n\to\infty} B_n(M) = 0$. For the case $i = 5,6$, the proof follows immediately from the imposed condition $r(F) < \infty$. On the other hand, for the cases $i = 2$ and for all $M > 1$, we get

$$B_n(M) = \frac{k}{b_n}\int_{M^{b_n}}^\infty y^{\frac{k}{b_n}-1}(1-F_{r_n:n}(y))dy \le \frac{k}{b_n}\int_1^\infty y^{\frac{k}{b_n}-1}(1-F_{r_n:n}(y))dy,$$

since $b_n \to \infty$. On the other hand the beta function $B_{\alpha,\beta}(x) = x^{\alpha-1}(1-x)^{\beta-1}, \alpha,\beta \ge 1,\ 0 \le x \le 1$, has its maximum at $x_0 = \frac{\alpha-1}{\beta+\alpha-2}$, we get

$$B_n(M) \le \frac{n!k}{(r_n-1)!(n-r_n)!b_n}\int_1^\infty y^{\frac{k}{b_n}-1}\int_{F(y)}^1 t^{r_n-1}(1-t)^{n-r_n} dt dy$$

$$\le C_n \int_1^\infty y^{\frac{k}{b_n}-1}(1-F(y))dy \le C_n \int_1^\infty y^{\varepsilon-1}(1-F(y))dy, \quad (8)$$

since $b_n \to \infty$, where

$$C_n = \frac{n!k}{(r_n-1)!(n-r_n)!b_n}\left(\frac{r_n-1}{n-1}\right)^{r_n-1}\left(\frac{n-r_n}{n-1}\right)^{n-r_n}.$$

Now, upon using Stirling's formula, we get

$$C_n \sim \frac{kn}{b_n\sqrt{2\pi r_n}}. \quad (9)$$

Therefore, by combining (8) and (9), the proof of (3) for $i = 2$, under the stated conditions, follows immediately. Since, for $i = 3$, $a_n = e^{-x_0}$ is a positive constant and $b_n \to \infty$, as $n \to \infty$, then the proof of this case is exactly the same as the case $i = 2$, with only the obvious changes. □

Remark 2.1. Clearly if $r(F) < \infty$, (5) will be satisfied for the cases $i = 2,3$. Moreover, the condition $\frac{n}{b_n\sqrt{r_n}} \to 0$, as $n \to \infty$, can be written in the form $\frac{n^{1-\frac{\alpha}{2}}}{b_n} \to 0$, as $n \to \infty$.

3 Conclusion

In the statistical modeling of order statistics values based on p-max-stable distributions. Two common approaches for statistical estimation are the method of moments and the method of maximum likelihood. The former estimation method involves moments, so it is important to know conditions under which the convergence of moments holds. In this paper, we study moments convergence of the intermediate order statistics under power normalization. We show that, among the possible nondegenerate limiting distributions, for the upper intermediate order statistics, only four of them have convergent moments. Therefore, we discuss the convergence of the moments of these‘four limiting distributions according to conditions imposed on the considered distribution and on the rank sequence.

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