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Singular solitons and bifurcation analysis of quadratic nonlinear Klein-Gordon equation

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Abstract: This paper studies the Klein-Gordon equation with quadratic nonlinearity. The ansatz approach is used to first obtain the singular soliton solution of the equation along with the corresponding domain restriction. The bifurcation analysis is also carried out. By this analysis, a few more traveling wave solutions are retrieved. The bifurcation phase portraits are also given.

Keywords: Bifurcation method, Klein-Gordon equation (KGE), traveling wave solution

1 Introduction

The Klein-Gordon equation (KGE) is one of the most important equations that is studied in the area of Quantum Physics. KGE appears in relativistic Quantum Mechanics. Therefore this equation is studied globally by several Physicists and Mathematicians. There are several forms of nonlinearity that goes with KGE. This paper will address the quadratic nonlinearity. In the past, various mathematicians and physicists have considered this equation in their research study and thus many interesting and useful results have been reported in several journals. This paper is going to address this KGE from an integration point of view.

There are several approaches to integrating any nonlinear evolution equation (NLEE) [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28]. Some of the familiar techniques that are commonly visible all across the research board are variational iteration method, Adomian decomposition method [1], Lie symmetry approach [4,5], ansatz method [21,22] and of course numerical simulations [13], just to name a few. KGE, which falls in the category of NLEE is, going to be integrated in this paper by using a couple of approaches. The wave solutions are of interest in this paper. First, the ansatz approach will be employed to obtain the singular solitons. This approach will lead to a domain restriction that will be listed. Continuing on, the bifurcation analysis will be carried out for this equation and thus the phase portraits will be exhibited and analyzed. This will lead to a few more wave solutions to this equation. Such results will be definitely useful in the area of Quantum Mechanics.

2 Governing Equation

The KGE with quadratic law nonlinearity that will be studied in this paper is given by

$$q_{tt} - k^2 q_{xx} = aq + bq^2. \tag{1}$$

In (1), $q(x,t)$ represents the dependent variable or the wave function while the independent variables x and t are the spatial and temporal variables respectively. On the right hand side, is the quadratic form with constant real-valued coefficients a and b.

This equation (1) was studied earlier on several occasions. The soliton perturbation theory was applied to it and the adiabatic parameter dynamics was obtained [2,3,6]. The exact soliton solution was also obtained for the perturbed KGE with quadratic nonlinearity by the aid of

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ansatz method as well as traveling wave hypothesis [4,8,12]. Additionally, the semi-inverse variational principle was utilized to obtain an analytical soliton solution to the perturbed KGE with quadratic nonlinearity [9,11]. In all of these occasions, it is the solitary wave that was studied. In no cases, the singular soliton was taken into consideration.

2.1 Singular Soliton Solution

The starting hypothesis for the singular soliton solution is given by

$$q(x,t) = A\operatorname{csch}^p \tau, \tag{2}$$

where

$$\tau = B(x - vt). \tag{3}$$

In (2), the constant parameters are A and B while v is the velocity of the soliton. Now, substituting (2) into (1) gives

$$\left(v^2 - k^2\right) p^2 A B^2 \operatorname{csch}^p \tau + \left(v^2 - k^2\right) p(p + 1) A B^2 \operatorname{csch}^{p+2} \tau = aA\operatorname{csch}^p \tau + bA^2 \operatorname{csch}^{2p} \tau. \tag{4}$$

It is well known that the solitons are the outcome of a delicate balance between dispersion and nonlinearity. Therefore by the aid of this balancing principle, equating the exponents $2p$ and $p+2$ implies

$$2p = p + 2, \tag{5}$$

that gives

$$p = 2, \tag{6}$$

Now the linearly independent functions in (4) are $\operatorname{csch}^{p+j}\tau$ for $j = 0, 2$. Therefore, setting their respective coefficients to zero implies

$$A = \frac{3a}{2b}, \tag{7}$$

and

$$B = \frac{1}{2}\sqrt{\frac{a}{v^2 - k^2}}. \tag{8}$$

Now (8) poses the restriction

$$a\left(v^2 - k^2\right) > 0. \tag{9}$$

Hence, finally, the singular 1-soliton solution to (1) is given by

$$q(x,t) = A\operatorname{csch}^2[B(x - vt)], \tag{10}$$

where the parameters A and B are given by (7) and (8) respectively. The domain restriction that is given by (9) must also hold in order for the singular soliton to exist.

3 Bifurcation Analysis

In this section, the aim is to study the traveling wave solutions and their relations for (1) by using the bifurcation method and qualitative theory of dynamical systems[15,16,17,18,19,20]. Through some special phase orbits, we obtain many smooth periodic wave solutions and periodic blow-up solutions. Their limits contain singular solitary wave solutions, periodic singular wave solutions and solitary wave solutions.

3.1 Phase Portraits and Qualitative Analysis

We assume that the traveling wave solutions of (1) is of the form

$$q(x,t) = \varphi(\xi), \quad \xi = x - ct, \tag{11}$$

we have

$$(c^2 - k^2)\varphi'' = a\varphi + b\varphi^2. \tag{12}$$

Letting $\varphi' = y$, then we get the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{a}{c^2-k^2}\varphi + \frac{b}{c^2-k^2}\varphi^2. \end{cases} \tag{13}$$

Obviously, the above system (13) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, y) = y^2 - \frac{a}{c^2 - k^2}\varphi^2 - \frac{2b}{3(c^2 - k^2)}\varphi^3. \tag{14}$$

In order to investigate the phase portrait of (13), set

$$f(\varphi) = \frac{a}{c^2 - k^2}\varphi + \frac{b}{c^2 - k^2}\varphi^2. \tag{15}$$

Obviously, $f(\varphi)$ has two zero points, φ_0 and φ_1, which are given as follows

$$\varphi_0 = 0, \qquad \varphi_1 = -\frac{a}{b}. \tag{16}$$

Letting $(\varphi_i, 0)$ be one of the singular points of system (13), then the characteristic values of the linearized system of system (13) at the singular points $(\varphi_i, 0)$ are

$$\lambda_{\pm} = \pm\sqrt{f'(\varphi_i)}. \tag{17}$$

From the qualitative theory of dynamical systems, we know that

(I) If $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.
(II) If $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.
(III) If $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (13) in Fig. 1.
Let

$$H(\varphi, y) = h, \tag{18}$$

where h is Hamiltonian.
Next, we consider the relations between the orbits of (13) and the Hamiltonian h.

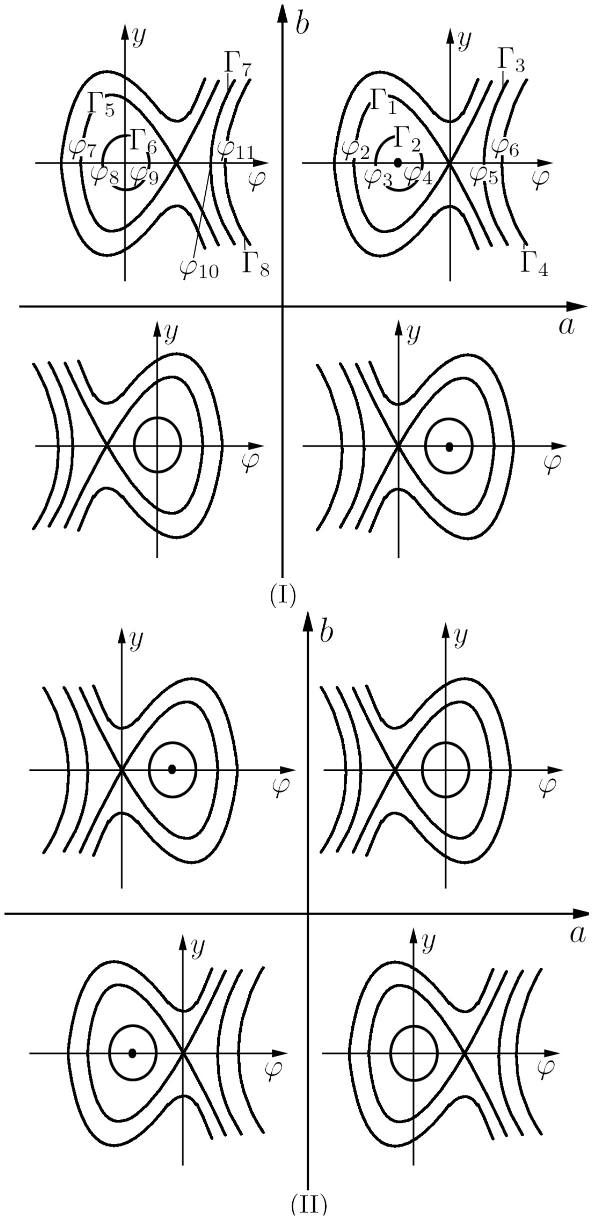

Fig. 1: The Bifurcation phase portraits of system (13), (I) $c^2 - k^2 > 0$, (II) $c^2 - k^2 < 0$

Set

$$h^* = H(\varphi_1, 0) = -\frac{a^3}{3b^2(c^2 - k^2)}. \quad (19)$$

According to Fig. 1, we get the following propositions.

Proposition 1. Suppose that $a > 0$, $b > 0$ and $c^2 - k^2 > 0$, we have

(I) When $h > 0$ or $h < h^*$, system (13) does not any closed orbit.

(II) When $h = 0$, system (13) has a homoclinic orbit Γ_1.

(III) When $h^* < h < 0$, system (13) has a periodic orbit Γ_2 and a special orbit Γ_3.

(IV) When $h = h^*$, system (13) has a special orbit Γ_4.

Proposition 2. Suppose that $a < 0$, $b > 0$ and $c^2 - k^2 > 0$, we have

(I) When $h < 0$ or $h > h^*$, system (13) does not any closed orbit.

(II) When $h = h^*$, system (13) has a homoclinic orbit Γ_5.

(III) When $0 < h < h^*$, system (13) has a periodic orbit Γ_6 and a special orbit Γ_7.

(IV) When $h = 0$, system (13) has a special orbit Γ_8.

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions.

Proposition 3. If $a > 0$, $b > 0$ and $c^2 - k^2 > 0$, we have

(I)When $h = 0$, (1) has a solitary wave solution and a singular solitary solution (corresponding to the homoclinic orbit Γ_1 in Fig. 1).

(II) When $h^* < h < 0$, (1) has a periodic wave solution and a singular wave solution(corresponding to the periodic orbit Γ_2 and the special orbit Γ_3 in Fig. 1).

(III) When $h = h^*$, (1) has a periodic singular wave solution (corresponding to the special orbit Γ_4 in Fig. 1).

Proposition 4. If $a < 0$ and $b > 0$, we have

(I) When $h = h^*$, (1) has a solitary wave solution and a singular solitary solution (corresponding to the homoclinic orbit Γ_5 in Fig. 1).

(II) When $0 < h < h^*$, (1) has a periodic wave solution and a singular wave solution(corresponding to the periodic orbit Γ_6 and the special orbit Γ_7 in Fig. 1).

(III) When $h = 0$, (1) has a periodic singular wave solution (corresponding to the special orbit Γ_8 in Fig. 1).

3.2 Traveling Wave Solutions and Their Relations

Firstly, we will obtain the explicit expressions of traveling wave solutions for the (1) when $a > 0$, $b > 0$ and $c^2 - k^2 > 0$.

(I) From the phase portrait, we note that there is a homoclinic orbit Γ_1 passing the point $(0,0)$. In (φ, y)-plane the expressions of the orbits are given as

$$y = \pm\sqrt{\frac{2b}{3(c^2 - k^2)}\varphi^2(\varphi - \varphi_2)}, \quad (20)$$

where $\varphi_2 = -\frac{3a}{2b}$.

Substituting (20) into $\frac{d\varphi}{d\xi}=y$ and integrating them along Γ_1,we have

$$\pm\int_{\varphi_2}^{\varphi}\frac{1}{\sqrt{s^2(s-\varphi_2)}}ds=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds, \quad (21)$$

$$\pm\int_{\varphi}^{\infty}\frac{1}{\sqrt{s^2(s-\varphi_2)}}ds=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds. \quad (22)$$

Completing above integrals we obtain

$$\varphi=-\frac{3a}{2b}\left(\text{sech}\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}\xi\right)^2, \quad (23)$$

$$\varphi=\frac{3a}{2b}\left(\text{csch}\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}\xi\right)^2. \quad (24)$$

Noting that (11), we get the following solitary wave solution

$$q_1(x,t)=-\frac{3a}{2b}\left(\text{sech}\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right)^2, \quad (25)$$

and singular solitary solution

$$q_2(x,t)=-\frac{3a}{2b}\left(\text{csch}\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right)^2. \quad (26)$$

(II) From the phase portrait, we note that there are a periodic orbit Γ_2 and a special orbit Γ_3 passing the points $(\varphi_3,0)$, $(\varphi_4,0)$ and $(\varphi_5,0)$. In (φ,y)-plane the expressions of the orbits are given as

$$y=\pm\sqrt{\frac{a}{c^2-k^2}\varphi^2+\frac{2b}{3(c^2-k^2)}\varphi^3+h}$$
$$=\pm\sqrt{\frac{2b(\varphi-\varphi_3)(\varphi-\varphi_4)(\varphi-\varphi_5)}{3(c^2-k^2)}}, \quad (27)$$

where $\varphi_3<\varphi_4<\varphi_5$.
Substituting (27) into $\frac{d\varphi}{d\xi}=y$ and integrating them along the two orbits Γ_2 and Γ_3, it follows that

$$\pm\int_{\varphi_3}^{\varphi}\frac{1}{\sqrt{(s-\varphi_3)(s-\varphi_4)(s-\varphi_5)}}ds$$
$$=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds, \quad (28)$$

$$\pm\int_{\varphi}^{+\infty}\frac{1}{\sqrt{(s-\varphi_3)(s-\varphi_4)(s-\varphi_5)}}ds$$
$$=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds. \quad (29)$$

Completing above integrals we obtain

$$\varphi=\varphi_3+(\varphi_4$$
$$-\varphi_3)\left(\text{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_5-\varphi_3)}{3(c^2-k^2)}}\xi,\sqrt{\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}}\right)\right)^2, \quad (30)$$

$$\varphi=\varphi_3+\frac{(\varphi_5-\varphi_3)}{\left(\text{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_5-\varphi_3)}{3(c^2-k^2)}}\xi,\sqrt{\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}}\right)\right)^2}. \quad (31)$$

Noting that (11), we get the following periodic wave solution

$$q_3(x,t)=\varphi_3+(\varphi_4-\varphi_3)$$
$$\left(\text{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_5-\varphi_3)}{3(c^2-k^2)}}(x-ct),\sqrt{\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}}\right)\right)^2, \quad (32)$$

and singular wave solution

$$q_4(x,t)=\varphi_3$$
$$+\frac{(\varphi_5-\varphi_3)}{\left(\text{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_5-\varphi_3)}{3(c^2-k^2)}}(x-ct),\sqrt{\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}}\right)\right)^2}. \quad (33)$$

(III)From the phase portrait, we note that there is a special orbit Γ_4, which has the same hamiltonian with that of the center point $(\varphi_1,0)$. In (φ,y)-plane the expressions of the heterclinic orbits are given as

$$y=\pm\sqrt{\frac{2b}{3(c^2-k^2)}(\varphi-\varphi_1)^2(\varphi-\varphi_6)}. \quad (34)$$

where $\varphi_6=\frac{3a}{2b}$.
Substituting (34) into $\frac{d\varphi}{d\xi}=y$ and integrating them along the special orbit Γ_4 , it follows that

$$\pm\int_{\varphi}^{+\infty}\frac{1}{(s-\varphi_1)\sqrt{(s-\varphi_6)}}ds=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds. \quad (35)$$

Completing above integral we obtain

$$\varphi=\frac{a}{2b}\left(1+3\left(\cot\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}\xi\right)^2\right). \quad (36)$$

Noting that (11), we get the following periodic singular wave solution

$$q_5(x,t)=\frac{a}{2b}\left(1+3\left(\cot\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct)\right)^2\right). \quad (37)$$

Secondly, we will obtain the explicit expressions of traveling wave solutions for the (1) when $a<0$, $b>0$ and $c^2-k^2>0$.
(I) From the phase portrait, we note that there is a homoclinic orbit Γ_5 passing the point $(\varphi_1,0)$. In (φ,y)-plane the expressions of the orbits are given as

$$y=\pm\sqrt{\frac{2b}{3(c^2-k^2)}(\varphi-\varphi_1)^2(\varphi-\varphi_7)}, \quad (38)$$

where $\varphi_7=\frac{3a}{2b}$.
Substituting (38) into $\frac{d\varphi}{d\xi}=y$ and integrating them along Γ_5, we have

$$\pm\int_{\varphi_7}^{\varphi}\frac{1}{\sqrt{(s-\varphi_1)^2(s-\varphi_7)}}ds=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}ds, \quad (39)$$

$$\pm\int_{\varphi}^{\infty}\frac{1}{\sqrt{(s-\varphi_1)^2(s-\varphi_7)}}\mathrm{d}s=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}\mathrm{d}s. \quad (40)$$

Completing above integrals we obtain

$$\varphi=\frac{a}{2b}\left(1-3\left(\tanh\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}\xi\right)^2\right), \quad (41)$$

$$\varphi=\frac{a}{2b}\left(1-3\left(\coth\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}\xi\right)^2\right). \quad (42)$$

Noting that (11), we get the following solitary wave solution

$$q_6(x,t)=\frac{a}{2b}\left(1-3\left(\tanh\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct)\right)^2\right) \quad (43)$$

and singular solitary wave solution

$$q_7(x,t)=\frac{a}{2b}\left(1-3\left(\coth\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct)\right)^2\right) \quad (44)$$

(II) From the phase portrait, we note that there are a periodic orbit Γ_6 and a special orbit Γ_7 passing the points $(\varphi_8,0)$, $(\varphi_9,0)$ and $(\varphi_{10},0)$. In (φ,y)-plane the expressions of the orbits are given as

$$y=\pm\sqrt{\frac{a}{c^2-k^2}\varphi^2+\frac{2b}{3(c^2-k^2)}\varphi^3+h}$$
$$=\pm\sqrt{\frac{2b(\varphi-\varphi_8)(\varphi-\varphi_9)(\varphi-\varphi_{10})}{3(c^2-k^2)}}, \quad (45)$$

where $\varphi_8<\varphi_9<\varphi_{10}$.
Substituting (45) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y$ and integrating them along the two orbits Γ_6 and Γ_7, it follows that

$$\pm\int_{\varphi_8}^{\varphi}\frac{1}{\sqrt{(s-\varphi_8)(s-\varphi_9)(s-\varphi_{10})}}\mathrm{d}s$$
$$=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}\mathrm{d}s, \quad (46)$$

$$\pm\int_{\varphi}^{+\infty}\frac{1}{\sqrt{(s-\varphi_8)(s-\varphi_9)(s-\varphi_{10})}}\mathrm{d}s$$
$$=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}\mathrm{d}s. \quad (47)$$

Completing above integrals we obtain

$$\varphi=\varphi_8+(\varphi_9-\varphi_8)\left(\mathrm{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_{10}-\varphi_8)}{3(c^2-k^2)}}\xi,\sqrt{\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8}}\right)\right)^2, \quad (48)$$

$$\varphi=\varphi_8+\frac{(\varphi_{10}-\varphi_8)}{\left(\mathrm{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_{10}-\varphi_8)}{3(c^2-k^2)}}\xi,\sqrt{\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8}}\right)\right)^2}. \quad (49)$$

Noting that (11), we get the following periodic wave solution

$$q_8(x,t)=\varphi_8+(\varphi_9-\varphi_8)\left(\mathrm{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_{10}-\varphi_8)}{3(c^2-k^2)}}(x-ct),\sqrt{\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8}}\right)\right)^2. \quad (50)$$

and singular wave solution

$$q_9(x,t)=\varphi_8+\frac{(\varphi_{10}-\varphi_8)}{\left(\mathrm{sn}\left(\frac{1}{2}\sqrt{\frac{2b(\varphi_{10}-\varphi_8)}{3(c^2-k^2)}}(x-ct),\sqrt{\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8}}\right)\right)^2}. \quad (51)$$

(III)From the phase portrait, we note that there is a special orbit Γ_8, which has the same hamiltonian with that of the center point $(0,0)$. In (φ,y)-plane the expressions of the special orbit are given as

$$y=\pm\sqrt{\frac{2b}{3(c^2-k^2)}\varphi^2(\varphi-\varphi_{11})}, \quad (52)$$

where $\varphi_{11}=-\frac{3a}{2b}$.
Substituting (52) into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y$ and integrating them along the special orbit Γ_8, it follows that

$$\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{(s-\varphi_{11})}}\mathrm{d}s=\sqrt{\frac{2b}{3(c^2-k^2)}}\int_0^{\xi}\mathrm{d}s. \quad (53)$$

Completing above integral we obtain

$$\varphi=-\frac{3a}{2b}\left(1+\left(\cot\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}\xi\right)^2\right). \quad (54)$$

Noting that (11), we get the following periodic singular wave solution

$$q_{10}(x,t)=-\frac{3a}{2b}\left(1+\left(\cot\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct)\right)^2\right) \quad (55)$$

Thirdly, we will give that relations of the traveling wave solutions.

(1) Letting $h\to 0-$, it follows that $\varphi_3\to-\frac{3a}{2b}$, $\varphi_4\to 0$, $\varphi_5\to 0$, $\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}\to 1$ and $\mathrm{sn}(\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct),1)=\tanh\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct)$. Therefore, we obtain $q_3(x,t)\to q_1(x,t)$ and $q_4(x,t)\to q_2(x,t)$.

(2) Letting $h\to h^*+$, it follows that $\varphi_3\to-\frac{a}{b}$, $\varphi_4\to-\frac{a}{b}$, $\varphi_5\to\frac{a}{2b}$, $\frac{\varphi_4-\varphi_3}{\varphi_5-\varphi_3}\to 0$ and $\mathrm{sn}(\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct),1)=\sin\frac{1}{2}\sqrt{\frac{a}{c^2-k^2}}(x-ct)$. Therefore, we obtain $q_4(x,t)\to q_5(x,t)$.

(3) Letting $h\to h^*-$, it follows that $\varphi_8\to\frac{a}{2b}$, $\varphi_9\to-\frac{a}{b}$, $\varphi_{10}\to-\frac{a}{b}$, $\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8}\to 1$ and $\mathrm{sn}(\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct),1)=\tanh\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct)$. Therefore, we obtain $q_8(x,t)\to q_6(x,t)$ and $q_9(x,t)\to q_7(x,t)$.

(4) Letting $h \to 0+$, it follows that $\varphi_8 \to 0$, $\varphi_9 \to 0$, $\varphi_{10} \to -\frac{3a}{2b}$, $\frac{\varphi_9-\varphi_8}{\varphi_{10}-\varphi_8} \to 0$ and $\mathrm{sn}(\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct),1) = \sin\frac{1}{2}\sqrt{-\frac{a}{c^2-k^2}}(x-ct)$. Therefore, we obtain $q_9(x,t) \to q_{10}(x,t)$.

Finally, we will show that the periodic wave solution $q_3(x,t)$ evolute into the solitary wave solution $q_1(x,t)$ when the Hamiltonian $h \to 0-$ (corresponding to the changes of phase orbits of Fig. 1 as h varies). We take some suitable choices of the parameters, such as

$$a = 2, \qquad b = 1, \qquad c = 2, \qquad k = 1, \tag{56}$$

as an illustrative sample and draw their plots (see Fig. 2).

Remark: One may find that we only consider two cases when $a > 0, b > 0, c^2 - k^2 > 0$ and $a < 0, b > 0, c^2 - k^2 > 0$ in Propositions 1-4. In fact, we may get exactly the similar conclusions in the other cases.

4 Conclusions

This paper studied the KGE with quadratic law nonlinearity. The singular solitary wave solution was first obtained directly by the ansatz method. Subsequently, the bifurcation analysis of the dynamical system was carried out, where the dynamical system of the KGE was obtained from the traveling wave hypothesis. This bifurcation analysis additionally obtained the phase portraits of the dynamical system. Furthermore, several nonlinear wave solutions were extracted from this analysis. These are the solitary wave solutions, topological solitons, cnoidal wave solutions, singular periodic waves and others. These solutions are going to be extremely useful in carrying out further investigation for this equation. For example, in future, the perturbed KGE with quadratic nonlinearity will be investigated. There are various other approaches that will be implemented to integrate these proposed extended models. The results of those researches will be published elsewhere.

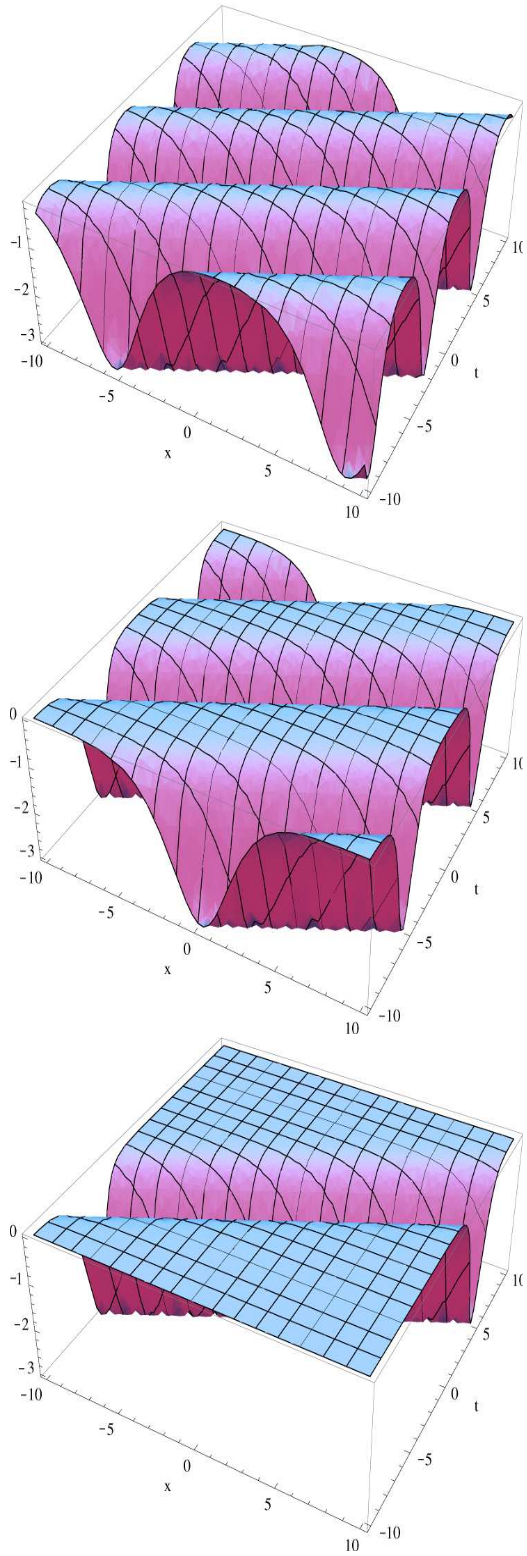

Fig. 2: The periodic wave solution $q_3(x,t)$ evolute into the solitary wave solution $q_3(x,t)$ at $t = 0$ with the conditions (56). (I) $h = -0.01$; (II) $h = -0.0001$; (III) $h = 0$.

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