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Solving Multi-Objective Linear plus Linear Fractional Programming Problem

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Abstract: Multi-objective Programming Problem consists of conflicting objective functions, therefore, it is not necessary that optimum solution for one objective is also optimum for other objectives. In such situation, we need to find out a solution which is optimum for all the objectives in some sense i.e. compromise solution. In this paper, Value function and Chebyshev Goal Programming approaches are suggested to derive the optimum solution of Multi-objective Linear plus Linear Fractional Programming Problem (MOLPLFPP). Illustrative numerical examples are presented for demonstration purpose and the obtained solutions are compared with some existing solutions.

Keywords: Multi-Objective Programming; Multi-Objective Linear plus Linear Fractional Programming; Compromise Solution.

1 Introduction

In mathematical optimization, linear-fractional programming (LFP) is a generalization of linear programming (LP). As the objective functions in linear programs are linear functions, the objective function in a linear-fractional program is a ratio of two linear functions. A linear program can be regarded as a special case of a linear-fractional program in which the denominator is the constant function one.

Interest of this subject was generated by the fact that various optimization problems from engineering and economics consider the minimization of a ratio between physical and/or economical functions, for example cost/time, cost/volume, cost/profit, or other quantities which measures the efficiency of a system. For example, the productivity of industrial systems, defined as the ratio between the realized services in a system within a given period of time and the utilized resources, is used as one of the best indicators of the quality of their operation. Such problems, where the objective function appears as a ratio of functions, constitute fractional programming problem. Due to its importance in modeling various decision processes in management science, operational research, and economics, and also due to its frequent appearance in other problems that are not necessarily economical, such as information theory, numerical analysis, stochastic programming, decomposition algorithms for large linear systems, etc., the fractional programming method has received particular attention in the last three decades.

Therefore, multi-objective linear plus linear fractional programming consists of multiple objectives having the combination of linear and linear fractional programming.

Multi-Objective Linear Fractional Programming (MOLFP) technique is a very important technique for decision making and is used in variety of problems having multiple objectives such as profit/cost, actual cost/standard cost, debt/equity , inventory sales etc, subject to the system constraints. It can also be used to solve various real life problems related to fields like planning problems in agriculture, production, inventory etc.

Charnes and Cooper [4] gave "Programming with Linear Fractional Functionals". Later, Zoints [21] and Schaible [16] also gave ideas which helped a lot in the development of Fractional Programming. In the beginning, Multi-objective Linear Fractional Programming Problem (MOLFPP) posed some difficulties, so, they were converted into single

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objectives and were solved by the methods of Charnes and Cooper [4] and Bitran and Novaes[1]. In 1981, Kornbluth and Steuer [12] introduced Goal Programming technique to solve multi-objective linear fractional programming problem based on variable change method. In 1982, Choo and Atkins [5] gave bi-criteria Linear Fractional Programming. Hirche [9] gave a note on Linear plus Linear Fractional Programming.

Nykowski and Zolkiewski [13], then Dutta *et al.* [6], Chadha [2], Chakraborty and Gupta [3], Pal *et al.* [14] , Guzel and Sivri [8], etc. also contributed in the field. In 2008, Mangal and Sangeeta studied alternative approach for solving LPLFPP based on branch and bound method. Jain *et al.* [10] presented a method for solving multi-objective linear plus linear fractional programming problem (MOLPLFPP) containing non-differentiable term in the constraints. Then Kheirfan [11] suggested the approach of sensitivity analysis to Linear plus Linear Fractional Programming Problems when the right hand side vector and the coefficients of the objective function are allowed to vary. Sharma and Kumar [17] solved LPLFPP subject to two sided linear inequality constraints. They transformed the problem into fractional programming problem by parametric approach and obtained the solution of the problem by using programming theorems.

Singh *et al.* [18] proposed fuzzy method for MOLPLFPP by transforming the problem into multi-objective linear programming problem using first order Taylor series. Then the problem is solved by reducing MOLPLFPP into single objective programming problem by assigning equal weight. Singh *et al.* [19] again studied MOLPLFPP but this time using the goal programming approach. Pramanik *et al.* [15] gave fuzzy goal programming approach to solve MOLFP based on Taylor series approximation. Many other methods have also been developed for solving MOLFP problems using the fuzzy approach and are available in the literature [Kornbluth and Steuer [12], Chakraborty and Gupta [3], Toksari [20] etc].

In this paper, an attempt is made to obtain the compromise solution of MOLPLFP. Formulation of the problem is introduced in section 2. Optimization methods to solve the problem are presented in section 3. Section 4 is devoted to illustrative examples to demonstrate the solution procedure. And finally section 5 gives some concluding remarks.

2 Multi-Objective Linear plus Linear Fractional Programming Problem

The purpose of this paper is to obtain the solution of the following Multi-objective Linear plus Linear Fractional Programming Problem (MOLPLFPP):

$$\left.\begin{array}{r}\text{Maximize } \; Z_i(\bar{x}) = (\bar{c}_i^T\bar{x}+d_i)+\dfrac{\bar{\gamma}_i^T\bar{x}+\alpha_i}{\bar{\delta}_i^T\bar{x}+\beta_i}\\ \text{subject to } \; \bar{x}\in S=\{\bar{x}\in\bar{R}\mid \bar{A}\bar{x}\le=\ge\bar{b},\bar{x}\ge\bar{0}\}\end{array}\right\}\; i=1,2,\ldots,k \tag{1}$$

where $\bar{c}_i^T, \bar{\gamma}_i^T, \bar{\delta}_i^T \in \bar{R}^N; i=1,2,\ldots,k$ and $\bar{A}\in\bar{R}^{N\times M}, \bar{b}\in\bar{R}^M, S$ is assumed to be non-empty, convex and compact in $\bar{R}^N$ and further we also assume that $\bar{x}\in S$ and $(\bar{\delta}_i^T\bar{x}+\beta_i)>0$ for $i=1,2,\ldots,k$.

Here we can see that the objective function, $Z_i(\bar{x})$ in problem (1) is a combination of two terms. The first term is linear and the second term is fractional with linear numerator and denominator.

3 Optimization Techniques

3.1 Value Function

A function which represents the preferences of the decision maker among the objectives is called a value function. Its totally a decision maker concept. Different decision makers have different values for same problem. It offers a total or complete ordering of objective functions.
Now the problem (1) under value function will be expressed as

$$\left.\begin{array}{r}\text{Maximize } \; \phi(Z_i(\bar{x}))\\ \text{subject to } \; \bar{x}\in S=\{\bar{x}\in\bar{R}\mid \bar{A}\bar{x}\le=\ge\bar{b},\bar{x}\ge\bar{0}\}\end{array}\right\} \tag{2}$$

where $\varphi(.)$ is a scalar function that summarizes the preference of each objective function. For different problems the value function $\varphi(.)$ takes a form which is appropriate to the nature of optimum problem. Under this, the given problem will be of the following form.

$$\left.\begin{array}{r}\text{Maximize } \lambda_i(Z_i(\overline{x}))\\ \text{subject to } \overline{x}\in S=\{\overline{x}\in \overline{R}\mid \overline{A}\overline{x}\leq=\geq \overline{b},\overline{x}\geq \overline{0}\}\end{array}\right\} \quad (3)$$

where $\lambda_i \geq 0$ are the weights according to the relative preference of the objective function and $\sum\lambda_i = 1; i = 1,2,\ldots,k$

3.2 Chebyshev Goal Programming

Chebyshev goal programming is one of the major variant of goal programming. It was introduced by Flavell [7]. It is known as Chebyshev goal programming, because it uses the Chebyshev (L_∞) means of measuring distance. That is, in CGP, the maximal deviation from the given goal is minimized as opposed to the sum of all deviations in GP. For this reason Chebyshev goal programming is sometimes referred as Minmax goal programming. Chebyshev goal programming is also the only major variant that can find optimal solutions for linear models that are not located at extreme points in decision space. All of the above points lead to the conclusion that Chebyshev goal programming is one of the powerful form of GP in the situations where a balance between the levels of satisfaction of the goals is required.
Let λ be the maximal deviation from amongst the set of goals then the Chebyshev goal programming has the following algebraic format:

$$\left.\begin{array}{l}\text{Minimize } \lambda\\ \text{subject to } Z_i(\overline{x})+\lambda \geq \alpha_i\\ \qquad\qquad \overline{x}\in S=\{\overline{x}\in \overline{R}\mid \overline{A}\overline{x}\leq=\geq \overline{b},\overline{x}\geq \overline{0}\}\end{array}\right\} \quad (4)$$

where $\alpha_i (i = 1,2,\ldots,k)$ are the most acceptable aspiration levels which are obtained by the following payoff matrix:

$$\begin{array}{c}\\ \overline{x}_1^*\\ \vdots\\ \overline{x}_i^*\end{array}\begin{array}{c}\begin{array}{ccc} Z_1(\overline{x}) & \cdots & Z_i(\overline{x})\end{array}\\ \begin{pmatrix} Z_1(\overline{x}_1^*) & \cdots & Z_i(\overline{x}_1^*)\\ \vdots & \ddots & \vdots\\ Z_1(\overline{x}_i^*) & \cdots & Z_i(\overline{x}_i^*)\end{pmatrix}\end{array}$$

where $\overline{x}_i^*;\ i = 1,2,\ldots,k$ are the individual optimal points of each objective function.
The maximum value of each column gives the best solution i.e., most acceptable aspiration value and the minimum value of each column gives the worst solution i.e., least acceptable value.

4 Illustrative Examples

4.1 Example 1

Consider the following MOLPLFPP objective functions studied by Singh *et al.* [18,19].

$$\left.\begin{array}{l}\text{Maximize } Z_1(\overline{x}) = (-x_1-1)+\dfrac{(-5x_1+4x_2)}{(2x_1+x_2+5)}\\ \text{Maximize } Z_2(\overline{x}) = (x_2+1)+\dfrac{(9x_1+2x_2)}{(7x_1+3x_2+1)}\\ \text{Maximize } Z_3(\overline{x}) = (x_1+1)+\dfrac{(3x_1+8x_2)}{(4x_1+5x_2+3)}\\ \qquad \text{subject to } x_1-x_2\geq 2,\\ \qquad\qquad 4x_1+5x_2\leq 25,\\ \qquad\qquad x_1+9x_2\geq 9,\\ \qquad\qquad x_1\geq 5,\ \ x_1,x_2\geq 0\end{array}\right\} \quad (5)$$

The individual best and worst solutions are obtained by payoff matrix as follows:

$$\begin{array}{r} \\ \bar{x}_1(5,1) \\ \bar{x}_2(5,1) \\ \bar{x}_3(5.8064,0.3548) \end{array} \begin{array}{c} \begin{array}{ccc} Z_1 & Z_2 & Z_3 \end{array} \\ \begin{pmatrix} -7.3125 & 3.205128 & 6.82149 \\ -7.3125 & 3.205128 & 6.82149 \\ -8.4333 & 2.59502 & 7.529954 \end{pmatrix} \end{array}$$

$Z_1^B = -7.312$ & $Z_1^W = 8.433$; $Z_2^B = 3.205$ & $Z_2^W = 2.595$; $Z_3^B = 7.53$ & $Z_3^W = 6.736$.

Value function technique:
To formulate the problem under value function approach, we give equal preference to first and second objective function i.e. $\lambda_1 = \lambda_2 = 0.3$ and more preference to third objective function i.e. $\lambda_3 = 0.4$ as follows:

$$\left.\begin{aligned} \text{Maximize} = & 0.3\left((-x_1-1)+\frac{(-5x_1+4x_2)}{(2x_1+x_2+5)}\right) \\ & +0.3\left((x_2+1)+\frac{(9x_1+2x_2)}{(7x_1+3x_2+1)}\right) \\ & +0.4\left((x_1+1)+\frac{(3x_1+8x_2)}{(4x_1+5x_2+3)}\right) \\ \text{subject to } & x_1-x_2 \geq 2, \\ & 4x_1+5x_2 \leq 25, \\ & x_1+9x_2 \geq 9, \\ & x_1 \geq 5, \;\; x_1,x_2 \geq 0 \end{aligned}\right\} \quad (6)$$

Above problem (6) is solved by an optimization software LINGO and derive the optimum compromise solution which is summarized in Table (1).

Chebyshev goal programming technique
Using the best and worst solutions, the compromise solution of problem (5) is obtained as follows:

$$\left.\begin{aligned} & \text{Minimize} = \lambda \\ & \text{subject to } \left((-x_1-1)+\frac{(-5x_1+4x_2)}{(2x_1+x_2+5)}\right)+\lambda \geq -7.312 \\ & \left((x_2+1)+\frac{(9x_1+2x_2)}{(7x_1+3x_2+1)}\right)+\lambda \geq 3.205 \\ & \left((x_1+1)+\frac{(3x_1+8x_2)}{(4x_1+5x_2+3)}\right)+\lambda \geq 7.53 \\ & x_1-x_2 \geq 2, \\ & 4x_1+5x_2 \leq 25, \\ & x_1+9x_2 \geq 9, \\ & x_1 \geq 5, \;\; x_1,x_2,\lambda \geq 0 \end{aligned}\right\} \quad (7)$$

Above problem (7) is solved by an optimization software LINGO and derive the optimum compromise solution which is summarized in Table (1).

Table 1: Compromise solution of MOLPLFFPP

Approach	Solution point		Objective values		
	x_1	x_2	Z_1	Z_2	Z_3
Value function	5	1	-7.312	3.025	6.821
Chebychev goal programming	5.3101	0.7519	-7.748	2.9713	7.0939
Pramanik et al. (2011)[model 1]	5.352	0.718	-7.807	3.025	6.821
Pramanik et al. (2011)[model 2]	5	1	-7.312	3.025	6.821
Singh et al. (2010)	5	1	-7.312	3.025	6.821
Singh et al. (2011)	5	1	-7.312	3.025	6.821

4.2 Example 2

Consider the following MOLPLFFPP objective functions studied by Singh *et al.* [18,19].

$$\left.\begin{aligned}
\text{Maximize } Z_1(\bar{x}) &= (-x_1-1)+\frac{(-x_1+2x_2-5)}{(7x_1+3x_2+1)}\\
\text{Maximize } Z_2(\bar{x}) &= (-2x_2-1)+\frac{(2x_1-3x_2-5)}{(x_1+1)}\\
\text{Maximize } Z_3(\bar{x}) &= (-3x_1-1)+\frac{(5x_1+2x_2-19)}{(-5x_1+20)}\\
\text{subject to } & x_1\le 6,\\
& x_2\le 6,\\
& 2x_1+x_2\le 9,\\
& -2x_1+x_2\le 5,\\
& x_1-x_2\le 5,\ \ x_1,x_2\ge 0
\end{aligned}\right\} \tag{8}$$

The individual best and worst solutions are obtained by payoff matrix as follows:

$$\begin{matrix} \\ \bar{x_1}(0,5) \\ \bar{x_2}(4.5,0) \\ \bar{x_3}(0,5) \end{matrix}
\begin{matrix} \begin{matrix} Z_1 & \quad Z_2 & \quad Z_3 \end{matrix} \\
\begin{pmatrix} -0.6875 & -31 & -1.45 \\ -5.7923 & -0.2727 & -15.9 \\ -0.6875 & -31 & -1.45 \end{pmatrix} \end{matrix}$$

$Z_1^B = -0.688$ & $Z_1^W = -5.792$; $Z_2^B = -0.272$ & $Z_2^W = -31$; $Z_3^B = -1.45$ & $Z_3^W = -15.9$.

The compromise solution of problem (8) is obtained on similar steps as problem (5) and summarized in Table (2).

Table 2: Compromise solution of MOLPLFFPP

Approach	Solution point		Objective values		
	x_1	x_2	Z_1	Z_2	Z_3
Value function	0.4806	0	-2.736409	-3.727813	-3.384972
Chebychev goal programming	0.7480	0	-2.669745	-3.004577	-4.182499
Pramanik et al. (2011)[model 1]	0	0.302	-3.306	-7.51	-1.92
Pramanik et al. (2011)[model 2]	0	0	-6	-6	-1.95
Singh et al. (2010)	0	0	-6	-6	-1.95
Singh et al. (2011)	0	0	-6	-6	-1.95

5 Conclusion

In the present paper, Value function and Chebyshev Goal Programming techniques are suggested to derive the optimum compromise solution of MOLPLFFPP. However, Pramanik in 2011 applied Fuzzy goal programming technique and formulate two models, whereas, Pitam Singh in 2010 proposed an algorithm using fuzzy set theory and Taylor series polynomial method and again in 2011, Goal programming technique to solve these types of problems but the techniques used in this manuscript are very easy to apply and produces the result in minimum number of steps. The numerical examples are solved to demonstrate the computational details and the result are compared with the result of Pramanik *et al.* [15] and Singh *et al.* [18,19] and it can be concluded from the computational results that our result is quite comparable with their results.

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References

[1] Bitran, G., and Novaes, A. Linear programming with a fractional objective function. *Operations Research 21*, 1 (1973), 22–29.
[2] Chadha, S. Dual of the sum of a linear and linear fractional program. *European journal of operational research 67*, 1 (1993), 136–139.
[3] Chakraborty, M., and Gupta, S. Fuzzy mathematical programming for multi objective linear fractional programming problem. *Fuzzy sets and systems 125*, 3 (2002), 335–342.
[4] Charnes, A., and Cooper, W. W. Programming with linear fractional functionals. *Naval Research logistics quarterly 9*, 3-4 (1962), 181–186.
[5] Choo, E., and Atkins, D. Bicriteria linear fractional programming. *Journal of Optimization Theory and Applications 36*, 2 (1982), 203–220.
[6] Dutta, D., Tiwari, R., and Rao, J. Multiple objective linear fractional programminga fuzzy set theoretic approach. *Fuzzy Sets and Systems 52*, 1 (1992), 39–45.
[7] Flavell, R. A new goal programming formulation. *Omega 4*, 6 (1976), 731–732.
[8] Guzel, N., and Sivri, M. Taylor series solution of multi objective linear fractional programming problem. *Trakya University Journal Science 6*, 2 (2005), 80–87.
[9] Hirche, J. A note on programming problems with linear-plus-linear-fractional objective functions. *European journal of operational research 89*, 1 (1996), 212–214.
[10] Jain, S., Mangal, A., and Parihar, P. Solution of a multi objective linear plus linear fractional programming problem containing non-differentiable term. *International J. of Math. Sci. and Engg. Appls 2* (2008), 221–229.
[11] Kheirfam, B. Senstivity analysis in linear plus linear fractional programming problems. *Iranian Journal of Optimization 1* (2009), 1–12.
[12] Kornbluth, J. S., and Steuer, R. E. Goal programming with linear fractional criteria. *European Journal of Operational Research 8*, 1 (1981), 58–65.
[13] Nykowski, I., and Zołkiewski, Z. A compromise procedure for the multiple objective linear fractional programming problem. *European Journal of Operational Research 19*, 1 (1985), 91–97.
[14] Pal, B. B., Moitra, B. N., and Maulik, U. A goal programming procedure for fuzzy multiobjective linear fractional programming problem. *Fuzzy Sets and Systems 139*, 2 (2003), 395–405.
[15] Pramanik, S., Dey, P. P., and Giri, B. C. Multi-objective linear plus linear fractional programming problem based on taylor series approximation. *International Journal of Computer Applications 32*, 8 (2011), 61–68.
[16] Schaible, S. A note on the sum of a linear and linear-fractional function. *Naval Research Logistics Quarterly 24*, 4 (1977), 691–693.
[17] Sharma, S., and Kumar, P. On linear plus linear fractional interval programming problem. *International Journal of Mathematical Sciences & Engineering Applications (IJMSEA) 5*, 1 (2011), 155–160.
[18] Singh, P., Kumar, S. D., and Singh, R. Fuzzy method for multi-objective linear plus linear fractional programming problem. In *International Mathematical Forum* (2010), vol. 5, pp. 2971–2983.
[19] Singh, P., Kumar, S. D., and Singh, R. Fuzzy multi-objective linear plus linear fractional programming problem: approximation and goal programming approach. *International Journal of Mathematics and Computers in Simulation 5*, 5 (2011), 395–404.
[20] Toksari, M. D. Taylor series approach to fuzzy multiobjective linear fractional programming. *Information Sciences 178*, 4 (2008), 1189–1204.
[21] Zionts, S. Programming with linear fractional functionals. *Naval Research Logistics Quarterly 15*, 3 (1968), 449–451.