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Solutions of Nonlinear Oscillators by Iteration Perturbation Method

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Abstract: In this paper, the iteration perturbation method is applied to solve nonlinear oscillations. Two examples are given to illustrate the effectiveness and convenience of this iteration procedure. Comparison with the numerical solutions is also presented, revealing that this iteration leads to accurate solutions.

Keywords: Iteration perturbation method, Nonlinear oscillators, Duffing oscillator, Van der Pol oscillator.

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1 Introduction

The most common and most widely studied methods for determining analytical approximate solutions of a nonlinear oscillatory system are the perturbation methods. These methods involve the expansion of a solution to an oscillation equation in a series in a small parameter. Several researchers have studied different nonlinear problems by means of iteration procedures [1,2,3,4,5,6,7,8].

The purpose of this paper is to apply the iteration procedure to determine analytical approximate solutions to the nonlinear oscillation equation. With this procedure, the analytical approximate period and the corresponding periodic solutions, valid for small as well as large amplitudes of oscillation, can be obtained. The nonlinear Duffing and Van der Pol oscillations will be taken as examples to illustrate the applicability and accuracy of the iteration procedure.

2 The iteration procedure

Consider a nonlinear conservative oscillator described as

$$\ddot{x} + f(x) = 0,\ x(0) = A,\ \dot{x}(0) = 0, \tag{1}$$

where $f(x)$ in a nonlinear function and has the property

$$f(-x)) = -f(x). \tag{2}$$

Eq. (1) can be rewritten as

$$\ddot{x} + \omega^2 x = \omega^2 x - f(x), \tag{3}$$

where the constant ω is a priori unknown frequency of the periodic solution $x(t)$ being sough. The original Mickens procedure is given as [1] .

$$\ddot{x}_k + \omega^2 x_k = g(\omega, x_{k-1}),\ k = 1, 2, \ldots \tag{4}$$

where the input of starting function is

$$x_0(t) = A\cos\omega t. \tag{5}$$

This iteration scheme was used to solve many nonlinear oscillating equations [9,10,11].

Lim et al. [3] proposed a modified iteration scheme

$$\begin{gathered} \ddot{x}_{k+1} + \omega^2 x_{k+1} = g(\omega, x_{k-1}) \\ + g_x(\omega, x_{k-1})(x_k - x_{k-1}),\ k = 0, 1, 2, \ldots \end{gathered} \tag{6}$$

with the imputes of starting functions as

$$x_{-1}(t) = x_0(t) = A\cos\omega t \tag{7}$$

where $g_x(\omega, x) = \partial g(\omega, x)/\partial x$. The modified procedure was also applied to solve many nonlinear oscillators [12,13,14,15]

Chen and Liu [5] proposed a new iteration scheme, considering ω as ω_k:

$$\ddot{x}_k + \omega_{k-1}^2 x_k = g(\omega_{k-1}, x_{k-1}),\ k = 1, 2, \ldots \tag{8}$$

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where the right hand side of Eq (8) can be expanded in the Fourier series

$$g(\omega_{k-1},x_{k-1})=\sum_{i=1}^{\varphi(k)} a_{k-1,i(\omega_{k-1})}\cos(i\omega_{k-1}t), \qquad (9)$$

where the coefficient $a_{k-1,j}$ are functions of ω_{k-1} and $\varphi(k)$ is a positive integer. The $(k-1)$ th-order approximation ω_{k-1} is obtain by eliminating the so-called secular terms, i.e., letting

$$a_{k-1,i}(\omega_{k-1})=0, \quad k=1,2,\dots \qquad (10)$$

Eq (10) is always a linear algebraic equation in ω^2_{k-1}

J. H. He [7] proposed a new iteration scheme considering the following nonlinear oscillator

$$\ddot{x}+x+\varepsilon f(x,\dot{x})=0,\ x(0)=A,\ \dot{x}(0)=0. \qquad (11)$$

We rewrite Eq (11) in the following form

$$\ddot{x}+x+\varepsilon x g(x,\dot{x})=0 \qquad (12)$$

where $g(x,\dot{x})=f(x,\dot{x})/x$

J H He has constructed an iteration formula for the above equation

$$\ddot{x}_{k+1}+x_{k+1}+\varepsilon x_{k+1}g(x_k,\dot{x}_k), \qquad (13)$$

Marinca and Herisanu [8] proposed a new iteration method by combining Mickens and He's iteration methods, considering the following nonlinear oscillator

$$\ddot{x}+\omega^2x=f(x,\dot{x},\ddot{x})=0,\ x(0)=A,\ \dot{x}(0)=0 \qquad (14)$$

We rewrite Eq. (14) in the following form

$$\ddot{x}+\Omega^2x=x\left(\Omega^2-\omega^2-\frac{f(x,\dot{x},\ddot{x})}{x}\right):=xg(x,\dot{x},\ddot{x}). \quad (15)$$

where Ω is a priori unknown frequency of the periodic solution $x(t)$ being sought.

The proposed iteration scheme is

$$\begin{aligned}\ddot{x}_{n+1}+\Omega^2x_{n+1}=x_{n-1}\,[&g(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})\\&+g_x(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})(x_n-x_{n-1})\\&+g_{\dot{x}}(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})(\dot{x}_n-\dot{x}_{n-1})\\&+g_{\ddot{x}}(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})(\ddot{x}_n-\ddot{x}_{n-1})],\ n=0,1,2,\dots\end{aligned} \qquad (16)$$

where the imputes of starting functions are [3]

$$x_{-1}(t)=x_0(t)=A\cos\Omega t. \qquad (17)$$

It is further required that for each n, the solution to Eq. (16), is to satisfy initial conditions

$$x_n(0)=A,\ \dot{x}_n(0)=0,\ n=1,2,3,\dots. \qquad (18)$$

Note that, for given $x_{n-1}(t)$ and $x_n(t)$ Eq. (16) is a second order inhomogeneous differential equation for $x_{n-1}(t)$. Its right side can be expanded into the following Fourier series:

$$\begin{aligned}&x_{n-1}\,[g(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})+g_x(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})(x_n-x_{n-1})\\&+g_{\dot{x}}(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})(\dot{x}_n-\dot{x}_{n-1})+g_{\ddot{x}}(x_{n-1},\dot{x}_{n-1},\ddot{x}_{n-1})\\&x(\ddot{x}_n-\ddot{x}_{n-1})]=a_1(A,\Omega,\omega)\cos\Omega t+b_1(A,\Omega,\omega)\sin\Omega t\\&+\textstyle\sum_{n=2}^{N}a_n(A,\Omega,\omega)\cos n\Omega t+\sum_{n=2}^{N}b_n(A,\Omega,\omega)\sin n\Omega t,\end{aligned} \qquad (19)$$

where the coefficients $a_n(A,\Omega,\omega)$ and $b_n(A,\Omega,\omega)$ are known functions of A and ω, and the integer N depends upon the function $g(x,\dot{x},\ddot{x})$ on the right hand side of Eq. (15). In view of Eq. (19), the solution to Eq. (16) is taken to be

$$\begin{aligned}x_{n+1}=A\cos\Omega t&-\sum_{n=2}^{N}\tfrac{a_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(\cos n\Omega t-\cos\Omega t)\\&-\sum_{n=2}^{N}\tfrac{b_n(A,\Omega,\omega)}{(n^2-1)\Omega^2}(\sin n\Omega t-\sin\Omega t),\end{aligned} \qquad (20)$$

where A is, tentatively, an arbitrary constant. In Eq. (20), the particular solution is chosen such that it contains no secular terms needs

$$a_1(A,\Omega,\omega)=0,\quad b_1(A,\Omega,\omega)=0. \qquad (21)$$

Eq. (21) allows the determination of the frequency Ω as a function of A and ω. this procedure can be performed to any desired iteration step n.

3 Applications

In order to illustrate the remarkable accuracy of this iteration, we compare the approximate results with numerical integration results for the following two examples.

3.1 *Duffing oscillator with high nonlinearity*

Consider the following nonlinear Duffing equation with high nonlinearity, which models many structural systems, it is regarded as one of the most important differential equations because it appears in various physical and engineering problems such that, nonlinear optics and plasma physics [16,17].

$$\ddot{x}+x+\alpha x^3+\beta x^5+\gamma x^7=0,\ x(0)=A,\ \dot{x}(0)=0, \quad (22)$$

where x is displacement and, α, β and γ are arbitrary constants.

We rewrite Eq. (12) in the form

$$\ddot{x}_1+\omega^2x_1=x_0\left(\omega^2-1-\alpha x_0^2-\beta x_0^4-\gamma x_0^6\right), \qquad (23)$$

where $g(x,\dot{x},\ddot{x})=\left(\omega^2-1-\alpha x_0^2-\beta x_0^4-\gamma x_0^6\right)$ and the inputs of the starting function are $x_{-1}(t)=x_0(t)=A\cos\omega t$.

The first iteration is given by the equation

$$\begin{aligned}\ddot{x}_1+\omega^2x_1=&-\left(\tfrac{64A+48\alpha A^3+40\beta A^5+35\gamma A^7-64A\omega^2}{64}\right)\cos\omega t\\&-\left(\tfrac{16\alpha A^3+20\beta A^5+21\gamma A^7}{64}\right)\cos 3\omega t\\&-\left(\tfrac{4\beta A^5+7\gamma A^7}{64}\right)\cos 5\omega t-\left(\tfrac{\gamma A^7}{64}\right)\cos 7\omega t.\end{aligned}\tag{24}$$

No secular terms in x_1 requires that

$$\omega=\omega_1=\sqrt{1+\frac{3}{4}\alpha A^2+\frac{5}{8}\beta A^4+\frac{35}{64}\gamma A^6}.\tag{25}$$

This equation is identical to Eq. (10) in Ref [16] and Eq. (25) in Ref [17]. Solving Eq. (24) with initial conditions (18), x_1 is obtained as

$$\begin{aligned}x_1=&A\cos\omega t+\left(\tfrac{16\alpha A^3+20\beta A^5+21\gamma A^7}{512\omega^2}\right)(\cos 3\omega t-\cos\omega t)\\&+\left(\tfrac{4\beta A^5+7\gamma A^7}{1536\omega^2}\right)(\cos 5\omega t-\cos\omega t)\\&+\left(\tfrac{\gamma A^7}{3072\omega^2}\right)(\cos 7\omega t-\cos\omega t),\end{aligned}\tag{26}$$

for $n=1$ into Eq. (16) with the initial functions (18) and x_1 given by Eq. (26) we obtain the following differential equation for x_2

$$\begin{aligned}\ddot{x}_2+\omega^2x_2=&-\Big(A+\tfrac{3\alpha A^3}{4}+\tfrac{5\beta A^5}{8}+\tfrac{35\gamma A^7}{64}-\tfrac{\alpha^2A^5}{32\omega^2}-\tfrac{21\alpha\beta A^7}{256\omega^2}\\&-\tfrac{7\beta^2A^9}{128\omega^2}-\tfrac{183\alpha\gamma A^9}{2048\omega^2}-\tfrac{185\beta\gamma A^{11}}{1536\omega^2}-\tfrac{1095\gamma 2A^{13}}{16384\omega^2}-A\omega^2\Big)\cos\omega t\\&-\Big(\tfrac{\alpha A^3}{4}+\tfrac{5\beta A^5}{16}+\tfrac{21\gamma A^7}{64}+\tfrac{\alpha^2A^5}{64\omega^2}+\tfrac{7\alpha\beta A^7}{256\omega^2}+\tfrac{7\beta^2A^9}{768\omega^2}+\tfrac{125\alpha\gamma A^9}{6144\omega^2}\\&+\tfrac{15\beta\gamma A^{11}}{2048\omega^2}-\tfrac{99\gamma 2A^{13}}{32768\omega^2}\Big)\cos 3\omega t-\Big(\tfrac{\beta A^5}{16}+\tfrac{7\gamma A^7}{64}+\tfrac{\alpha^2A^5}{64\omega^2}\\&+\tfrac{35\alpha\beta A^7}{768\omega^2}+\tfrac{25\beta^2A^9}{768\omega^2}+\tfrac{299\alpha\gamma A^9}{6144\omega^2}+\tfrac{425\beta\gamma A^{11}}{6144\omega^2}+\tfrac{599\gamma 2A^{13}}{16384\omega^2}\Big)\cos 5\omega t\\&-\Big(\tfrac{\gamma A^7}{64}+\tfrac{7\alpha\beta A^7}{768\omega^2}+\tfrac{19\beta^2A^9}{1536\omega^2}+\tfrac{53\alpha\gamma A^9}{3072\omega^2}+\tfrac{455\beta\gamma A^{11}}{12288\omega^2}+\tfrac{845\gamma^2A^{13}}{32768\omega^2}\Big)\\&\mathrm{x}\cos 7\omega t-\Big(\tfrac{\beta^2A^9}{1536\omega^2}+\tfrac{19\alpha\gamma A^9}{6144\omega^2}+\tfrac{27\beta\gamma A^{11}}{4096\omega^2}+\tfrac{225\gamma^2A^{13}}{32768\omega^2}\Big)\cos 9\omega t\\&-\Big(\tfrac{\beta\gamma A^{11}}{3072\omega^2}+\tfrac{5\gamma^2A^{13}}{8192\omega^2}\Big)\cos 11\omega t-\tfrac{\gamma^2A^{13}}{32768\omega^2}\cos 13\omega t.\end{aligned}\tag{27}$$

The absence of secular term gives the following equation for ω^2

$$\begin{aligned}\omega^4-&\left(\tfrac{64+48\alpha A^2+40\beta A^4+35\gamma A^6}{64}\right)\omega^2+\Big(\tfrac{786432\alpha^2A^4+2064384\alpha\beta A^6}{25165824}\\&\tfrac{+1376256\beta^2A^8+2248704\alpha\gamma A^8+3031040\beta\gamma A^{10}+1681920\gamma^2A^{12}}{25165824}\Big)=0.\end{aligned}\tag{28}$$

Solving Eq. (28) for ω yields

$$\omega=\omega_2=\sqrt{\frac{64+48\alpha A^4+40\beta A^4+35\gamma A^6}{128}+\frac{\sqrt{\Delta_1+\Delta_2}}{64\sqrt{6}}},\tag{29}$$

where

$$\begin{aligned}\Delta_1&=6144+9216\alpha A^2+2688\alpha^2A^4\\&\quad+7680\beta A^4+3744\alpha\beta A^6;\\\Delta_2&=1056\beta^2A^8+6720\gamma A^6+2844\alpha\gamma A^8\\&\quad+1240\beta\gamma A^{10}+195\gamma^2A^{12}.\end{aligned}$$

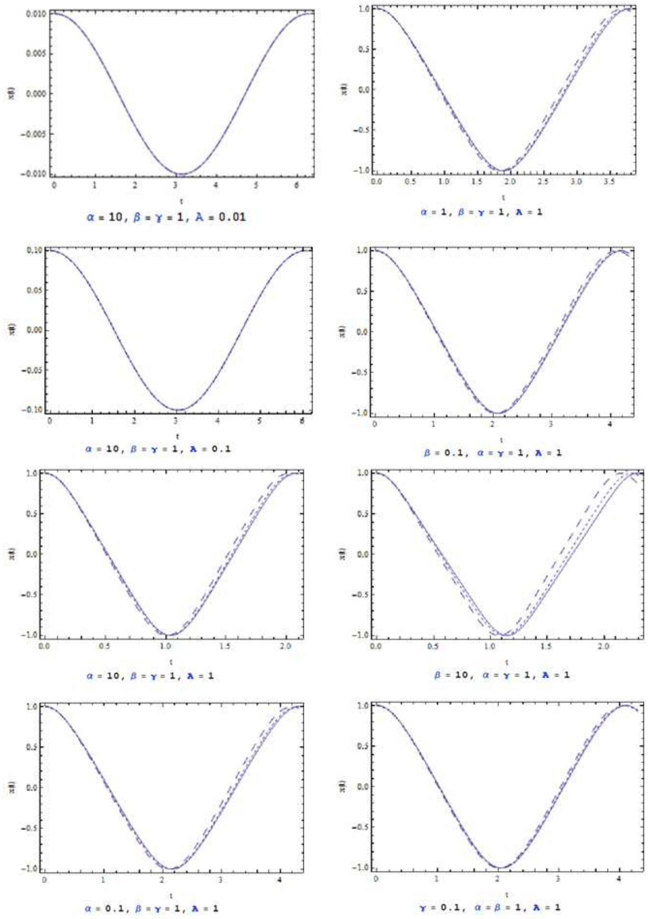

Fig. 1: Comparison of the approximate periodic solution with the numerical solution. Numerical; ..... ; $x_1(t)$ - - - - [16]; $x_2(t)$ —

Solving Eq. (27) with the initial condition (18), we obtain

$$\begin{aligned}x_2=&A\cos\omega t+\Big(\tfrac{\alpha A^3}{32\omega^2}+\tfrac{5\beta A^5}{128\omega^2}+\tfrac{21\gamma A^7}{512\omega^2}+\tfrac{\alpha^2A^5}{512\omega^4}+\tfrac{7\alpha\beta A^7}{2048\omega^4}\\&+\tfrac{7\beta^2A^9}{6144\omega^4}+\tfrac{125\alpha\gamma A^9}{49152\omega^4}+\tfrac{15\beta\gamma A^{11}}{16384\omega^4}-\tfrac{99\gamma 2A^{13}}{262144\omega^4}\Big)\ (\cos 3\omega t-\cos\omega t)\\&+\Big(\tfrac{\beta A^5}{384\omega^2}+\tfrac{7\gamma A^7}{1536\omega^2}+\tfrac{\alpha^2A^5}{1536\omega^4}+\tfrac{35\alpha\beta A^7}{18432\omega^4}+\tfrac{25\beta^2A^9}{18432\omega^4}\\&+\tfrac{299\alpha\gamma A^9}{147456\omega^4}+\tfrac{425\beta\gamma A^{11}}{147456\omega^4}+\tfrac{599\gamma 2A^{13}}{393216\omega^4}\Big)\ (\cos 5\omega t-\cos\omega t)\\&+\Big(\tfrac{\gamma A^7}{3072\omega^2}+\tfrac{7\alpha\beta A^7}{36864\omega^4}+\tfrac{19\beta^2A^9}{73728\omega^4}+\tfrac{53\alpha\gamma A^9}{147456\omega^4}+\tfrac{455\beta\gamma A^{11}}{589824\omega^4}\\&+\tfrac{845\gamma 2A^{13}}{1572864\omega^4}\Big)\ (\cos 7\omega t-\cos\omega t)+\Big(\tfrac{\beta^2A^9}{122880\omega^4}\\&+\tfrac{19\alpha\gamma A^9}{491520\omega^4}+\tfrac{27\beta\gamma A^{11}}{327680\omega^4}+\tfrac{225\gamma 2A^{13}}{2621440\omega^4}\Big)(\cos 9\omega t-\cos\omega t)\\&+\ \Big(\tfrac{\beta\gamma A^{11}}{368640\omega^4}+\tfrac{5\gamma 2A^{13}}{983040\omega^4}\Big)(\cos 11\omega t-\cos\omega t)\\&+\tfrac{\gamma 2A^{13}}{5505024\omega^4}(\cos 13\omega t-\cos\omega t).\end{aligned}\tag{30}$$

Fig. 1 shows a comparison between the present solution obtained from formulae (29) and (30) and the numerical integration results obtained by using the Runge-Kutta method. From the results presented here and the results of Ref. [16], it is shown that the present results are in a good agreement with those presented in Ref. [16].

3.2 Autonomous modified Van der Pol oscillator

One of the classical equations of non-linear dynamics was formulated by Dutch physicist Van der Pol. Originally it was a model for an electrical circuit with a triode valve, and was later extensively studied as a host of a rich class of dynamical behavior, including relaxation oscillations, quasi periodicity, elementary bifurcations, and chaos [18, 19]. A modified Van der Pol oscillator has been proposed to describe a self-excited body sliding on a periodic potential. This autonomous modified Van der Pol oscillator is described by the following equation [20].

$$\ddot{x}+x+\varepsilon\left(x^2-1\right)\dot{x}+P\sin x=0,\ x(0)=A,\ \dot{x}(0)=0. \quad (31)$$

In this case we have $g(x,\dot{x},\ddot{x}) = \left(\omega^2-1-\frac{\varepsilon\left(x_0^2-1\right)\dot{x}_0-P\sin x_0}{x_0}\right)$ and $x_{-1}(t) = x_0(t) = A\cos\omega t$. The first iteration can be written in the form

$$\ddot{x}_1+\omega^2 x_1 = x_0\left(\omega^2-1-\frac{\varepsilon\left(x_0^2-1\right)\dot{x}_0-P\sin x_0}{x_0}\right). \quad (32)$$

The term $\sin x_0 = \sin(A\cos\omega t)$ can be expanded in the power series

$$\begin{aligned}&\sin(A\cos\omega t) = A\cos\omega t-\tfrac{A^3\cos^3\omega t}{3!}\\&+\tfrac{A^5\cos^5\omega t}{5!}-\tfrac{A^7\cos^7\omega t}{7!}+\tfrac{A^9\cos^9\omega t}{9!}+\ldots\end{aligned} \quad (33)$$

We rewrite powers $\cos\omega t$ in Eq. (33) in terms of the cosine of multiples of ωt with the aid of the identity [21].

$$\cos^{2n+1}\omega t = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{n-k}\cos(2k+1)\omega t, \quad (34)$$

where

$$\binom{n}{p} = \frac{n!}{p!(n-p)!};\ \binom{n}{0} = 1;\ k! = 1.2.3.....k; k\varepsilon N\ .$$

By using Eq. (34), Eq. (33) may be expressed in the form

$$\begin{aligned}&\sin(A\cos\omega t) = A\cos\omega t-\tfrac{A^3}{24}(\cos 3\omega t+3\cos\omega t)\\&+\tfrac{A^5}{1920}(\cos 5\omega t+5\cos 3\omega t+10\cos\omega t)\\&-\tfrac{A^7}{322560}(\cos 7\omega t+7\cos 5\omega t+21\cos 3\omega t+35\cos\omega t)\\&+\tfrac{A^9}{92897280}(\cos 9\omega t+9\cos 7\omega t+36\cos 5\omega t\\&+84\cos 3\omega t+126\cos\omega t).\end{aligned} \quad (35)$$

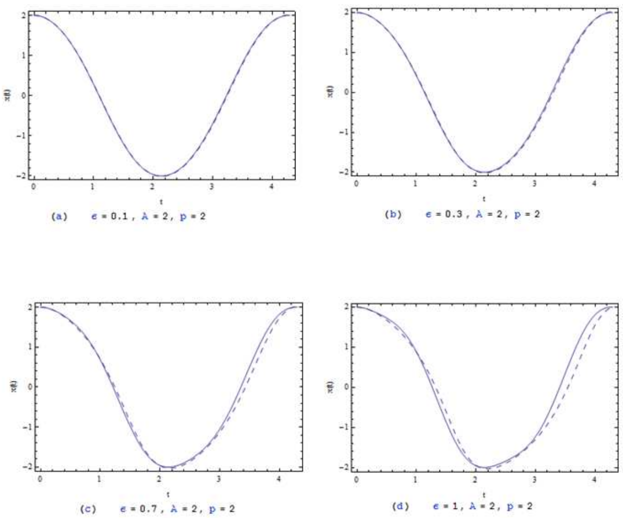

Fig. 2: Comparison of the approximate periodic solution (—) with the numerical solution (- - -).

Substituting Eq. (35) into Eq. (32), this can be rewritten as:

$$\begin{aligned}&\ddot{x}_1+\omega^2 x_1 = \left(-A-p+\tfrac{A^3 p}{8}-\tfrac{A^5 p}{192}+\tfrac{A^7 p}{9216}-\tfrac{A^9 p}{737280}+A\omega^2\right)\\&\times\cos\omega t-\left(\varepsilon A\omega-\tfrac{1}{4}\varepsilon A^3\omega\right)\sin\omega t+\tfrac{1}{4}\varepsilon A^3\omega\sin 3\omega t+\\&\left(\tfrac{A^3 p}{24}-\tfrac{A^5 p}{384}+\tfrac{A^7 p}{15360}-\tfrac{A^9 p}{1105920}\right)\cos 3\omega t-\left(\tfrac{A^5 p}{1920}-\tfrac{A^7 p}{46080}\right.\\&\left.+\tfrac{A^9 p}{2580480}\right)\cos 5\omega t+\left(\tfrac{A^7 p}{322560}-\tfrac{A^9 p}{10321920}\right)\cos 7\omega t\\&-\tfrac{A^9 p}{92897280}\cos 9\omega t.\end{aligned} \quad (36)$$

No secular terms in x_1 requires that

$$A=2,\quad \omega_1=\sqrt{1+p-\frac{A^2 p}{8}+\frac{A^4 p}{192}-\frac{A^6 p}{9216}+\frac{A^8 p}{737280}}. \quad (37)$$

Solving Eq. (36) with initial conditions (18), x_1 is obtained as

$$\begin{aligned}&x_1 = A\cos\omega t+\tfrac{\varepsilon}{4\omega}(3\sin\omega t-\sin 3\omega t)\\&+\tfrac{557p}{17280\omega^2}(\cos\omega t-\cos 3\omega t)\\&-\tfrac{71p}{120960\omega^2}(\cos\omega t-\cos 5\omega t)\\&+\tfrac{7p}{967680\omega^2}(\cos\omega t-\cos 7\omega t)\\&-\tfrac{p}{14515200\omega^2}(\cos\omega t-\cos 9\omega t).\end{aligned} \quad (38)$$

Fig. 2 shows a comparison between the analytical solution obtained from formulae (37) and (38) and the numerical integration results obtained by using the Runge-Kutta method. It is seen that the solution obtained by the iteration procedure is very close to that obtained by the numerical method. One concludes that adopting present technique to analyze the solutions of the modified Van der pol equation, a satisfactory results are obtained for small values of parameter ε.

4 Conclusion

In this paper, the iteration perturbation method has been successfully used to study the nonlinear oscillators. The examples of nonlinear oscillations has illustrated that the iteration procedure can give excellent approximate results. The first approximate frequency ω_1 given in equation (25) is identical to equation (10) in Ref [16]. The examples of nonlinear oscillations has illustrated that the present method can give excellent approximate results. The second approximate frequency ω_2 obtained by the second iteration gives very accurate solutions. Also, the second approximate periodic solution x_2 is in good agreement with the numerical integration results obtained by using a fourth order Runge-Kutta method.

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