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H. I. Abdel-Gawad
Department of Mathematics, Faculty of Science, Cairo University, Egypt, hamdy@sci.cu.edu.eg

Mohamed Tantawy
Department of Mathematics, Faculty of Science, Cairo University, Egypt, mtantawy20@yahoo.com

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Exact Solutions of The Shamel-Korteweg-de Vries Equation With Time Dependent Coefficients

*Hamdy I. Abdel-Gawad and Mohamed Tantawy**

Department of Mathematics, Faculty of Science, Cairo University, Egypt

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Abstract: In this paper, we present the Shamel-Korteweg-de Vries (S-KdV) equation which play an important role in studying the effect of electron trapping on the nonlinear interaction of ion-acoustic waves by including a quasi-potential. Here, a wide class of exact solutions to this equation with time dependent coefficients is found. It is shown that, the traveling wave solutions exist and they travel with time-dependent speed along the characteristic curves. The class of the obtained solutions is classified to different wave structures: periodic, elliptic or interaction of soliton, kink and anti-kink waves. The method used in this work is the extended unified method which was presented by one of the authors.

Keywords: Variable coefficients, Shamel-Korteweg-de Vries equation, The extended unified method, Inner and outer conoidal waves, Kink and anti-kink waves

1 Introduction

A huge number of works on the exact traveling wave solutions of nonlinear evolution equations with constant coefficients have been carried out in the literature. This may be argued to:
– A wide class of these equations is completely integrable
– Searching for the traveling wave solutions investigate many physical aspects of the problem under consideration.

On the other hand, they may be considered as "asymptotic" or steady state solutions to evolution equations. Different methods were constructed to find some exact solutions of evolution equations by using the painleve' test for integrability and the auto-Bäcklund transformation [1–15].

Recently, the unified method has been suggested by the first author in [16]. Indeed, the unified method suggests a new classification to the different types of solutions, that is polynomial or rational function solutions in some "auxiliary" function with an appropriate auxiliary equation. Furthermore, the necessary conditions for the existence of each type of solutions may be constructed. In a subsequent paper, the unified method was extended to find exact solutions of nonlinear evolution equations with variable coefficients [17]. This later method is used here to find exact solutions to Shamel-Korteweg-de Vries (S-KdV) equation with time-dependent coefficients.

The Shamel-Korteweg-de Vries with constant coefficients is

$$u_t + (\alpha u^{\frac{1}{2}} + \beta u) u_x + \delta u_{xxx} = 0, \; \alpha\beta \neq 0, \qquad (1)$$

where α, β and δ are constants which they are refer to the activation trapping, the convection and the dispersion coefficients respectively. We mention that, when $\alpha = 0$ or $\beta = 0$ equation (1) reduces to the KdV or Shamel equation respectively. Equation (1) describes many phenomena in plasma physics. In particular, it describes the nonlinear interaction of ion-acoustic waves (ICW) in plasma physics by including a quasi potential effect. That is, by taking into consideration the electron trapping effect [18–24]. We mention that, the equation (1) is a particular case of the generalized Gardner equations [25]. In this context, the study of the ICW in the presence of drag force acted by the waves on the particles issues to the Burger's-KdV equation [26]. So that, in a future work the exact solutions for Shamel-Burger-Korteweg-de Vries equation will be studied. We bear in mind that, the exact solutions of (1) were studied in [19–21]. It is worthy to mention that, in [27–41] soliton (coupled to kink and anti-kink) and periodic solutions were only found to the

* Corresponding author e-mail: mtantawy20@yahoo.com

equation (1). In the present work, elliptic waves are found. They exhibit outer or inner conidal waves to the soliton one's.

An interesting case physically, rather than mathematically arise when these parameters are time dependent. To this end we consider the equation

$$(v^2)_t + (\alpha(t)\,v + \beta(t)\,v^2)\,(v^2)_x + \delta(t)\,(v^2)_{xxx} = 0, \tag{2}$$

where $u = v^2$. Then (2) is dealt with by using the extended unified method. The application of (2) are practical interest in the propagation of soliton waves in fibre optics when t is replaced by z. In the view of the unified method, the solutions to the nonlinear evolution equation can be classified to polynomial or rational in an "auxiliary" function. First, we give here a brief account to the case of polynomial solutions.

1.1 Polynomial solutions

To search for polynomial solutions of (2), the unified method suggests the solution in the form

$$v(x,t) = \sum_{j=0}^{n} a_j(x,t)\,\varphi^j(x,t), \tag{3}$$

where the auxiliary function φ satisfies the auxiliary and the compatibility equations which are given by

$$(\varphi_x(x,t))^p = \textstyle\sum_{j=0}^{p\ ,k} c_j(x,t)\ ,\varphi^j(x,t),\ ,(\varphi_t(x,t))^p = \\ dd\textstyle\sum_{j=0}^{pk} b_j(x,t)\,\varphi^j(x,t), \varphi_{xt}(x,t) = \varphi_{tx}(x,t),\ p = 1,2. \tag{4}$$

For instance, when $p = 1$, the necessary conditions for finding the exact polynomial solutions of equation (2) are;
(i) The balance condition is $n = k - 1$.
(ii) The consistency condition for the existence of solutions is $k \le \frac{11}{3}$. For details see [16].
Thus, the polynomial solutions exist when $k = 2,3$. We mention that, the consistency conditions is constructed by using the number of principle and compatibility equations namely; $(2k-1)$ equations and the number of unknown functions $a_i\,b_i$ and c_i. By bearing in mind the complete integrability of (1), we set the difference between them to be $(\le m)$, where m is the highest order partial derivative.

When substituting from (3) and (4) into (2), we get an equation which is splitting to a set of equations, namely the "principle" equations.

Steps of computation:
1- Solving the principle equations.
2- Solving the compatibility equation $(4)_3$.
3- Solving the auxiliary equations.
4- Find the exact solution.

1.2 Rational solutions

The case of rational solutions can be treated by the same way. So, we assume the rational function solution of (2) in the form

$$v(x,t) = \textstyle\sum_{i=0}^{n} p_i(x,t)\,\varphi^i(x,t) / \sum_{i=0}^{r} q_i(x,t)\,\varphi^i(x,t), \\ (\varphi_x(x,t))^p = \textstyle\sum_{i=0}^{pk} c_i(x,t)\,\varphi^i(x,t),\ (\varphi_t(x,t))^p = \\ \textstyle\sum_{i=0}^{pk} b_i(x,t)\,\varphi^i(x,t), \varphi_{xt}(x,t) = \varphi_{tx}(x,t),\ p = 1,2. \tag{5}$$

where the denominator in $(5)_1$ does not vanish for all $-\infty < x < \infty$ and $t \ge 0$.
Here, (5) will be considered when $n > r$, $n = r$ separately. In each case, there are appropriated auxiliary equations. Indeed, the balance and consistency conditions could be constructed in each case according to the relation between n and r.

In the next section, we study the case in which the coefficients of (2) are proportional.

2 The case when the S-KdV is integrable

By using the polynomial solution when $k = 2,3$ and the rational solution when $n = r = 1$, $k = 1$ and $n - r = 1$, $k = 2$, we found that the solutions of (2) exist only when $\alpha(t) = \mu\,\beta(t)$ and $\delta(t) = \delta_0\,\beta(t)$, where μ and δ_0 are constants.
Under the last conditions, equation (2) reduces to ones with constant coefficients

$$(v^2(x,\tau))_\tau + (\mu\,v(x,\tau) + v^2(x,\tau))\,(v^2(x,\tau))_x + \\ \delta_0\,(v^2(x,\tau))_{xxx} = 0,\ \tau = \textstyle\int_0^t \beta(t_1)\,dt_1 \tag{6}$$

We mention that, (6) admits a traveling wave solutions where the details of these solutions as they follow;

2.1 Polynomial solutions

We write $v(x,t) = w(z)$, $z = \sigma_1\,x + \sigma_2\,\tau$, then equation (6) reduces to

$$\sigma_2\,(w^2(z))' + (\mu\,w(z) + w^2(z))\,\sigma_1(w^2(z))' + \\ \sigma_1^3\,\delta_0\,(w^2(z))''' = 0,\ ()' = \frac{d}{dz}(). \tag{7}$$

For the polynomial solutions, we have

$$w(z) = \sum_{j=0}^{n} a_j\,\varphi^j(z),\ (\varphi'(z))^j = \sum_{j=0}^{pk} c_j\,\varphi^j(z),\ p = 1,2, \tag{8}$$

where a_j and c_j are arbitrary constants.
The exact solution of (7) are elementary (when $p = 1$, $k = 2,3$) or elliptic solutions (when $p = 2$, $k = 2,3$) and they are classified as follows;

(i) When $p = 1$ and $k = 2,3$.
(i_1) When $k = 2$, and $n = k-1$, we have

$$\begin{array}{c} w(z) = a_1\,\varphi(z) + a_0, \\ \varphi'(z) = c_2\,\varphi^2(z) + c_1\,\varphi(z) + c_0 \end{array} \tag{9}$$

By using any package, the solution of (2) is given by

$$\begin{array}{c} u(z) = w^2(z) = \frac{4}{25}\mu^2(-1+\tanh(\frac{R\,(z+A)}{2}))^2, \\ \sigma_2 = \frac{16\,\mu^2\,\sigma_1}{75} \end{array} \tag{10}$$

where $\sigma_1 = \frac{32\,\mu^3}{375\,R\sqrt{-3\,\delta_0}}$, $R^2 = c_1^2 - 4\,c_2\,c_0$, $\delta_0 < 0$ and A are arbitrary constants. Due to the translation symmetry, we take $A = 0$.
(i_2) When $k = 3$, $n = 2$. By a similar way as we did in the last case, the solution of (2) is given by

$$u(z) = \frac{4}{25}\mu^2(-1+\tanh(\frac{R_1\,z}{3\,c_3}))^2,\ \sigma_2 = \frac{16\,\mu^2\,\sigma_1}{75} \tag{11}$$

where $\sigma_1 = \frac{\sqrt{3}\,c_3\,\mu}{5\,R_1\sqrt{-\delta_0}}$, $R_1 = c_2^2 - 3\,c_1\,c_3 < 0$ and μ are arbitrary constants. Indeed, this solutions is a solitary wave solution.
(ii) When $p = 2$. In this case, the solutions are elliptic and they may be given in Jacobi elliptic functions or as elliptic integrals of the first and third kinds.
(ii_1) When $k = 2$. In this case, the solution take the form

$$\begin{array}{c} w(z) = a_1\,\varphi(z) + a_0, \\ \varphi'(z)^2 = \sum_{j=0}^{4} c_j\,\varphi^j(z). \end{array} \tag{12}$$

When subtitling from (12) into (2), we get

$$a_1 = -\frac{2\,\mu}{5} - \frac{\sqrt{3\,\delta_0}\,c_3\,\sigma_1}{2\sqrt{-c_4}},$$

$$a_0 = 2\sqrt{-3\,c_4\,\delta_0}\,\sigma_1,$$

$$\begin{array}{l} \sigma_2 = \frac{32\sqrt{-3\,c_4}\,c_4\,\mu^3 + 450\sqrt{-3\,c_4}\,c_3^2\,\delta_0\,\mu\,\sigma_1^2}{1200\sqrt{-3\,c_4}\,c_4\,\delta_0\,\mu\,\sigma_1^2 + 4500\,c_3\,\sigma_0^{3/2}\,\sigma_1^3} + \\ \frac{1125\,c_3^2\,\delta_0^{3/2}\,\sigma_1^3 + 9000\,c_1\,c_4^2\,\delta_0^{3/2}\,\sigma_1^3}{1200\sqrt{-3\,c_4}\,c_4\,\delta_0\,\mu\,\sigma_1^2 + 4500\,c_3\,\sigma_0^{3/2}\,\sigma_1^3}, \\ c_2 = \frac{\sigma_1(1125\,c_3^3\sigma_0^{3/2}\,\sigma_1^3 - 18000\,c_1\,c_4^2\,\sigma_0^{3/2}\,\sigma_1^3}{150\,c_4(4\sqrt{-3\,c_4}\,\mu + 15\,c_3\sqrt{\delta_0}\,\sigma_1)} + \\ \frac{16\,c_4\,\mu^2(8\sqrt{-3\,c_4}\,\mu + 45\,c_3\sqrt{\delta_0}\,\sigma_1))}{150\,c_4(4\sqrt{-3\,c_4}\,\mu + 15\,c_3\sqrt{\delta_0}\,\sigma_1)} \end{array} \tag{13}$$

where c_j, $j = 0,1,3,4$ are arbitrary constants and $\delta_0\,c_4 < 0$.
For particular values of c_j, $j = 0,...,4$, we get different solutions in Jacobi elliptic functions.

Here, if we take (according to the classification in [30])

$$\begin{array}{c} c_0 = -\frac{(1-m^2)^2}{4},\ c_2 = \frac{1+m^2}{2},\ c_4 = -\frac{1}{4}, \\ c_1 = c_3 = 0, \end{array} \tag{14}$$

and substituting into (12), we get

$$\varphi(z) = m\,\mathrm{cn}(z,m) \pm \mathrm{dn}(z,m), \tag{15}$$

where $\sigma_2 = \frac{16}{75}\mu^2\,\sigma_1$, $\sigma_1 = \frac{2\,\mu}{5\sqrt{3\,\delta_0\,(1+m^2)}}$. By substituting from (15) into (12), the solution of (2) is given by

$$u(z) = \frac{4\mu^2}{25}(-1+\frac{m\,\mathrm{cn}(z,m) \pm \mathrm{dn}(z,m)}{\sqrt{1+m^2}})^2, \tag{16}$$

where μ is a constant and m $(0 < m < 1)$ denotes the modulus of the Jacobi elliptic function.

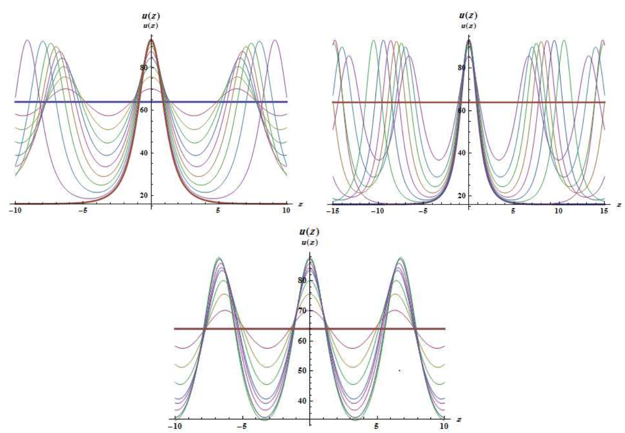

Figure 1: (a)-(c) $(a_1 = \frac{4}{\sqrt{1+m^2}},\ a_0 = -4,\ \mu = 10)$.

In Fig.1(a), the solution given by (16) is displayed against z for different value of m, the thick straight line corresponds to $m = 0$ and the thick curve (soliton wave) corresponds to $m = 1$. Conoidal waves are inner when $m < 0.8$ and are outer when $m \geq 0.8$.
In Fig.1(b), the same caption as in Fig.1(a) but for different values of $m_0 = \sqrt{1-m^2}$, the thick soliton wave correspond to $m = 0$ and the thick straight line correspond to $m = 1$.
In Fig.1(c), the same caption as in Fig.1(a) but for different value of $m_1 = m\sqrt{1-m^2}$, the thick straight line correspond to $m = 0,1$.

To inspect the physics revealed by Fig.1, we mention that $\mu = \frac{\alpha(t)}{\beta(t)}$, where $\alpha(t)$ and $\beta(t)$ are the trapping and nonlinear coefficients. In these figures, we take $\mu \gg 1$. So

the trapping prevails the nonlinearity.
Fig.1(a) shows that all elliptic waves are "outer waves" where their amplitudes are higher than the amplitude of the soliton wave in the limit case when $m = 1$.
Fig.1(b) shows that all elliptic waves are "inner waves" when $m \geq 0.8$ that is, their amplitudes are smaller than the amplitude of the soliton wave in the limit case when $m = 0$.
Fig.1(c) shows that elliptic waves as larger than the straight line ($m = 0$ or 1).
We remark that the corner wave (upper curve when $m = 0.9$) steepens which may be argued to the fact that the trapping coefficient prevails the non linearity as $\mu = \frac{\alpha(t)}{\beta(t)} \gg 1$. This may agree with the results in [31, 32] in different applications.
(ii_2) When $k = 3$. In this case, the solution take the form

$$w(z) = a_2\,\varphi(z)^2 + a_1\,\varphi(z) + a_o, \tag{17}$$

$$(\varphi'(z))^2 = \sum_{j=0}^{j=6} c_j\,\varphi^j(z), \tag{18}$$

By substituting from (17) and (18) into (2), the principle equation solves to

$$a_2 = 4\sigma_1\sqrt{-3c_6\,\delta_0},\; a_1 = \frac{4c_5\sqrt{-\delta_0}\,\sigma_1}{\sqrt{3c_6}},$$
$$a_0 = -\frac{2\mu}{5} + \frac{(-11c_5^2 + 36c_4c_6)\sqrt{-\delta_0}\,\sigma_1}{12\sqrt{3}c_6^{3/2}},$$
$$\sigma_2 = \frac{8\mu^2\sigma_1}{25} + \frac{(25c_5^4 - 504c_4c_5^2c_6}{216c_6^3} - \frac{432c_6^2(-3c_4^2 + 8c_2c_6))\delta_0\,\sigma_1^3)}{216c_6^3},$$
$$c_3 = \frac{-5c_5^3 + 18c_4c_5c_6}{27c_6^2},\; c_1 = \frac{c_5^5 - 3c_4c_5^3c_6 + 27c_2c_5c_6^3}{81c_6^4}, \tag{19}$$

and c_0 which is too lengthy to be written here. In equation (19) $\delta_0 < 0, c_6 > 0,\ \mu$ and $c_j,\ j = 2,4,5$ are arbitrary constants.

According to the classification in [30], If we take

$$c_0 = c_1 = c_3 = c_5 = 0,\; c_2 > 0, \tag{20}$$

and substituting in (18), we get

$$\varphi(z) = \sqrt{\frac{c_2\,\mathrm{csch}^2(\sqrt{c_2}\,z)}{c_4 + 2\sqrt{c_2c_6}\,\coth(\sqrt{c_2}\,z)}} \tag{21}$$

By substituting from (21) into equation (18), the solution of (2) is given by

$$u(z) = (-\frac{2\mu}{5} + \frac{c_4\sigma_1\sqrt{-3c_6\,\delta_0}}{\sqrt{c_6}} + 4\sigma_1\sqrt{-3c_6\,\delta_0}(\frac{c_2\,\mathrm{csch}^2(\sqrt{c_2}\,z)}{c_4 + 2\sqrt{c_2c_6}\,\coth(\sqrt{c_2}\,z)}))^2. \tag{22}$$

2.2 Rational function solutions

In this section, we find rational solutions for some different values of n, r and p.
We mention that, k is found by using the balance condition which is given by $n - r = k - 1$.
(i) When $p = 1$. By taking $n - r = 1$ (when $k = 2$) and by using (5), the solution of (2) has the form

$$w(z) = \frac{p_2\varphi^2(z) + p_1\varphi(z) + p_0}{q_1\varphi(z) + q_0},\quad \varphi'(z) = c_2\varphi^2(z) + c_1\varphi(z) + c_0. \tag{23}$$

By a direct calculation, the solution of (2) is given by

$$u(z) = (\frac{15\sqrt{3}\,\sigma_2\,\mathrm{sech}^2(\frac{\sqrt{\sigma_2}\,z}{4\sqrt{-\delta_0}\,\sigma_1^{3/2}})}{8\sqrt{3}\mu\sigma_1 - 30\sqrt{\sigma_1\sigma_2}\tanh(\frac{\sqrt{\sigma_2}\,z}{4\sqrt{-\delta_0}\,\sigma_1^{3/2}})})^2, \tag{24}$$

where $\sigma_1, \sigma_2, \delta_0 < 0, \mu$ are arbitrary constants.
When $(\sigma_2 < 0)$, say $(\sigma_2 = -\rho^2)$, in equation (24), we find that

$$u(z) = (\frac{-15\sqrt{3}\rho^2\sec^2(\frac{\rho z}{4\sqrt{-\delta_0}\,\sigma_1^{3/2}})}{8\sqrt{3}\mu\sigma_1 + 30\rho\sqrt{\sigma_1}\tan(\frac{\rho z}{4\sqrt{-\delta_0}\,\sigma_1^{3/2}})})^2, \tag{25}$$

where σ_1, μ and ρ are arbitrary constants.
(ii) When $p = 2$. By taking $n = r = 1$ (reduced to $k = 1$) and by using (5), the solution of (2) has the form

$$w(z) = \frac{p_1\varphi(z) + p_0}{q_1\varphi(z) + q_0},\; \varphi'(z) = \sqrt{c_2\varphi^2(z) + c_1\varphi(z) + c_0}. \tag{26}$$

By a direct calculation, the solution of (2) is given by

$$u(z) = (\frac{1 + A_1e^{\sqrt{c_2}z} + A_2e^{2\sqrt{c_2}z}}{1 + A_3e^{\sqrt{c_2}z} + A_2e^{2\sqrt{c_2}z}})^2, \tag{27}$$

where $A_i, i = 1,2,3$ are arbitrary constants, that are functions in c_i, p_i, q_i and μ.
When $(c_2 = -\rho^2)$, then (27) will give rise to a periodic solution

$$u(z)(\frac{1 + B_1\cos(\rho z)}{B_2 + B_3\cos(\rho z)})^2, \tag{28}$$

where B_i, $i = 1,2,3$ are arbitrary constants which depend on A_i.

It is worth to be noticing that, (28) is obtained from (27) by separating the real and imaginary part into (27) and by setting the coefficients of imaginary part equal zero.

The solutions which are given by (24) and (27) are displayed against x and t in figures 2(a) and 2(b) respectively. We bearing in mind that, $z = \sigma_1 x + \sigma_2\tau$, $\tau = \int_0^t \beta(t_1)\,dt_1$.

At these figures, we find that
In Fig.2(a), a single soliton which is moving along the

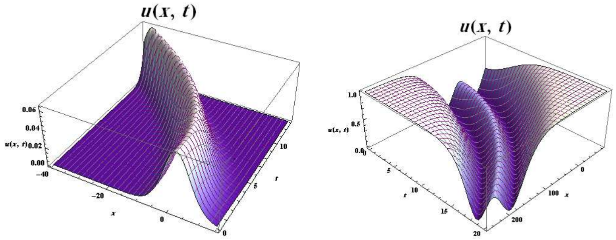

Figure 2: (a) $\sigma_2 = \frac{1}{4}, \sigma_1 = 1, \delta_0 = -1, \mu = 2, \tau = t^2 - t$ or $\beta(t) = 2t - 1$. (b) $\sigma_1 = \frac{2}{65}, \sigma_2 = \frac{32}{975}, A_1 = -8, A_2 = \frac{8}{7}, A_3 = \frac{48}{7}, \tau = t + \frac{1}{2}t^2$ or $\beta(t) = 1 + t$.

characteristic curve $\sigma_1 x + \sigma_2(t^2 - t) = \text{constant}$.
In Fig.2(b), the solution shows the interaction between soliton, kink and ant-kink waves and they are moving along the characteristic curve $\sigma_1 x + \sigma_2(t + \frac{1}{2}t^2) = \text{constant}$.

We mention that, the solutions which are found by using this method cover all the solutions that could be obtained by using the well-known methods namely; the tanh-method, Jacobic-elliptic function expansion, Exp-function method and G'/G expansion method [33–35]. Indeed, the work done in [16] unifies all the methods known in the literature. On the other hand the results for exact solutions obtained by this method cover all solutions that could be found by the pre-mentioned approaches.
For the case when the coefficients are not linearly dependent, No exact solution were found by using extended unified method. We think that, this result can be justified by using the painleve' test for integrability of the S-KdV equation with variable coefficients. But this lies outside the scope of this paper.

3 Conclusions

The extended unified method was used to obtain a class of different solutions structures to the of the S-KdV with time dependent coefficients. This method allowed us to find a wide class of exact solutions that may be classified into different types of wave geometries namely: periodic, soliton waves or elliptic waves that are propagating along the characteristics curves. On the other hand, they show the interaction between soliton, kink and anti-kink waves. The inner or outer conoidal waves to the soliton wave solutions were shown. In a future work, the S-KdV equation with space dependent coefficients will be studied which is more realistic. This case reflects the inhomogeneity of the medium that has an impact on the dispersion and the dusty plasma coefficients. the study will be carried via the method used here.

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Hamdy I. Abdel-Gawad is a Professor of Applied Mathematics at Cairo University, Cairo, Egypt. He obtained his Ph.D from University-Paris XI, Paris, France in 1984. His research interest include:Fractional Calculus-q-Calculus-Mathematical Modeling in Biology, Medicine Chemistry and Physics-Stability analysis of Dynamical systems. Also, He published many papers in international journals.

Mohamed Tantawy received the B.Sc in Mathematics from AL-Azhar University-Egypt in 2007. Now he is pursuing master in Applied Mathematics from Cairo-University, Egypt. Area of research interests include: partial Differential equations in Biological Modeling, Medicine and Chemistry.