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Entropy and Average Cost of AUH Codes

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Abstract: In this paper we address the class of anti-uniform Huffman (AUH) codes, also named unary codes, for sources with finite and infinite alphabet, respectively. Geometric, quasi-geometric, Fibonacci and exponential distributions lead to anti-uniform sources for some ranges of their parameters. Huffman coding of these sources results in AUH codes. We prove that, in general, sources with memory are obtained as result of this encoding. For these sources we attach the graph and determine the transition matrix between states, the state probabilities and the entropy. We also compute the average cost for these AUH codes.

Keywords: Huffman coding, average codeword length, average cost, entropy.

1. Introduction

Let $(p_1, p_2, \dots, p_n)$ be the probability distribution of a n message source $\xi_n = \{s_1, s_2, \dots, s_n\}$. It is well known that the Huffman encoding algorithm [1] produces an optimal binary prefix-free code for ξ_n. A binary Huffman code is usually represented by a binary tree, whose leaves correspond to the source messages. The two edges emanating from each intermediate tree node (father) are labeled either 0 or 1.The length between the root and a leaf is the length of the binary codeword associated with the corresponding message. We denote by $v_i, i = 1, 2, \dots, n$, the codeword representing the message s_i, and the length of v_i by l_i. The optimality of Huffman coding implies that $l_i \leq l_j$, if $p_i > p_j$. For related literature on Huffman coding and Huffman trees, we refer the reader to [2]$-$[6].

Anti-uniform Huffman (AUH) codes were firstly introduced in [7]. A Huffman code representing a finite source ξ_n satisfying $p_1 \geq p_2 \geq \dots \geq p_n > 0$ is an anti-uniform code, if $l_i = i, i = 1, 2, \dots, n-1$ and $l_n = l_{n-1}$. A source ξ_n having an anti-uniform Huffman code is called an anti-uniform source. These sources were extensively analysed, concerning bounds on average codeword length, entropy and redundancy for different types of probability distributions [7]$-$[10]. The AUH sources appear in a wide variety of situations in the real world, because this class of sources have the property of achieving minimum redundancy in different situations and minimum average cost in highly unbalanced cost regime [11], [12]. These properties determine a wide range of applications and motivate us to study these sources from an information theoretic perspective. One example is the telegraph channel with the alphabet $\{. -\}$ in which dashes are twice as long as dots [13]. Another is the $\{a, b\}$ run $-$ length $-$ limited codes used in magnetic and optical storage, in which the binary codewords are constrained so that each 1 must be preceded by at least a, and at most b, $0'$s [14]. The binary Huffman codes, constrained so that all codewords must end in a 1, are used for group testing and self-synchronizing codes [15], [16]. As another example, binary codes whose codewords contain at most a specified number of $1'$s are used for energy minimization of transmissions in mobile environments [17]. AUH sources can be generated by several probability distributions. It has been shown that sources with geometric, quasi-geometric, Fibonacci and exponential distributions lie in the class of AUH sources for some regimes of their parameters [7], [18] - [20]. Related topics were addressed in [21], where the authors studied weakly super increasing (WSI) and partial WSI sources in connection with Fibonacci numbers and golden mean, which appeared extensively in modern science and have applications in coding and information theory.

The paper is organized as follows. In Section 2 we present the Huffman encoding of an antiuniform source and the graph of the resulting memory source. We show

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that, in general, by employing Huffman coding, a source with memory results. The entropy and the average cost of the code are also derived. In Sections 3 we compute the code entropy, as well as the average cost for AUH codes corresponding to sources with geometric, quasi-geometric, Fibonacci and exponential distributions, respectively. We conclude the paper in Section 4.

2. Entropy and average cost of AUH codes

Let us consider a discrete and memoryless source, characterized by the distribution:

$$\xi_n : \begin{pmatrix} s_1 \; s_2 \; \dots \; s_n \\ p_1 \; p_2 \; \dots \; p_n \end{pmatrix}, \tag{1}$$

$$p_1 \geq p_2 \geq \dots \geq p_n \tag{2}$$

$$\sum_{i=1}^{n} p_i = 1 \tag{3}$$

If [7]

$$\sum_{k \geq i+2}^{n} p_k \leq p_i, 1 \leq i \leq n-3 \tag{4}$$

the source becomes anti-uniform.

After a binary Huffman encoding of the source with the distribution in (1), that fulfils (4), the graph in Fig. 1 is obtained. The structure of codewords resulting from binary Huffman encoding is:

$$\begin{gathered} s_0 \rightarrow v_0 \rightarrow 1 \\ s_1 \rightarrow v_1 \rightarrow 01 \\ s_2 \rightarrow v_2 \rightarrow 001 \\ \dots\dots\dots\dots\dots\dots \\ s_{n-1} \rightarrow v_{n-1} \rightarrow \underbrace{00\dots0}_{n-2}1 \\ s_n \rightarrow v_n \rightarrow \underbrace{00\dots00}_{n-1} \end{gathered}$$

The length l_i of the codeword associated with the message s_i is the number of edges on the path between the root and the node s_i in the Huffman tree.

$$l_i = i, i = 1, 2, \dots, n-1 \tag{5}$$

$$l_n = l_{n-1} \tag{6}$$

The average codeword length is determined with

$$\overline{l_n} = \sum_{i=1}^{n} p_i l_i = \sum_{i=1}^{n-1} i p_i + (n-1) p_n \tag{7}$$

In Fig. 1, $s_{n+i}, i = 1, 2, \dots, n-2$ denote the intermediate nodes in the graph, also called parents. The probability of a parent is obtained as the sum of the two sibling probabilities. Denoting by p_{n+i} the probabilities of intermediate nodes, we have

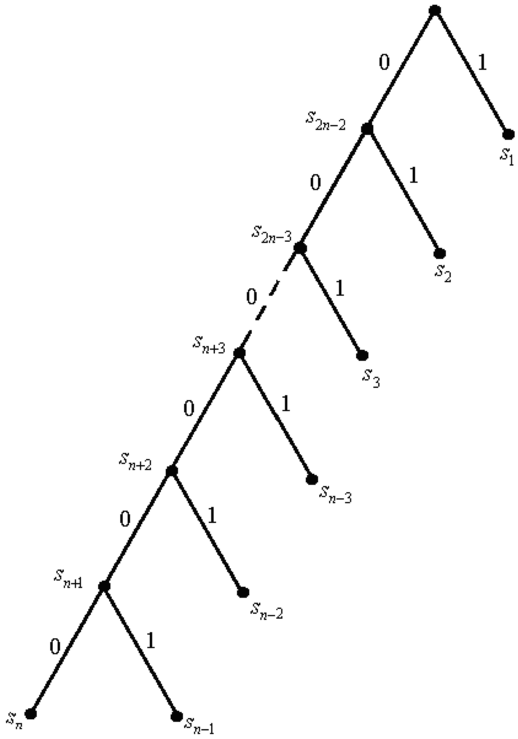

Figure 1 The graph of binary Huffman encoding for the source ξ_n with distribution in (1)

$$p_{n+i} = \sum_{k=n-i}^{n} p_k; i = 1, 2, \dots, n-2 \tag{8}$$

The branches between succesive nodes have the probabilities equal to the ratio between the probability of the node in which the branch ends and the probability of the node from which it starts.

For a sequence of messages s_i, the source delivers a string of binary symbols from the code alphabet $X = \{x_0 = 0, x_1 = 1\}$. As the probabilities of the symbols in the binary string depend on the node from which they are generated, the set X, which is the output bitstream obtained as result of binary Huffman coding, is a source with memory.

When a terminal node $s_i, i = 1, 2, \dots, n$ is reached, the source ξ_n will deliver another message and the source with memory X will generate another string of binary symbols.

The graph attached to the source with memory X can be obtained from the Huffman encoding graph in Fig. 1, as follows:

$a)$ We link the terminal nodes with the graph root;

$b)$ Each terminal node $s_i, i = 1, 2, \dots, n$, or intermediate ones $s_{n+i}, i = 1, 2, \dots, n-2$, (excepting the root of the encoding graph) will represent the states $S_i, i =$

$1, 2, \ldots, 2n-2$, of the souce with memory X. The set of these states is denoted by $S = \{S_1, S_2, \ldots, S_{2n-2}\}$. The graph of the source X is shown in Fig. 2.

The transition probabilities from the state S_i, to the state S_j, that is, $p(S_j|S_i)$, is equal to the probability of the branch between the node s_i and the node s_j. When the source enters the state S_j from the state S_i, it generates either the symbol $x_0 = 0$, or $x_1 = 1$. Therefore, the probability of delivering the symbol $x_j, j = 0, 1$, from the state S_i is the same as the probability to reach the state S_j, starting from the state S_i, that is:

$$p(x_1|S_i) = p(S_1|S_i) = p_1, i = 1, 2, \ldots, n \tag{9}$$

$$p(x_0|S_i) = p(S_{2n-2}|S_i) = \sum_{k=2}^{n} p_k, i = 1, 2, \ldots, n \tag{10}$$

$$p(x_1|S_{n+i}) = p(S_{n-i}|S_{n+i}) = \frac{p_{n-i}}{\sum_{k=n-i}^{n} p_k}, \quad i = 1, 2, \ldots, n-2 \tag{11}$$

$$p(x_0|S_{n+i}) = p(S_{n+i-1}|S_{n+i}) = \frac{\sum_{k=n-i+1}^{n} p_k}{\sum_{k=n-i}^{n} p_k}, \quad i = 1, 2, \ldots, n-2 \tag{12}$$

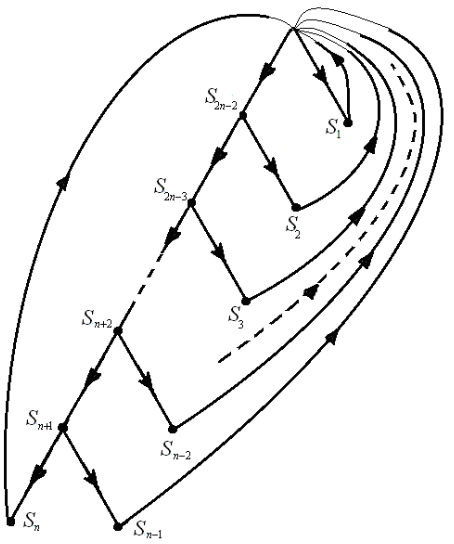

Figure 2 The graph of the source with memory X

Let $\pi_i, i = 1, 2, \ldots, 2n-2$, denote the stationary state probabilities of the source with memory. They can be determined by means of [22]

$$[\pi_1, \pi_2, \ldots, \pi_{2n-2}] = [\pi_1, \pi_2, \ldots, \pi_{2n-2}]\mathbf{T} \tag{13}$$

$$\sum_{i=1}^{2n-2} \pi_i = 1 \tag{14}$$

Considering (7), from (13) and (14) we obtain the state probabilities as:

$$\pi_i = \frac{p_i}{\overline{l_n}}, i = 1, 2, \ldots, n \tag{15}$$

$$\pi_{n+i} = \frac{1}{\overline{l_n}} \sum_{k=n-i}^{n} p_k, i = 1, 2, \ldots, n-2 \tag{16}$$

Generally, the entropy of the source with memory is computed by [23]

$$H(X) = -\sum_{i=1}^{2n-2} \sum_{j=1}^{2} \pi_i p(x_j|S_i) \log p(x_j|S_i) \tag{17}$$

where π_i are given in (15) and (16) and $p(x_j|S_i)$ are given in (9), (10), (11) and (12).

Considering (9), (10), (11) and (12) as well as the graph in Fig. 2, the transition matrix between states is given in relation (18), where the entries of the matrix $\mathbf{T}$ are $t_{ij} = p(S_j|S_i)$.

Let c_0 and c_1 be the costs of storing or transmitting symbols "0" and "1", respectively, resulted after the binary Huffman encoding of the source ξ_n. The average cost is determined by [9]

$$\overline{C} = \sum_{i=1}^{n} p_i[n_0(i)c_0 + n_1(i)c_1], \tag{19}$$

where $n_0(i)$ and $n_1(i)$ denote the number of $0'$s and $1'$s in the codeword c_i.

Considering the structure of the codewords for AUH sources, (5) and (6), the average cost is

$$\overline{C} = \sum_{i=1}^{n} p_i[(i-1)c_0 + c_1] + (n-1)c_0 p_n \tag{20}$$

3. Case studies for distributions leading to AUH sources

In the following we will analyze several source distribution types. More precisely, we will determine the conditions that should be respected by the distributions parameters so that the resulting sources are AUH. We also compute the entropy and average cost of the resulting codes.

$$\mathbf{T} = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\cdots & 0 & \sum\limits_{k=2}^{n} p_k \\ p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\cdots & 0 & \sum\limits_{k=2}^{n} p_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\;\vdots & \vdots & \vdots \\ p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0\cdots & 0 & \sum\limits_{k=2}^{n} p_k \\ 0 & 0 & \cdots & 0 & \frac{p_{n-1}}{\sum\limits_{k=n-1}^{n} p_k} & \frac{\sum\limits_{k=n}^{n} p_k}{\sum\limits_{k=n-1}^{n} p_k} & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & \cdots & \frac{p_{n-2}}{\sum\limits_{k=n-2}^{n} p_k} & 0 & 0 & \frac{\sum\limits_{k=n-1}^{n} p_k}{\sum\limits_{k=n-2}^{n} p_k} & 0\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\;\vdots & \vdots & \vdots \\ 0 & \frac{p_2}{\sum\limits_{k=2}^{n} p_k} & \cdots & 0 & 0 & 0 & 0 & 0\cdots & \frac{\sum\limits_{k=3}^{n} p_k}{\sum\limits_{k=2}^{n} p_k} & 0 \end{bmatrix} \quad (18)$$

3.1. AUH sources with geometric distribution

Let there be a discrete source characterized by the geometric distribution:

$$\xi_n : \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ p_1 = 1-p & p_2 = p(1-p) & \cdots & p_n = p^{n-1}(1-p) \end{pmatrix}, \quad (21)$$

In this case the source is not complete, because

$$\sum_{i=1}^{n} p_i = 1 - p^n. \quad (22)$$

In order to get a complete source, we normalize the probabilities p_i given in (21), obtaining:

$$\xi_n^{norm} : \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ p_1 = \frac{1-p}{1-p^n} & p_2 = \frac{p(1-p)}{1-p^n} & \cdots & p_n = \frac{p^{n-1}(1-p)}{1-p^n} \end{pmatrix}, \quad (23)$$

For this source to become AUH, relation (4) is required to be fulfilled. Replacing the probabilities from (23) in (4), we obtain, as in [18]:

$$0 < p < \frac{\sqrt{5}-1}{2} \quad (24)$$

The average codeword length results by replacing the probabilities p_i from (23) into (7).

$$\overline{l_n^{norm}} = \frac{1}{(1-p)(1-p^n)} \cdot \cdot \left[1 - p^{n-1} - (n-1)p^n + (n-1)p^{n+1}\right] \quad (25)$$

Theorem 1 *The entropy and the average cost of the source with memory resulted by binary encoding of the AUH source with geometric distribution are determined by:*

$$H_n(X) = -\frac{1}{\overline{l_n^{norm}}}\left[\log(1-p^n) - \log(1-p) - \right. \left. - \frac{1 - np^{n-1} + (n-1)p^n}{(1-p^n)(1-p)} p \log p\right] \quad (26)$$

$$\overline{C}_n = \frac{1}{1-p^n}\left[(1-p^{n-1})c_0 + \frac{1 - np^{n-1} + (n-1)p^n}{1-p} pc_1\right] \quad (27)$$

Proof. The stationary state probabilities are obtained by replacing the probabilities p_i from (23) into (16) and (17), and considering (25):

$$\begin{cases} \pi_i = \frac{p^{i-1}(1-p)}{(1-p^n)\overline{l_n^{norm}}}, i = 1, 2, \ldots, n \\ \pi_{n+i} = \frac{p^{n-i-1}(1-p^{i+1})}{(1-p^n)\overline{l_n^{norm}}}, i = 1, 2, \ldots, n-2 \end{cases} \quad (28)$$

The probabilities to deliver the symbols $x_0 = 0$ and $x_1 = 1$, from the states $S_i, i = 1, 2, \dots, 2n-2$, result by replacing the probabilities p_i from (23) into (9) − (12):

$$\begin{cases} p(x_1|S_i) = \frac{1-p}{1-p^n},\ i = 1,2,\dots,n \\ p(x_0|S_i) = \frac{p(1-p^{n-1})}{1-p^n},\ i = 1,2,\dots,n \\ p(x_1|S_{n+i}) = \frac{1-p}{1-p^{i+1}},\ i = 1,2,\dots,n-2 \\ p(x_0|S_{n+i}) = \frac{p(1-p^i)}{1-p^{i+1}},\ i = 1,2,\dots,n-2 \end{cases} \quad (29)$$

Substituting (28) and (29) into (18), the relation (26) results.

Substituting (23) into (20), the relation (27) results.

When the number of messages, n, of the source ξ_n increases, when $n \to \infty$, we obtain

$$\bar{l}_\infty = \frac{1}{1-p} \quad (30)$$

$$H_\infty(X) = -p\log p - (1-p)\log(1-p) \quad (31)$$

$$\overline{C}_\infty = c_0 + \frac{p}{1-p}c_1 \quad (32)$$

3.2. AUH sources with quasi-geometric distribution

Let there be a discrete source characterized by the quasi-geometric distribution:

$$\xi_n : \begin{pmatrix} s_1 & s_2 & \cdots & s_{n-1} & s_n \\ p_1 = 1-p & p_2 = \frac{p}{2} & \cdots & p_{n-1} = \frac{p}{2^{n-2}} & p_n = \frac{p}{2^{n-2}} \end{pmatrix}, \quad (33)$$

Following similar procedures as in the previous case, we get:

– The range for the source parameter for the source to be AUH

$$0 < p \le \frac{2}{3} \quad (34)$$

– The average codeword length

$$\bar{l}_n = 1 + \frac{2^{n-2}-1}{2^{n-3}}p \quad (35)$$

– The entropy of the source with memory X

$$\left.\begin{array}{r} H_n(X) = \frac{A(p)}{B(p)}, \\ A(p) = (1-p)^{n+1}\log(1-p)+ \\ +[1-np^{n-1}+(n-1)p^n]p\log p, \\ B(p) = 1-p^{n-1}-(n-1)p^n+(n-1)p^{n+1} \end{array}\right\} \quad (36)$$

– The average cost of the source with memory X

$$\overline{C}_n = \frac{(2^{n-1}-1)pc_1 + (2^{n-2}-p)c_0}{2^{n-2}} \quad (37)$$

When $n \to \infty$, we obtain

$$H_\infty(X) = -\frac{1}{1+2p}[(1-p)\log(1-p) + p\log p - 2p] \quad (38)$$

$$\overline{C}_\infty = 2pc_1 + c_0 \quad (39)$$

In the special case, when $p = \frac{1}{2}$, the source ξ_n becomes dyadic one. In this case X becomes memoryless and then:

$$p(x_j|S_i) = \frac{1}{2}, j = 0,1; i = 1,2,\dots,2n-2 \quad (40)$$

$$\bar{l}_{dn} = 2 - \frac{1}{2^{n-2}} \quad (41)$$

$$H_{dn}(X) = 1 \quad (42)$$

$$\overline{C}_{dn} = \frac{(2^{n-1}-1)(c_1+c_0)}{2^{n-1}} \quad (43)$$

Imposing $n \to \infty$ in (41) and (43), we obtain

$$\bar{l}_{d\infty} = 2 \quad (44)$$

$$\overline{C}_{d\infty} = c_0 + c_1 \quad (45)$$

3.3. AUH sources with Fibonacci distribution

Let there be a discrete and finite AUH source characterized by the Fibonacci distribution

$$\xi_n : \begin{pmatrix} s_1 & \cdots & s_{n-1} & s_n \\ p_1 = \frac{f_{n-1}}{f_{n+1}} & \cdots & p_{n-1} = \frac{f_1}{f_{n+1}} & p_n = \frac{f_1}{f_{n+1}} \end{pmatrix}, \quad (46)$$

where f_n is the n^{th} Fibonacci number defined as

$$\begin{cases} f_1 = f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, n \ge 3 \end{cases} \quad (47)$$

The source ξ_n is also AUH, because relation (4) is fulfilled. For this case we obtain:

– The average codeword length

$$\bar{l}_n = \frac{f_{n+3}-3}{f_{n+1}} \quad (48)$$

– The entropy of the source with memory X

$$H_n(X) = \frac{f_{n+1}}{f_{n+3}-3}\log f_{n+1} - \frac{1}{f_{n+3}-3}\sum_{i=1}^{n-1} f_i \log f_i \quad (49)$$

– The average cost of the source with memory X

$$\overline{C}_n = \frac{f_{n+1}-f_1}{f_{n+1}}c_0 + \frac{f_{n+2}}{f_{n+1}}c_1 \quad (50)$$

3.4. AUH sources with exponential distribution

The density probability function of exponential distribution is

$$f(x,\lambda) = \begin{cases} \lambda e^{-\lambda x}, \text{if } x \geq 0 \\ 0, \text{if } x < 0 \end{cases} \quad (51)$$

We define the cumulative density function

$$F(i) = 1 - e^{-\lambda i} \quad (52)$$

and the probabilities

$$p_i = F(i) - F(i-1) = e^{-(i-1)\lambda}(1-e^{-\lambda}) \quad (53)$$

Let us consider the source ξ_n with exponential distribution

$$\xi_n : \begin{pmatrix} s_1 & \cdots & s_n \\ p_1 = 1-e^{-\lambda} & \cdots & p_n = e^{-(n-1)\lambda}(1-e^{-\lambda}) \end{pmatrix} \quad (54)$$

For this case we obtain:

– The range for the source parameter for the source to be AUH [8]

$$\lambda \geq \ln\left(\frac{\sqrt{5}+1}{2}\right) \quad (55)$$

– The average codeword length

$$\overline{l_n^{norm}} = \frac{1}{(1-e^{-\lambda})(1-e^{-n\lambda})} \cdot \cdot \left[1 - e^{-(n-1)\lambda} - (n-1)e^{-n\lambda} + (n-1)e^{-(n+1)\lambda}\right] \quad (56)$$

– The entropy of the source with memory X

$$H_n(X) = -\frac{1}{\overline{l_n^{norm}}}\left[\log(1-e^{-n\lambda}) - \log(1-e^{-\lambda}) + \right. \left. + \frac{1 - ne^{-(n-1)\lambda} + (n-1)e^{-n\lambda}}{(1-e^{-n\lambda})(1-e^{-\lambda})}\lambda e^{-\lambda}\log e\right] \quad (57)$$

– The average cost of the source with memory X

$$\overline{C}_n = \frac{1}{1-e^{-n\lambda}}\left[(1-e^{-(n-1)\lambda})c_0 + \right. \left. + \frac{1 - ne^{-(n-1)\lambda} + (n-1)e^{-n\lambda}}{1-e^{-\lambda}}e^{-\lambda}c_1\right] \quad (58)$$

When $n \to \infty$, we obtain

$$H_\infty(X) = -(1-e^{-\lambda})\left[\log(1-e^{-\lambda}) + \frac{e^{-\lambda}\log e^{-\lambda}}{1-e^{-\lambda}}\right] \quad (59)$$

$$\overline{C}_\infty = c_0\frac{e^{-\lambda}}{1-e^{-\lambda}} + c_1 \quad (60)$$

$$\bar{l}_\infty = \frac{1}{1-e^{-\lambda}} \quad (61)$$

4. Conclusion

In this paper we have considered the class of AUH sources with finite and infinite alphabets. Performing a binary Huffman encoding of these sources, we show that sources with memory result. For these sources we build the encoding graph and the graph of the source with memory X. The graph of the source with memory is obtained from the encoding graph by linking the terminal nodes with the graph root. The states of the source with memory correspond to the terminal or intermediate nodes in the encoding graph, excepting the root. We determined in the general case the state probabilities of the source with memory, as well as the transition probabilities between states. The entropy of this source with memory is computed. We assumed the costs c_0 and c_1 for the symbols "0" and "1", respectively, and compute the average cost for these Huffman codes. Obviously, the Huffman encoding procedure assures minimum average length, but the average cost has not a minimum value. If the costs of symbols "0" and "1" are equal to unity, the average cost becomes equal to the average length. We applied the results for several AUH sources with geometric, quasi-geometric, Fibonacci and exponential distributions. We have also analyzed the limit cases, when the source alphabet increases unlimited, that is, $n \to \infty$.

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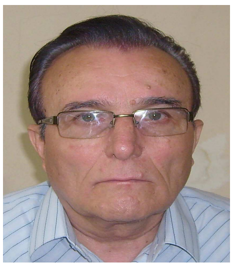

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