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Absolute Tauberian constants for H-J Hausdorff matrices

F. Aydin Akgun[1] *and B. E. Rhoades*[2,*]

[1]Department of Mathematical Engineering, Yildiz Technical University, 34210 Esenler, Istanbul, Turkey
[2]Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, U.S.A.

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Abstract: Let C^k denote the Ces'aro matrix of order $k>0$, $\sum a_k$ a series with partial sums s_n. Then, with $C_n^k := \binom{n+k}{n}\sum_{\upsilon=1}^{n}\binom{n-\upsilon+k}{n-\upsilon}a_\upsilon$, Sherif [5] obtained estimates of the form $\sum|\tau_n - a_n| \leqslant K\sum|\Delta(na_n)|$ and $\sum|\tau_n - a_n| \leqslant K'\sum_n|\Delta\tau_{n-1}|$, under the assumption that $\sum_n|\Delta\tau_{n-1}|$ is finite where Δ is the forward diference operator and $\tau_n := C_n^k - C_{n-1}^k$. The constants K and k' he names absolute Tauberian constants. In a later paper [6] he obtained analogous results for regular Hausdorff matrices In this paper we obtain results similar to [6] for the H-J and E-J generalized Hausdorff matrices.

Keywords: Tauberian constants, generalized Hausdorff matrices.

1. Introduction

Let $\{\lambda_n\}$ be a sequence satisfying

$$0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n \cdots$$

such that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n} = \infty. \tag{1}$$

For any sequence $\{\mu_n\}$, an H-J generalized Hausdorff matrix is a lower triangular matrix with entries

$$h_{nk} = \lambda_{k+1}\cdots\lambda_n[\mu_k,\ldots,\mu_n],$$

where the divided difference is defined by $[\mu_n] = \mu_n$,

$$[\mu_k,\ldots,\mu_n] = \frac{[\mu_k,\ldots,\mu_{n-1}] - [\mu_{k+1},\ldots,\mu_n]}{\lambda_n - \lambda_k}.$$

and where it is understood that $\lambda_{k+1}\cdots\lambda_n = 1$ when $k = n$.

Hausdorff [2] made this definition for $\lambda_0 = 0$, and Jakimovski [3] extended it to the cases in which $\lambda_0 > 0$. If $\lambda_n = n$, then the definition reduces to that of an ordinary Hausdorff matrix. If $\lambda_n = n + \alpha$, then the definition reduces to the E-J generalization developed independently by Endl [1] and Jakimovski [3] in 1958.

An infinite matrix $A = (a_{nk})$ is called conservative if it maps c into c, where c denotes the space of convergent real or complex sequences. Necessary and sufficient conditions for this to happen are the well-known Silverman-Toeplitz conditions:
(i) $\|A\| = \sum_k |a_{nk}| < \infty$,
(ii) $a_k := \lim_n a_{nk}$ exists for each k,
(iii) $t := \lim t_n = lim_n \sum_k a_{nk}$ exists.

An infinite matrix $A = (a_{nk})$ is called regular if it is limit preserving over c. In this case the Silverman-Toeplitz conditions take the form
(i) $\|A\| = \sum_k |a_{nk}| < \infty$,
(ii) $a_k := \lim_n a_{nk} = 0$ for each k,
(iii) $t = 1$.

A conservative H-J matrix has the property that the moment generating sequence $\{\mu_n\}$ takes the form

$$\mu_n = \int_0^1 t^{\lambda_n} d\chi(t),$$

where $\chi(t)$ is a function of bounded variation over [0, 1].

A conservative Hausdorff matrix also has the property that each of the column limits, except possibly for the first, has limit zero, and is regular if and only if

$$\chi(0+) = \chi(0) = 0, \quad \chi(1) = 1.$$

* Corresponding author e-mail: rhoades@indiana.edu

If $\{\mu_n\}$ is totally monotone then the corresponding H-J matrix enjoys the same properties. (A sequence $\{\mu_n\}$ is called totally monotone if $[\mu_0,\ldots,\mu_n]\geq 0$ for each $n\geq 0$.)

Let $\sum a_k$ denote a series with $a_0=0$ and partial sums s_n, and let $H=(h_{nk})$ be an H-J matrix. Set

$$u_n=\sum_{k=0}^{n} h_{nk}s_k,$$

and define $\{b_n\}$ by

$$u_n=\sum_{k=0}^{n} b_k.$$

Then $b_0=\mu_0 a_0=0$, and, for $n>0$, using the definition of divided difference, and the fact that $a_0=0$,

$$\begin{aligned}
b_n &= u_n-u_{n-1}\\
&=\sum_{k=0}^{n}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n]s_k\\
&-\sum_{k=0}^{n-1}\lambda_{k+1}\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_{n-1}]s_k\\
&=\sum_{k=0}^{n}\{(\lambda_{k+1}\ldots\lambda_{n-1})\\
&\times(\lambda_n-\lambda_k)[\mu_k,\ldots,\mu_n]\}s_k\\
&+\sum_{k=0}^{n}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]s_k\\
&-\sum_{k=0}^{n-1}\lambda_{k+1}\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_{n-1}]s_k\\
&=\sum_{k=0}^{n-1}\lambda_{k+1}\ldots\lambda_{n-1}\{[\mu_k,\ldots,\mu_{n-1}]-[\mu_{k+1},\ldots\mu_n]\}s_k\\
&+\sum_{k=0}^{n-1}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]s_k\\
&-\sum_{k=0}^{n-1}\lambda_{k+1}\ldots\lambda_{n-1}[\mu_{k+1},\ldots,\mu_{n-1}]s_k\\
&=-\sum_{j=1}^{n}\lambda_j\ldots\lambda_{n-1}[\mu_j,\ldots,\mu_n]s_{j-1}\\
&+\sum_{k=0}^{n}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]s_k\\
&=\sum_{k=1}^{n}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]a_k\\
&+\lambda_0\ldots\lambda_{n-1}[\mu_0,\ldots,\mu_n]s_0\\
&=\sum_{k=0}^{n}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]a_k,
\end{aligned}$$

since $s_0=a_0=0$.

With Δ the forward difference operator defined by $\Delta w_n=w_n-w_{n+1}$ for any sequence $\{w_n\}$, a straightforward calculation verifies that

$$a_n=-\frac{1}{\lambda_n}\sum_{\nu=0}^{n-1}\Delta(\lambda_\nu a_\nu)\quad\text{for}\quad n>0.$$

Thus

$$\begin{aligned}
b_n&=\sum_{k=0}^{n}\lambda_k\ldots\lambda_{n-1}[\mu_k,\ldots,\mu_n]\Big(-\frac{1}{\lambda_k}\sum_{\nu=0}^{k-1}\Delta(\lambda_\nu a_\nu)\Big)\\
&=-\frac{1}{\lambda_n}\sum_{k=0}^{n}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n]\sum_{\nu=0}^{k-1}\Delta(\lambda_\nu a_\nu)\\
&=-\sum_{\nu=0}^{n-1}\Delta(\lambda_\nu a_\nu)\frac{1}{\lambda_n}\sum_{k=\nu+1}^{n}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n],
\end{aligned}$$

and

$$\begin{aligned}
b_n-a_n&= \qquad (2)\\
&=\sum_{\nu=0}^{n-1}\Delta(\lambda_\nu a_\nu)\frac{1}{\lambda_n}\Big\{1-\sum_{k=\nu+1}^{n}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n]\Big\}\\
&=\sum_{\nu=0}^{n-1}\Delta(\lambda_\nu a_\nu)\frac{1}{\lambda_n}\Big\{1-t_n\\
&\qquad+\sum_{k=0}^{\nu}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n]\Big\}, \qquad (3)
\end{aligned}$$

where

$$t_n:=\sum_{k=0}^{n}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n].$$

A moment sequence $\{\mu_n\}$ is called regular if the corresponding Hausdorff matrix, or generalized Hausdorff matrix, is regular.

We shall need the following two lemmas for the proofs of the theorems of this paper.

Lemma 1.

$$\frac{d}{dt}[t^{\lambda_k},\ldots,t^{\lambda_n}]=\frac{\lambda_k}{t}[t^{\lambda_k},\ldots,t^{\lambda_n}]-\frac{1}{t}[t^{\lambda_{k+1}},\ldots,t^{\lambda_n}].$$

Proof The proof is by induction. For $n=k+1$,

$$\begin{aligned}
\frac{d}{dt}[t^{\lambda_k},t^{\lambda_{k+1}}]&=\frac{1}{\lambda_{k+1}-\lambda_k}\frac{d}{dt}(t^{\lambda_k}-t^{\lambda_{k+1}})\\
&=\frac{1}{\lambda_{k+1}-\lambda_k}(\lambda_k t^{\lambda_k-1}-\lambda_{k+1}t^{\lambda_{k+1}-1})\\
&=\frac{1}{t(\lambda_{k+1}-\lambda_k)}(\lambda_k t^{\lambda_k}-\lambda_{k+1}t^{\lambda_{k+1}})\\
&=\frac{1}{t(\lambda_{k+1}-\lambda_k)}\{\lambda_k(t^{\lambda_k}-t^{\lambda_{k+1}})\}\\
&+\frac{1}{t(\lambda_{k+1}-\lambda_k)}\{(\lambda_k-\lambda_{k+1})t^{\lambda_{k+1}}\}\\
&=\frac{\lambda_k}{t}[t^{\lambda_k},t^{\lambda_{k+1}}]-\frac{1}{t}t^{\lambda_{k+1}}.
\end{aligned}$$

Assume the induction hypothesis.

$$\begin{aligned}
&\frac{d}{dt}[t^{\lambda_k},\dots,t^{\lambda_{n+1}}]\\
&=\frac{1}{\lambda_{n+1}-\lambda_k}\frac{d}{dt}([t^{\lambda_k},\dots,t^{\lambda_n}]-[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}])\\
&=\frac{1}{\lambda_{n+1}-\lambda_k}\left\{\frac{\lambda_k}{t}[t^{\lambda_k},\dots,t^{\lambda_n}]-\frac{1}{t}[t^{\lambda_{k+1}},\dots,t^{\lambda_n}]\right.\\
&\left.-\frac{\lambda_{k+1}}{t}[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}]+\frac{1}{t}[t^{\lambda_{k+2}},\dots,t^{\lambda_{n+1}}]\right\}\\
&=\frac{1}{t(\lambda_{n+1}-\lambda_k)}\left\{\lambda_k([t^{\lambda_k},\dots,t^{\lambda_n}]\right.\\
&-[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}])\\
&\left.+\frac{\lambda_k-\lambda_{k+1}}{t(\lambda_{n+1}-\lambda_k)}[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}]\right\}\\
&-\frac{1}{t(\lambda_{n+1}-\lambda_k)}\Big([t^{\lambda_{k+1}},\dots,t^{\lambda_n}]\\
&-[t^{\lambda_{k+2}},\dots,t^{\lambda_{n+1}}]\Big)\\
&=\frac{\lambda_k}{t}[t^{\lambda_k},\dots,t^{\lambda_{n+1}}]\\
&+\frac{1}{t(\lambda_{n+1}-\lambda_k)}\Big((\lambda_k-\lambda_{k+1})[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}]\\
&-(\lambda_{n+1}-\lambda_{k+1})[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}]\Big)\\
&=\frac{\lambda_k}{t}[t^{\lambda_k},\dots,t^{\lambda_{n+1}}]-\frac{1}{t}[t^{\lambda_{k+1}},\dots,t^{\lambda_{n+1}}].
\end{aligned}$$

Lemma 2. *Let $A=(\alpha_{nk})$ be an infinite matrix, $\{f_n\}$ a sequence. Let*

$$A_n:=\sum_\nu \alpha_{n\nu}f_\nu.$$

Suppose that

$$\sum_n|\alpha_{n\nu}|\quad \text{is bounded}.$$

Let

$$K:=\sup_\nu\sum_n|\alpha_{n\nu}|.$$

Then

$$\sum_n|A_n|\le K\sum_\nu|f_\nu|, \tag{4}$$

and this constant is the best possible in the sense that (3) becomes false if K is replaced by any smaller constant.

Lemma 2 is Theorem 5, which appears on page 167 of [4].

2. Main results

Theorem 1. *Let $\{\mu_n\}$ be a regular moment sequence generated by the real function of bounded variation χ on $0\le t\le 1$ so that*

$$\mu_n=\int_0^1 t^{\lambda_n}d\chi(t)$$

where $\{\lambda_n\}$ satisfies (1) with $\lambda_0>0$, and is such that

$$L:=\sup_\nu\sum_{n=\nu+1}^\infty\frac{\lambda_{\nu+1}\dots\lambda_{n-1}}{\prod_{i=\nu+1}^n(\lambda_i+1)}<\infty, \tag{5}$$

$$\chi(0+)=\chi(0)=0,\quad \chi(1)=1,$$

$$\int_0^1\frac{|\chi(t)|}{t}dt<\infty,$$

and

$$\sum\frac{|1-t_n|}{\lambda_n}<\infty. \tag{6}$$

Let $\sum a_n$ be a series with partial sums s_n and $a_0=0$. Then

$$\sum|b_n-a_n|\le(K+M)\sum|\Delta(\lambda_na_n)|, \tag{7}$$

where

$$K=2L\int_0^1\frac{|\chi(t)|}{t}dt,$$

and

$$M=\sum\frac{|1-t_n|}{\lambda_n}.$$

Proof Applying Lemma 2 to (2), and using the definition of an H-J matrix,

$$\alpha_{n\nu}=\begin{cases}0, & \nu>n,\\ \dfrac{1}{\lambda_n}\left\{1-t_n+\displaystyle\sum_{k=0}^{\nu}\lambda_{k+1}\dots\lambda_n[\mu_k,\dots,\mu_n]\right\}, & \\ & 0\le\nu<n.\end{cases}$$

Since the H-J matrix in this theorem is regular, with mass function x, the series part of $\alpha_{n\nu}$ can be written in the form

$$\int_0^1\sum_{k=0}^n\lambda_{k+1}\cdots\lambda_n\left[t^{\lambda k},\dots,t^{\lambda n}\right]dX(t). \tag{8}$$

Using Lemma 1,

$$\begin{aligned}
&\frac{d}{dt}\sum_{k=0}^\nu\lambda_{k+1}\dots\lambda_n[t^{\lambda_k},\dots,t^{\lambda_n}]\\
&=\sum_{k=0}^\nu\lambda_{k+1}\dots\lambda_n\left(\frac{\lambda_k}{t}[t^{\lambda_k},\dots,t^{\lambda_n}]-\frac{1}{t}[t^{\lambda_{k+1}},\dots,t^{\lambda_n}]\right)\\
&=\frac{1}{t}\left\{\lambda_0\dots\lambda_n[t^{\lambda_0},\dots,t^{\lambda_n}]-\lambda_{\nu+1}\dots\lambda_n[t^{\lambda_{\nu+1}},\dots,t^{\lambda_n}]\right\}.
\end{aligned}$$

Integrating (8) by parts we have

$$\begin{aligned}
&\int_0^1\sum_{k=0}^\nu\lambda_{k+1}\dots\lambda_n[t^{\lambda_k},\dots,t^{\lambda_n}]d\chi(t)\\
&=\sum_{k=0}^\nu\lambda_{k+1}\dots\lambda_n[t^{\lambda_k},\dots,t^{\lambda_n}]\chi(t)\Big|_0^1\\
&-\int_0^1\Big\{\lambda_0\dots\lambda_n[t^{\lambda_0},\dots,t^{\lambda_n}]\\
&-\lambda_{\nu+1}\dots\lambda_n[t^{\lambda_{\nu+1}},\dots,t^{\lambda_n}]\Big\}\frac{\chi(t)}{t}dt
\end{aligned}$$

It then follows that

$$\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\sum_{k=0}^{\nu}\lambda_{k+1}\ldots\lambda_n[\mu_k,\ldots,\mu_n]\Big|$$
$$\leq \sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\int_0^1\Big\{\lambda_0\ldots\lambda_n[t^{\lambda_0},\ldots,t^{\lambda_n}]$$
$$-\lambda_{\nu+1}\ldots\lambda_n[t^{\lambda_{\nu+1}},\ldots,t^{\lambda_n}]\Big\}\frac{\chi(t)}{t}\Big|$$
$$\leq \int_0^1\Big\{\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\lambda_0\ldots\lambda_n[t^{\lambda_0},\ldots,t^{\lambda_n}]\Big|$$
$$+\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\lambda_{\nu+1}\ldots\lambda_n[t^{\lambda_{\nu+1}},\ldots,t^{\lambda_n}]\Big|\Big\}|dt|$$
$$\times\int_0^1\Big|\frac{\chi(t)}{t}\Big|dt$$

Let $H=(h_{nk})$ denote the H-J Cesáro matrix of order one. Then

$$h_{nn}=\frac{1}{\lambda_n+1},$$

$$h_{n+1,n}=\frac{\lambda_n}{(1+\lambda_n)(1+\lambda_{n+1})},$$

and, by induction,

$$h_{nk}=\frac{\lambda_{k+1}\ldots\lambda_n}{\prod_{i=k}^{n}(\lambda_i+1)}.$$

Thus

$$\lambda_0\int_0^1\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\lambda_1\ldots\lambda_n[t^{\lambda_0},\ldots,t^{\lambda_n}]\Big|dt$$

$$=\lambda_0\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}h_{n0},$$

$$\int_0^1\sum_{n=\nu+1}^{\infty}\frac{1}{\lambda_n}\Big|\lambda_{\nu+1}\ldots\lambda_n[t^{\lambda_{\nu+1}},\ldots,t^{\lambda_n}]dt\Big|$$

$$=\sum_{n=\nu+1}^{\infty}\frac{\lambda_{\nu+1}}{\lambda_n}h_{n,\nu+1},$$

and both expressions are finite by condition (4). Condition (6) now follows by using M.

For the E-J generalized Hausdorff matrices one has the following result.

Corollary 1. *Let $\{\mu_n\}$ be a regular moment sequence generated by the real function of bounded variation χ on $0\leq t\leq 1$ so that*

$$\mu_n=\int_0^1 t^{n+\alpha}d\chi(t)\quad for\ some\quad \alpha\geq 0,$$

$$\int_0^1\frac{|\chi(t)|}{t}dt<\infty,$$

and

$$\sum\frac{|1-t_n|}{n+\alpha}<\infty.$$

Let $\sum a_n$ be a series with partial sums s_n and $a_0=0$. Then

$$\sum|b_n-a_n|\leq(K+M)\sum|\Delta((n+\alpha)a_n)|,$$

where

$$K=2\int_0^1\frac{|\chi(t)|}{t}dt,$$

and

$$M=\sum\frac{|1-t_n|}{n+\alpha}.$$

Proof For the E-J matrices, (7) becomes

$$\frac{d}{dt}\sum_{k=0}^{\nu}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}$$
$$=\frac{1}{n+\alpha}\sum_{k=0}^{\nu}\binom{n+\alpha}{n-k}[(k+\alpha)t^{t+\alpha-1}(1-t)^{n-k}$$
$$-(n-k)t^{k+\alpha}(1-t)^{n-k-1}]$$
$$=\sum_{k=0}^{\nu}\binom{n+\alpha-1}{n-k}t^{k+\alpha-1}(1-t)^{n-k}$$
$$-\sum_{k=0}^{\nu}\binom{n+\alpha-1}{n-k}t^{k+\alpha}(1-t)^{n-k-1}$$
$$=\binom{n+\alpha-1}{n}t^{\alpha-1}(1-t)^n$$
$$-\binom{n+\alpha-1}{n-\nu}t^{\nu+\alpha}(1-t)^{n-\nu-1}.$$

Therefore, integrating by parts,

$$\int_0^1\sum_{k=0}^{\nu}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}d\chi(t)dt$$
$$=\sum_{k=0}^{\nu}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}\chi(t)\Big|_0^1$$
$$-\int_0^1\Big[\binom{n+\alpha-1}{n}t^{\alpha-1}(1-t)^n$$
$$-\binom{n+\alpha-1}{n-\nu-1}t^{\nu+\alpha}(1-t)^{n-\nu-1}\Big]\chi(t)dt$$

Consequently

$$\int_0^1 \sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n} t^{\alpha-1}(1-t)^n |\chi(t)| dt$$
$$+\int_0^1 \sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n-\nu-1} t^{\nu+\alpha}(1-t)^{n-\nu-1} |\chi(t)| dt$$
$$\leq \Big\{ \int_0^1 \sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n} t^{\alpha}(1-t)^n dt +$$
$$+\sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n-\nu-1} t^{\nu+\alpha+1}(1-t)^{n-\nu-1} dt \Big\} \times$$
$$\times \int_0^1 \frac{|\chi(t)|}{t} dt.$$

But

$$\sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n} t^{\alpha}(1-t)^n \leq \sum_{n=0}^{\infty} \binom{n+\alpha-1}{n} t^{\alpha}(1-t)^n = \frac{t^{\alpha}}{(1-(1-t))^{\alpha}} = 1,$$

and

$$\sum_{n=\nu+1}^{\infty} \binom{n+\alpha-1}{n-\nu-1} t^{\nu+\alpha+1}(1-t)^{n-\nu-1} = t^{\nu+\alpha+1} \sum_{i=0}^{\infty} \binom{\nu+i+\alpha}{i} (1-t)^i = \left(\frac{t}{1-(1-t)}\right)^{\nu+\alpha+1} = 1.$$

With $\lambda_n = n+\alpha$,

$$\sum_{n=\nu+1}^{\infty} \frac{\lambda_{\nu+1}\ldots\lambda_{n-1}}{\prod_{i=\nu+1}^{n}(\lambda_i+1)} = \sum_{n=\nu+1}^{\infty} \frac{(\nu+\alpha+1)\cdots(n+\alpha-1)}{(\nu+\alpha+2)\cdots(n+\alpha+1)} = \sum_{n=\nu+1}^{\infty} \frac{\nu+\alpha+1}{(n+\alpha)(n+\alpha+1)} = 1,$$

and (4) is automatically satisfied.

Corollary 2. *Let $\{\mu_n\}$ be a regular moment sequence generated by the real function of bounded variation χ on $0 \leq t \leq 1$ so that*

$$\mu_n = \int_0^1 t^{\lambda_n} d\chi(t),$$

where $\{\lambda_n\}$ satisfies (1) *with $\lambda_0 = 0$,*

$$\chi(0+) = \chi(0) = 0, \quad \chi(1) = 1,$$

condition (3) *is satisfied, and*

$$\int_0^1 \frac{|\chi(t)|}{t} dt < \infty.$$

Let $\sum a_n$ be a series with partial sums s_n. Then

$$\sum |b_n - a_n| \leq K \sum |\Delta(\lambda_n a_n)|,$$

where

$$K = L \int_0^1 \frac{|\chi(t)|}{t} dt,$$

and where L satisfies (4).

Proof For the H-J matrices with $\lambda_0 = 0$, each row sum t_n is equal to μ_0. For a regular H-J matrix, $\mu_0 = 1$. Therefore, from condition (5) of Theorem 1, $M = 0$.
For ordinary Hausdorff matrices we have the following.

Corollary 3. *Let $\{\mu_n\}$ be a regular moment sequence generated by the real function of bounded variation χ on $0 \leq t \leq 1$ so that*

$$\mu_n = \int_0^1 t^n d\chi(t),$$

where

$$\int_0^1 \frac{|\chi(t)|}{t} dt < \infty.$$

Let $\sum a_n$ be a series with partial sums s_n. Then

$$\sum |b_n - a_n| \leq K \sum |\Delta(\lambda_n a_n)|,$$

where

$$K = \int_0^1 \frac{|\chi(t)|}{t} dt.$$

Proof This result follows from Corollary 1 by setting $\alpha = 0$.

Corollary 3 is the sufficiency part of Theorem 2.1 of [6].
Results for the E-J matrices.

Theorem 2. *Let $\{\mu_n\}$ be a regular moment sequence generated by the mass function χ such that $\chi(t)/t$ is also of bounded variation on $0 \leq t \leq 1$ and so that*

$$\mu_n = \int_0^1 t^{n+\alpha} d\chi(t),$$

for some $\alpha \geq 0$. If , in addition,

$$\sum_{n=\nu}^{\infty} \frac{|t_n - 1|}{n+\alpha}$$

is satisfied, then

$$\sum |b_n - a_n| \leq A \sum \left| \Delta \left(\frac{1}{n+\alpha+1} \sum_{\nu=1}^{n-1} (\nu+\alpha) a_\nu \right) \right|, \quad (9)$$

where

$$A=\sum_{n=\nu}^{\infty}\frac{|t_n-1|}{n+\alpha}+\int_0^1 t^{-1}|\chi(t)|dt+\int_0^1 t\left|\left(\frac{\chi(t)}{t}\right)\right|.$$

Proof Define

$$w_n=\begin{cases}0, & n=0,\\ \dfrac{1}{n+\alpha+1}\sum_{\nu=1}^{n}(\nu+\alpha)a_\nu, & n\geq 1,\end{cases}$$

and let $\phi_n=-\Delta w_{n-1}=w_n-w_{n-1}$ for $n\geq 1$. Then

$$\begin{aligned}(n+\alpha)a_n&=(n+\alpha+1)w_n-(n+\alpha)w_{n-1}\\&=(n+\alpha)(w_n-w_{n-1})+w_n\\&=(n+\alpha)\phi_n+\sum_{\nu=1}^{n}\phi_\nu.\end{aligned}$$

Note that

$$\sum_{\nu=1}^{n}\phi_\nu=\sum_{\nu=1}^{n}(w_\nu-w_{\nu-1})=w_n-w_0=w_n.$$

Hence

$$a_n=\frac{1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu+\phi_n.$$

Using the expression for b_n in the proof of Corollary 1, and using the fact that $a_0=0$,

$$\begin{aligned}b_n&=\frac{1}{n+\alpha}\sum_{k=1}^{n}\binom{n+\alpha}{n-k}(k+\alpha)\Delta^{n-k}\mu_k a_k\\&=\frac{1}{n+\alpha}\sum_{k=1}^{n}\left\{\binom{n+\alpha}{n-k}(k+\alpha)\Delta^{n-k}\mu_k\times\right.\\&\left.\times\left[\frac{1}{(k+\alpha)}\sum_{\nu=1}^{k}\phi_\nu+\phi_k\right]\right\}\\&=B+D,\quad\text{say.}\end{aligned}$$

$$\begin{aligned}B&=\frac{1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu\sum_{k=\nu}^{n}\binom{n+\alpha}{n-k}\Delta^{n-k}\mu_k\\&=\frac{1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu\left(t_n-\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}\Delta^{n-k}\mu_k\right)\\&=\frac{t_n}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu\\&-\frac{1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu\int_0^1\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}d\chi(t)\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}\right\}\\&=\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}[(k+\alpha)t^{k+\alpha-1}(1-t)^{n-k}\\&-(n-k)t^{k+\alpha}(1-t)^{n-k-1}]\\&=\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}(k+\alpha)t^{k+\alpha-1}(1-t)^{n-k}\\&-\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}(n-k)t^{k+\alpha}(1-t)^{n-k-1}\\&=\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}(k+\alpha)t^{k+\alpha-1}(1-t)^{n-k}\\&-\frac{1}{n+\alpha}\sum_{j=1}^{\nu}\binom{n+\alpha}{n-j+1}(n-j+1)t^{j+\alpha-1}(1-t)^{n-j}\\&=\frac{1}{n+\alpha}\binom{n+\alpha}{n}\alpha t^{\alpha-1}(1-t)^n\\&-\binom{n+\alpha-1}{n-\nu}t^{\nu+\alpha-1}(1-t)^{n-\nu}\end{aligned}$$

since

$$\begin{aligned}\frac{1}{n+\alpha}\binom{n+\alpha}{n-k}(k+\alpha)&=\frac{\Gamma(n+\alpha+1)(k+\alpha)}{(n+\alpha)(n-k)!\Gamma(k+\alpha+1)}\\&=\frac{\Gamma(n+\alpha)}{(n-k)!\Gamma(k+\alpha)}=\binom{n+\alpha-1}{n-k},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n+\alpha}\binom{n+\alpha}{n-j+1}(n-j+1)&=\frac{\Gamma(n+\alpha)}{(n-j)!\Gamma(j+\alpha)}\\&=\binom{n+\alpha-1}{n-j}.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\frac{1}{n+\alpha}\int_0^1\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}d\chi(t)\\&=\frac{1}{n+\alpha}\sum_{k=0}^{\nu-1}\binom{n+\alpha}{n-k}t^{k+\alpha}(1-t)^{n-k}\chi(t)\Big|_0^1\\&+\binom{n+\alpha-1}{n-\nu}\int_0^1 t^{\nu+\alpha}(1-t)^{n-\nu}\frac{\chi(t)}{t}dt\\&-\binom{n+\alpha-1}{n}\int_0^1 t^{\alpha}(1-t)^n\frac{\chi(t)}{t}dt\end{aligned}$$

Therefore

$$B = \frac{t_n}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu + \sum_{\nu=1}^{n}\phi_\nu \int_0^1 \left[\binom{n+\alpha-1}{n} t^{\alpha}(1-t)^n - \binom{n+\alpha-1}{n-\nu} t^{\nu+\alpha}(1-t)^{n-\nu}\right]\frac{\chi(t)}{t}dt$$

$$D = \frac{1}{n+\alpha}\sum_{k=1}^{n}\binom{n+\alpha}{n-k}(k+\alpha)\Delta^{n-k}\mu_k\phi_k = \frac{1}{n+\alpha}\sum_{\nu=1}^{n}(\nu+\alpha)\phi_\nu \int_0^1 \binom{n+\alpha}{n-\nu} t^{\nu+\alpha}(1-t)^{n-\nu}d\chi(t)$$

$$\frac{d}{dt}\left(\binom{n+\alpha}{n-\nu}t^{\nu+\alpha}(1-t)^{n-\nu}\right) = \binom{n+\alpha}{n-\nu}(\nu+\alpha)t^{\nu+\alpha-1}(1-t)^{n-\nu} - \binom{n+\alpha}{n-\nu}(n-\nu)t^{\nu+\alpha}(1-t)^{n-\nu-1}$$

Using integration by parts,

$$D = \frac{1}{n+\alpha}\sum_{\nu=1}^{n}(\nu+\alpha)\phi_\nu\left[\chi(t)\binom{n+\alpha}{n-\nu}t^{\nu+\alpha}(1-t)^{n-\nu}\Big|_0^1 + \int_0^1\binom{n+\alpha}{n-\nu}(n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1}\frac{\chi(t)}{t}dt - \int_0^1\binom{n+\alpha}{n-\nu}(\nu+\alpha)t^{\nu+\alpha}(1-t)^{n-\nu}\frac{\chi(t)}{t}dt\right]$$
$$= \phi_n + \frac{1}{n+\alpha}\sum_{\nu=1}^{n}\Big\{(\nu+\alpha)\phi_\nu \times \int_0^1\left[\binom{n+\alpha}{n-\nu}(n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1} - \binom{n+\alpha}{n-\nu}(\nu+\alpha)t^{\nu+\alpha}(1-t)^{n-\nu}\right]\frac{\chi(t)}{t}dt\Big\}$$

Thus

$$b_n - a_n = \frac{t_n}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu + \sum_{\nu=1}^{n}\phi_\nu\int_0^1\left[\binom{n+\alpha+1}{n}t^{\alpha}(1-t)^n - \binom{n+\alpha-1}{n-\nu}t^{\nu+\alpha}(1-t)^{n-\nu}\right]\frac{\chi(t)}{t}dt + \phi_n$$
$$+ \sum_{\nu=1}^{n}\phi_\nu\int_0^1\left[\binom{n+\alpha-1}{n-\nu}(n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1} - \binom{n+\alpha-1}{n-\nu}(\nu+\alpha)t^{\nu+\alpha}(1-t)^{n-\nu}\right]\frac{\chi(t)}{t}dt - \frac{1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu - \phi_n$$
$$= \frac{t_n-1}{n+\alpha}\sum_{\nu=1}^{n}\phi_\nu + \sum_{\nu=1}^{n}\phi_\nu\int_0^1\left[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n - \binom{n+\alpha-1}{n-\nu}(\nu+\alpha+1)t^{\nu+\alpha}(1-t)^{n-\nu} + \binom{n+\alpha-1}{n-\nu}(n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1}\right]\frac{\chi(t)}{t}dt$$

Applying Lemma 2,

$$\alpha_{n\nu} = \begin{cases} 0, & \nu > n, \\ \frac{t_n-1}{n+\alpha} + \int_0^1\Big[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n - \binom{n+\alpha-1}{n-\nu}(\nu+\alpha+1)t^{\nu+\alpha}(1-t)^{n-\nu} + \binom{n+\alpha-1}{n-\nu}(n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1}\Big]\frac{\chi(t)}{t}dt, & 0 \le \nu < n, \\ \frac{t_n-1}{n+\alpha} + \int_0^1\Big[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n + (n+\alpha+1)t^{n+\alpha}\Big]\frac{\chi(t)}{t}dt, & \nu = n. \end{cases}$$

For $\nu < n$,

$$-\binom{n+\alpha-1}{n-\nu}\int_0^1\left[(\nu+\alpha+1)t^{\nu+\alpha}(1-t)^{n-\nu} - (n-\nu)t^{\nu+\alpha+1}(1-t)^{n-\nu-1}\right]\frac{\chi(t)}{t}dt$$
$$= -\binom{n+\alpha-1}{n-\nu}\int_0^1\left[t^{\nu+\alpha+1}(1-t)^{n-\nu}\frac{\chi(t)}{t}\Big|_0^1 + \binom{n+\alpha-1}{n-\nu}\int_0^1 t^{\nu+\alpha+1}(1-t)^{n-\nu}d\left(\frac{\chi(t)}{t}\right)\right].$$

Therefore, for $\nu < n$,

$$\alpha_{n\nu} = \frac{t_n-1}{n+\alpha} + \int_0^1\left[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n\frac{\chi(t)}{t}dt + \binom{n+\alpha-1}{n-\nu}\int_0^1 t^{\nu+\alpha+1}(1-t)^{n-\nu}d\left(\frac{\chi(t)}{t}\right)\right]. \quad (10)$$

$$\int_0^1 (n+\alpha+1)t^{n+\alpha}\frac{\chi(t)}{t}dt = t^{n+\alpha+1}\frac{\chi(t)}{t}\Big|_0^1 - \int_0^1 t^{n+\alpha+1}d\left(\frac{\chi(t)}{t}\right),$$

so that

$$\alpha_{nn} = \frac{t_n-1}{n+\alpha} + \int_0^1 \left[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n\frac{\chi(t)}{t}dt - \int_0^1 t^{n+\alpha+1}d\left(\frac{\chi(t)}{t}\right)\right. \quad (11)$$

Consequently, the conditions of Lemma 2 are satisfied with

$$A = \sup_{\nu}\psi_{\nu},$$

where

$$\begin{aligned}\psi_{\nu} &= \sum_{n=\nu}^{\infty}|\alpha_{n\nu}| \\ &\le \sum_{n=\nu}^{\infty}\frac{|t_n-1|}{n+\alpha} \\ &+ \int_0^1\sum_{n=\nu}^{\infty}\left[\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n\left|\frac{\chi(t)}{t}\right|dt\right. \\ &+ \int_0^1\sum_{n=\nu+1}^{\infty}\binom{n+\alpha-1}{n-\nu}t^{\nu+\alpha+1}(1-t)^{n-\nu}d\left|\left(\frac{\chi(t)}{t}\right)\right| \\ &+ \int_0^1 t^{\nu+\alpha+1}\left|\left(\frac{\chi(t)}{t}\right)\right|.\end{aligned}$$

$$\sum_{n=\nu}^{\infty}\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n \le \sum_{n=0}^{\infty}\binom{n+\alpha-1}{n}t^{\alpha}(1-t)^n = t^{\alpha}[1-(1-t)]^{-\alpha} = 1.$$

$$\begin{aligned}\sum_{n=\nu+1}^{\infty}\binom{n+\alpha-1}{n-\nu}t^{\nu+\alpha+1}(1-t)^{n-\nu} &= \\ &= \sum_{j=1}^{\infty}\binom{j+\nu+\alpha-1}{j}t^{\nu+\alpha+1}(1-t)^j \\ &= t^{\nu+\alpha+1}\left\{[1-(1-t)]^{-(\nu+\alpha)}-1\right\} \\ &= t(1-t^{\nu+\alpha}).\end{aligned}$$

Therefore we can choose

$$\begin{aligned}A &= \sum_{n=\nu}^{\infty}\frac{|t_n-1|}{n+\alpha} + \int_0^1\left|\frac{\chi(t)}{t}\right|dt + \int_0^1 t(1-t^{\nu+\alpha})\left|\left(\frac{\chi(t)}{t}\right)\right| \\ &+ \int_0^1 t^{\nu+\alpha+1}\left|\left(\frac{\chi(t)}{t}\right)\right| \\ &= \sum_{n=\nu}^{\infty}\frac{|t_n-1|}{n+\alpha} + \int_0^1 t^{-1}|\chi(t)|dt + \int_0^1 t\left|\left(\frac{\chi(t)}{t}\right)\right|.\end{aligned}$$

Corollary 4. *Let $\{\mu_n\}$ be a moment sequence for a regular Hausdorff matrix generated by a mass function $\chi(t)$ such that $\chi(t)/t$ is also of bounded variation on $0 \le t \le 1$.*
Then (8) *is satisfied with*

$$A = \int_0^1 t^{-1}|\chi(t)|dt + \int_0^1 t\left|\left(\frac{\chi(t)}{t}\right)\right|.$$

Proof
The proof follows from Theorem 2 by observing that, for any regular Hausdorff matrix, each $t_n = 1$.
Corollary 4 is an improvement on condition (3.1) of Theorem 3.1 of [6], since it is independent of ν.

3. Conclusion

Any sequence $\{\lambda_n\}$ in the range $n \leqslant \lambda_n \leqslant (n+1\,[\log(n+1)])^{\beta}$, for any $\beta \geqslant 0$, satisfies condition (1). Therefore the set of all regular H-J matrices is a significant generalization of both the ordinary Hausdorff matrices and the E-J generalized Hausdorff matrices. Consequently, Theorem 1 of this paper provides a substantial generalization of the corresponding result in [6].

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F. Aydin Akgun received a PhD in Istanbul, Turkiy from Yildiz Technical University in 2008. The thesis dealt with boundary value problems. Several papers on boundary value problems were published in 1999-2009. In 2010 she spend several months at Indiana University and worked with Professor B. E. Rhoades on summability and Hausdorff matrices. Recently, several papers on Summability have been published.

B. E. Rhoades is a leading world-known figure and is presently an Emeritus Professor at Indiana University, USA. He obtained his PhD from Lehigh University. He continues to be an active researcher on Analysis and Fixed Point Theory. He has been an invited speaker at a number of conferences and has published more than 390 research articles in reputed international journals. In addition to his individual studies he has carried out joint studies with a number of mathematicians, and has provided mathematical support and encouragement to researchers all over the world.