

Characterization and Estimation of Transmuted Rayleigh Distribution

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Abstract: In this paper, a new class of Transmuted Rayleigh distribution is introduced. The estimates of parameters of Transmuted Rayleigh distribution are obtained by using new method of moments. A new distribution which contains as a special case is introduced. The characterizing properties of the model are also derived.

Keywords: Transmuted Rayleigh distribution, Sample coefficient of variation, New Moment method.

1 Introduction

Rayleigh distribution (RD) is considered to be a very useful life distribution. Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. One major application of this model is used in analyzing wind speed data. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. This model was first introduced by Rayleigh (1980), Siddiqui (1962) discussed the origin and properties of the Rayleigh distribution. Inference for model Rayleigh model has been considered by Sinha and Howlader (1993), Lalitha et al. (1996) and Abd Elfattah et al. (2006). Faton Merovci (2013) generalizes the Rayleigh distribution using the quadratic rank transmutation map studied by Shaw et al. (2009) and named it Transmuted Rayleigh distribution. Ahmad et al. (2014) develops the Transmuted Inverse Rayleigh distribution and discussed its properties. The probability density function (pdf) of Rayleigh distribution is given as:

$$g(x, \theta) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right), \quad x > 0, \theta > 0 \quad (1.1)$$

And its corresponding cumulative distribution function (cdf) is given by

$$G(x, \theta) = 1 - \exp\left(-\frac{x^2}{2\theta^2}\right), \quad x > 0, \theta > 0 \quad (1.2)$$

Merovci (2013) used the quadratic rank transmutation map (QRTM) for a pair of distributions $F(x)$ and $G(x)$ where $G(x)$ is a sub model of $F(x)$. Therefore, a random variable X is said to have transmuted probability distribution with cdf $F(x)$ if

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x), \quad |\lambda| \leq 1$$

Which on differentiation yields

$$f(x) = g(x)[1 + \lambda - 2\lambda G(x)]$$

Where $G(x)$ and $g(x)$ is the cdf and pdf of the base distribution. Observe that at $\lambda = 0$, we have the distribution of the base random variable.

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Hence, the pdf of transmuted Rayleigh distribution with parameters θ and λ is

$$f(x, \theta, \lambda) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right) \quad (1.3)$$

And the corresponding cdf is given by

$$F(x, \theta, \lambda) = \left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right) \left(1 + \lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right) \quad (1.4)$$

The Rayleigh distribution is clearly a special case for $\lambda = 0$.

Figure 1 and 2 shows some of the possible shapes of the pdf and cdf of transmuted Rayleigh distribution for selected values of parameters θ and λ respectively.

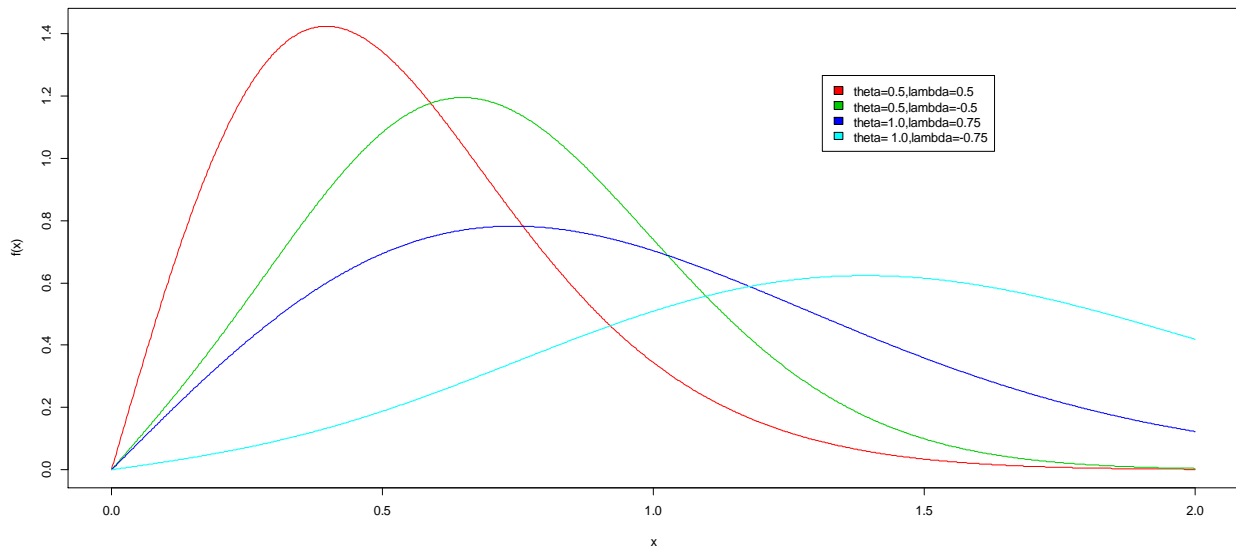


Figure 1: The pdf of transmuted Rayleigh distribution under various values of Parameters

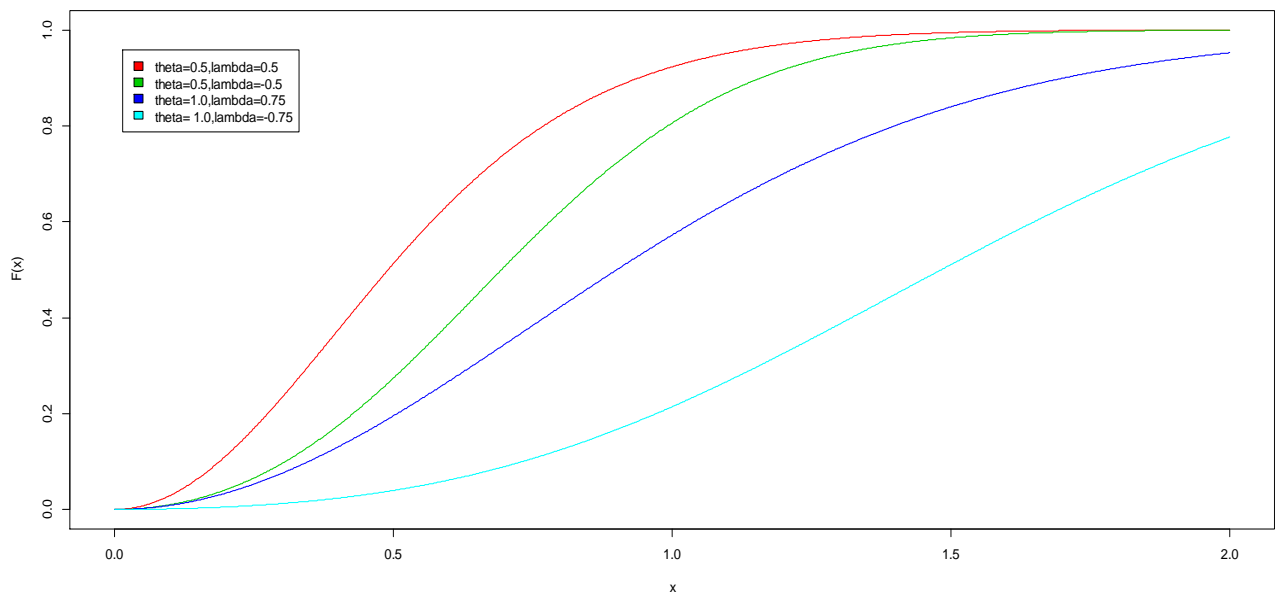


Figure 2: The cdf of transmuted Rayleigh distribution under various values of Parameters

2 Statistical Properties of Transmuted Rayleigh Distribution (TRD)

In this section, we present the statistical properties of Transmuted Rayleigh Distribution throughout computing the mean, variance, coefficient of variation, harmonic mean, moments, mode, coefficient of skewness, and coefficient of kurtosis as follow:

2.1 Moments of TRD

The r th moment of TRD is given by

$$\begin{aligned}
 E(X^r) &= \mu'_r = \int_0^\infty x^r f(x; \theta, \lambda) dx \\
 &= \int_0^\infty x^r \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right) dx \\
 &= \frac{(1-\lambda)}{\theta^2} \int_0^\infty x^{r+1} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx + \frac{2\lambda}{\theta^2} \int_0^\infty x^{r+1} \exp\left(-\frac{x^2}{\theta^2}\right) dx \\
 &= 2^{\frac{r-2}{2}} (1-\lambda) \theta^r r \Gamma\left(\frac{r}{2}\right) + \frac{\lambda r}{2} \theta^r \Gamma\left(\frac{r}{2}\right) \\
 \Rightarrow \mu'_r &= \frac{1}{2} \theta^r r \Gamma\left(\frac{r}{2}\right) \left(\lambda + 2^{\frac{r}{2}} (1-\lambda)\right) \tag{2.1}
 \end{aligned}$$

Put $r=1$ in eq. (2.1), we get mean of the Transmuted Rayleigh distribution which is given by

$$\mu = \mu'_1 = \frac{1}{2} \theta \sqrt{\pi} \left(\lambda + \sqrt{2} (1-\lambda)\right) \tag{2.2}$$

Put $r=2, 3$ and 4 in eq. (2.1), we get second third and fourth moment of TRD as given below

$$\mu'_2 = \theta^2 (\lambda + 2(1-\lambda))$$

$$\mu'_3 = \frac{3}{4} \theta^3 \sqrt{\pi} \left(\lambda + 2^{\frac{3}{2}} (1-\lambda)\right)$$

$$\mu'_4 = 2\theta^4 (\lambda + 4(1-\lambda))$$

2.2 Variance of Transmuted Rayleigh distribution

The variance of transmuted Weibull distribution is given by

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \theta^2 \left(2 - \lambda - \frac{\pi}{4} \left(\lambda + \sqrt{2} (1-\lambda)\right)^2\right) \tag{2.3}$$

2.3 Third and fourth moments Transmuted Rayleigh distribution

$$\mu_3 = \mu'_3 - 2\mu'_2 \mu'_1 + 2(\mu'_1)^3 = \frac{3}{4} \theta^3 \sqrt{\pi} \left(\sigma_3 - 2\sigma_2 \sigma_1 + \frac{\pi}{3} \sigma_1^3\right)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_2 + 6\mu'_2\mu'_1 - 3(\mu'_1)^4 = 2\theta^4\sigma_4 - 3\theta^4\sqrt{\pi}\sigma_3\sigma_2 + 3\theta^3\sqrt{\pi}\sigma_2\sigma_1 - \frac{3}{16}\theta^4\pi^2\sigma_1^4$$

2.4 Standard Deviation

$$\sigma = \sqrt{\mu_2} = \theta \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1-\lambda))^2 \right)^{\frac{1}{2}} \quad (2.4)$$

2.5 Coefficient of variation

$$C.V = \frac{\sigma}{\mu} = \frac{\left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1-\lambda))^2 \right)^{\frac{1}{2}}}{\frac{\sqrt{\pi}}{2} (\lambda + \sqrt{2}(1-\lambda))} \quad (2.5)$$

2.6 Skewness and Kurtosis

The most popular way to measure the skewness and kurtosis of a distribution function rests upon ratios of moments. Lack of symmetry of tails (about mean) of frequency distribution curve is known as skewness. The formula for measure of skewness and kurtosis given by Karl Pearson in terms of moments of frequency distribution is given by

$$\text{Skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad \text{Kurtosis} = \beta_2 = \frac{\mu_4^2}{\mu_2^2}$$

$$\beta_1 = \frac{\left(\frac{3}{4}\theta^3\sqrt{\pi} \left(\sigma_3 - 2\sigma_2\sigma_1 + \frac{\pi}{3}\sigma_1^3 \right) \right)^2}{\left(\theta^2 \left(2 - \lambda - \frac{\pi}{4}\sigma_2^2 \right) \right)^3}$$

$$\beta_2 = \frac{\left(2\theta^4\sigma_4 - 3\theta^4\sqrt{\pi}\sigma_3\sigma_2 + 3\theta^3\sqrt{\pi}\sigma_2\sigma_1 - \frac{3}{16}\theta^4\pi^2\sigma_1^4 \right)}{\left(\theta^2 \left(2 - \lambda - \frac{\pi}{4}\sigma_2^2 \right) \right)^2}$$

3 New Moment Estimator of parameters of Transmuted Rayleigh distribution.

In this section, we recall the new method based on moments, using its characterization for estimation of parameters of transmuted Rayleigh distribution. The result shows that this new method is easy and more efficient than MLE method in small sample. For deriving new moment estimators of the parameters of the Transmuted Rayleigh distribution, we need the following theorem obtained by using the similar approach of Hwang (2000), Hwang and Hu (1999) and Huang and Hwang (2006).

Theorem 3.1: Let $n \geq 3$ and let $X_1, X_2, X_3, \dots, X_n$ be a n positive identical independently distributed random variables having a probability density function $f(x)$. Then the independence of the sample mean \bar{X}_n and the sample coefficient of variation $V_n = \frac{S_n}{\bar{X}_n}$ is equivalent to that $f(x)$ is a transmuted Rayleigh density where S_n is the sample standard deviation.

The next result and theorem 3.1 are useful in deriving the expectation and the variance of $V_n^2 = \left(\frac{S_n}{\bar{X}_n}\right)^2$, where \bar{X}_n and S_n are respectively the sample mean and the sample standard deviation.

Theorem 3.2: Let $n \geq 3$ and let $X_1, X_2, X_3 \dots X_n$ be a n positive identical independently distributed random samples drawn from a population having a transmuted Rayleigh density

$$f(x, \theta, \lambda) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right)$$

Then

$$E(\bar{X}_n^2) = \frac{\theta^2}{n} \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2 (1 - n)\right)$$

$$E(S_n^2) = \theta^2 \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2\right)$$

Where \bar{X}_n and S_n^2 are respectively their sample mean and sample variance.

Proof: It is easy to prove that

$$E(X) = \frac{1}{2} \theta \sqrt{\pi} (\lambda + \sqrt{2}(1 - \lambda)), \quad \text{Var}(X) = \theta^2 \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2\right)$$

$$E(X^m) = \frac{1}{2} \theta^m m \Gamma\left(\frac{m}{2}\right) \left(\lambda + 2^{\frac{m}{2}} (1 - \lambda)\right)$$

$$E(\bar{X}_n) = \frac{1}{2} \theta \sqrt{\pi} (\lambda + \sqrt{2}(1 - \lambda))$$

$$\text{Var}(\bar{X}_n) = \frac{\theta^2 \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2\right)}{n}$$

$$E(\bar{X}_n^2) = \frac{\theta^2}{n} \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2 (n - 1)\right) \tag{3.1}$$

Now, $E(S_n^2) = n \text{Var}(\bar{X}_n)$

$$E(S_n^2) = \theta^2 \left(2 - \lambda - \frac{\pi}{4} (\lambda + \sqrt{2}(1 - \lambda))^2\right) \tag{3.2}$$

Theorem 3.3: Let $n \geq 3$ and let $X_1, X_2, X_3 \dots X_n$ be a n positive identical independently distributed random samples drawn from a population having transmuted Rayleigh density

$$f(x, \theta, \lambda) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right)$$

$$\text{Then } E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)}{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)(n-1)}$$

Where \bar{X}_n and S_n^2 are respectively their sample mean and sample variance.

Proof: By theorem 2.1, we have

$$E(S_n^2) = E\left(\frac{S_n^2}{\bar{X}_n^2} \cdot \bar{X}_n^2\right) = E\left(\frac{S_n^2}{\bar{X}_n^2}\right) E(\bar{X}_n^2)$$

And hence

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{E(S_n^2)}{E(\bar{X}_n^2)}$$

Applying theorem 3.2 to the above identity yields that

$$E\left(\frac{S_n^2}{\bar{X}_n^2}\right) = \frac{n\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)}{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)(n-1)} \quad (3.3)$$

Thus theorem 3.3 is established.

Note that $E\left(\frac{S_n^2}{\bar{X}_n^2}\right) \rightarrow \frac{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)}{\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2}$ as $n \rightarrow \infty$ and that this limit is the square of the coefficient of

variation. Thus $\left(\frac{S_n^2}{\bar{X}_n^2}\right)$ is an asymptotically unbiased estimator of the square of the coefficient of variation.

Based on Theorems 3.2 and 3.3 we set, by using moment estimation approach, two equations for finding two estimators $(\hat{\theta}, \hat{\lambda})$ of parameters (θ, λ) respectively as follows:

$$\frac{\sum_{i=1}^n x_i}{n} = \frac{1}{2} \theta \sqrt{\pi} (\lambda + \sqrt{2}(1-\lambda)) \quad (3.4)$$

$$\frac{S_n^2}{n \bar{X}_n^2} = \frac{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)}{\left(2-\lambda-\frac{\pi}{4}(\lambda+\sqrt{2}(1-\lambda))^2\right)(n-1)} \quad (3.5)$$

Thus the solutions of (θ, λ) are obtained by solving the two equations (3.4) and (3.5) simultaneously are proposed for their estimators.

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