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New Class of Boundary Value Problems

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Abstract: A category of singularly perturbed boundary value problem for nonlinear equation of fourth order with two parameters is considered here. The Adomian decomposition and Variational iteration techniques are used to propose an asymptotic solution to this problem. Some examples are given to demonstrate the effectiveness of the two techniques for solving the boundary value problems of nonlinear equation of fourth order with two parameters. The two solutions are compared for small and large *x*.

Keywords: nonlinear; two parameters; singular perturbation; variational iteration and Adomian decomposition method.

I. Introduction

The singularly perturbed problem with a turning point is very striking subject of study in international academic circle [1]. During the past decade, many approximate methods have been introduced for example, averaging method, the boundary layer method, the matched asymptotic expansion method, and multiple scales methods and WKB methods. in recent times many researchers including, Ni and Wei [2], Bartier [3] Libre et al [4], Guarguaglini and Natalini [5] and Jia-qi and Zhao-Hui have done must work for this concern. Our concern in thus work is to consider the nonlinear equation of fourth order with two parameters of form:

$$\epsilon\frac{d^4y(x)}{dx^4}+\mu p(x)\frac{d^3y(x)}{dx^3}+q(x)\frac{d^2y(x)}{dx^2}+\ c(x)\frac{dy(x)}{dx}=f(x) \qquad (1.1)$$

$$0\le x<\infty$$

Subject to the initial condition $y(0)=A$. Where ϵ and μ are positives small parameters. Here it is assumed that:

- $c(x)\neq 0$, $p(x), q(x)$ and $f(x)$ are sufficiently smooth with regard to their variables in corresponding domain.

- The functions, $\frac{f(x)}{c(x)}$, $\frac{p(x)}{c(x)}$ and $\frac{q(x)}{c(x)}$ are integrable in [0, ∞ [
- $p(x_0) = 0$ with x_0

This paper is presented as follows; we start by presenting Adomian decomposition method, following by an applications where the turning point occurs at the origin and at $x = \frac{\pi}{2}$. This is then followed by the presentation of Variational Iteration method with an application.

II-Adomain Decomposition methods

Dividing equation (1.1) in both sides with $c(x)$ we have the following equation

$$\frac{\epsilon}{c(x)}\frac{d^4y(x)}{dx^4} + \mu\frac{p(x)}{c(x)}\frac{d^3y(x)}{dx^3} + \frac{q(x)}{c(x)}\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} = \frac{f(x)}{c(x)} \tag{1.2}$$

The method used here is based on applying the operator $\int dx$ which the inverse operator of $\frac{d}{dx}$ on both sides of equation (1.2) to obtain:

$$y(x) - y(0) = \int_0^x \frac{\epsilon}{c(v)}\frac{d^4y(v)}{dv^4} - \mu\frac{p(v)}{c(v)}\frac{d^3y(v)}{dv^3} - \frac{q(v)}{c(v)}\frac{d^2y(v)}{dv^2} + \frac{f(v)}{c(v)}dv \tag{1.3}$$

The Adomian decomposition method [1, 2] suggests a series solution $y(x,\epsilon,\mu)$ for equation (1.3) given below as:

$$y(x,\epsilon,\mu) = \sum_{n=0}^{\infty} y_n(x,\epsilon,\mu) \tag{1.4}$$

Here the components $y_n(x,\epsilon,\mu)$ are to be determined recursively. Thus by replacing equation (1.4) into both sides of equation (1.3) provides:

$$\sum_{n=0}^{\infty} y_n(x,\epsilon,\mu) - y(0) = \int_0^x \frac{\epsilon}{c(v)}\frac{d^4}{dv^4}\left[\sum_{n=0}^{\infty} y_n(x,\epsilon,\mu)\right] - \mu\frac{p(v)}{c(v)}\frac{d^3}{dv^3}\left[\sum_{n=0}^{\infty} y_n(x,\epsilon,\mu)\right] - \frac{q(v)}{c(v)}\frac{d^2}{dv^2}\left[\sum_{n=0}^{\infty} y_n(x,\epsilon,\mu)\right] + \frac{f(v)}{c(v)}dv$$

By following the decomposition method, we find the recurrence formula as:

$$y_0(x,\epsilon,\mu) = y(0) - \int_0^x \frac{f(v)}{c(v)}$$

and

$$y_n(x,\epsilon,\mu) = \int_0^x \frac{\epsilon}{c(v)}\frac{d^4y_n(v,\epsilon,\mu)}{dv^4} - \mu\frac{p(v)}{c(v)}\frac{d^3y_n(v,\epsilon,\mu)}{dv^3} - \frac{q(v)}{c(v)}\frac{d^2y_n(v,\epsilon,\mu)}{dv^2}dv \tag{1.5}$$

From the above equation, if the first component $y_0(x,\epsilon,\mu)$ is known, the remaining components, $y_n(x,\epsilon,\mu), n \geq 1$ can be easily calculated in the way that every term is calculated by employing the preceding terms, and thus the series solution are entirely calculated. For this purpose we estimated the solution $y(x,\epsilon,\mu)$ by reduced the series to:

$$y_N(x,\epsilon,\mu) = \sum_{n=0}^{N-1} y_n(x,\epsilon,\mu) \text{ with } \lim_{N\to\infty} y_N(x,\epsilon,\mu) = y(x,\epsilon,\mu) \tag{1.6}$$

III-Variational Iteration method

The values of the variational iteration method and its applications for a range of categories of differentials equations can be viewed in [3, 4, and 5]. To solve equation (1.1) by means of variational iteration method, we put equation (1.1) in the form:

$$(\epsilon y(x))_{4x} + p(x)(\mu y(x))_{3x} + q(x)\left(y(x)\right)_{2x} + c(x)(y(x))_x - f(x) = 0 \tag{2.1}$$

The correction functional for equation (2.1) can be approximately expresses as follows for this matter as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(v)\left[\frac{d^m y(v)}{dv^m} - \bar{f}(v) + c(v)(\bar{y}(v))_v + q(v)(\bar{y}(v))_{2v} + p(v)\mu(\bar{y}(v))_{3v} + \epsilon(\bar{y}(v))_{4v}dv\right] \tag{2.2}$$

Where $\lambda(v)$ is a general Lagrange multiplier [6], and this is can be determined most selectively via variational hypothesis [7, 8,9]. Here $(\bar{y}(v))_{4v}$, $(\bar{y}(v))_{2v}$, $(\bar{y}(v))_{3v}$ and $\bar{f}(v)$
are considered here as restricted variations. Now making the above functional stationary

$$\delta y_{n+1}(x) = \delta \int_0^x \lambda(v)\left[\frac{d^m y(v)dv}{dv^m}\right]$$

giving up to the following Lagrange multipliers $\lambda = -1$ for the case where and $\lambda = v - x$ for $m = 2$. For these matter if $m = 1$, we obtained the following iteration formula:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[c(v)\frac{dy_n(v)}{dv} - f(v) + q(v)\frac{d^2y_n(v)}{dv^2} + p(v)\mu\frac{d^3y_n(v)}{dv^3} + \epsilon\frac{d^4y_n(v)}{dv^4}\right]dv$$

In this case, we begin the initial approximation $y_0(x) = y(0) = A$. And for $m = 2$, we obtained the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x (v-x)\left[c(v)\frac{dy_n(v)}{dv} - f(v) + q(v)\frac{d^2y_n(v)}{dv^2} + p(v)\mu\frac{d^3y_n(v)}{dv^3} + \epsilon\frac{d^4y_n(v)}{dv^4}\right]dv$$

In this case we begin with the initial $y_0(x) = y(0) + xy'(0) = A + xy'(0)$

IV-Applications And Results

Example 1

An application of the two approaches introduced earlier is performed here by considering the following boundary value problem for nonlinear equation of fourth order with parameters of form:

$$\epsilon\frac{d^4y(x)}{dx^4}+\mu x\frac{d^2y(x)}{dx^2}+x\frac{d^2y(x)}{dx^2}+\frac{dy(x)}{dx}=x^3 \tag{3.1}$$

subject to the initial condition introduced earlier. We start this application here with Adomian decomposition method.

3.1 Adomain Decomposition Method

Here equation (3.1) can be written as:

$$\frac{dy(x)}{dx}=x^3-\left[\epsilon\frac{d^4y(x)}{dx^4}+\mu x\frac{d^2y(x)}{dx^2}+x\frac{d^2y(x)}{dx^2}\right]$$

Applying the integral on both sides of the above equation we have the following:

$$y(x)-y(0)=-\int_0^x\left[\epsilon\frac{d^4y(v)}{dv^4}+\mu v\frac{d^2y(v)}{dv^2}+x\frac{d^2y(v)}{dv^2}-v^3\right]dv \tag{3.2}$$

Following the discussion presented earlier, a series solution of the above equation is suggested here as:

$$y(x,\epsilon,\mu)=\sum_{n=0}^{\infty}y_n(x.\epsilon.\mu)$$

Substituting the above into equation (3.2) and following the decomposition technique we introduced the following recursive formula:

$$y_{n+1}(x,\epsilon,\mu)=-\int_0^x\left[\epsilon\frac{d^4y_n(v,\epsilon,\mu)}{dv^4}+\mu v\frac{d^2y_n(v,\epsilon,\mu)}{dv^2}+x\frac{d^2y_n(v,\epsilon,\mu)}{dv^2}\right]dv \tag{3.3}$$

$$y_0(x,\epsilon,\mu)=y(0)+\int_0^x v^3\,dv=A+\frac{x^4}{4}$$

$$y_1(x,\epsilon,\mu)=-6\epsilon x-2\mu x^3-\frac{3}{4}x^4$$

$$y_2(x,\epsilon,\mu)=18\epsilon x+6\mu^2x^2+10\mu x^3+\frac{3}{4}x^4$$

$$y_3(x,\epsilon,\mu)=-6\epsilon x-30\,\mu^2x^2-6\mu^2x^2-20\mu x^3-\frac{9}{4}x^4$$

For this matter we stopped at N = 3 and the asymptotic solution of equation (3.1) was given by:

$$y(x,\epsilon,\mu) = y_0(x,\epsilon,\mu) + y_1(x,\epsilon,\mu) + y_2(x,\epsilon,\mu) + y_3(x,\epsilon,\mu) + \cdots + \tag{3.4}$$

3.2 Variational Iteartion Method

Following the discussion presented earlier we put equation (3.1) in the form:

$$(\epsilon y(x))_{4x} + x(\mu y(x))_{3x} + x\big(y(x)\big)_{2x} + (y(x))_x - x^3 = 0 \tag{3.5}$$

The correction functional for equation (3.5) can be roughly expresses as follows for this matter as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(v)\left[\frac{d^m y(v)}{dv^m} - \overline{x^3} + \epsilon(\bar{y}(v))_{4v} + v(\bar{y}(v))_{2v} + \ v\mu(\bar{y}(v))_{3v} + \qquad \epsilon(\bar{y}(v))_{4v} dv\right]$$

Now making the above functional stationary

$$\delta y_{n+1}(x) = \delta \int_0^x \lambda(v)\left[\frac{d^m y(v) dv}{dv^m}\right]$$

giving up to the following Lagrange multipliers $\lambda = -1$ for the case where and $\lambda = v - x$ for $m = 2$. For these matter if $m = 1$, we obtained the following iteration formula:

$$y_{n+1}(x) = y_n(x) - \int_0^x \left[\frac{dy_n(v)}{dv} - v^3 + v\frac{d^2 y_n(v)}{dv^2} + \ v\mu\frac{d^3 y_n(v)}{dv^3} + \ \epsilon\frac{d^4 y_n(v)}{dv^4}\right] dv$$

In this case, we begin the initial approximation $y_0(x) = A + \frac{x^4}{4}$, the case of $m = 2$ will not be investigated in this work.

$$y_1(x,\epsilon,\mu) = A - \frac{x^4}{2} - 2\mu x^3 - 6\epsilon\, x$$

$$y_2(x,\epsilon,\mu) = A + 12\epsilon\, x + 6\mu^2 x^2 + 10\mu x^3 + \frac{7}{4}x^4$$

$$y_3(x,\epsilon,\mu) = A - 42\epsilon\, x - 36\mu x^2 - 34\mu x^3 - 5x^4$$

For this purpose we stopped for N = 3 and the series solution of equation (3.1) can be approximated to:

$$y(x,\epsilon,\mu) = y_0(x,\epsilon,\mu) + y_1(x,\epsilon,\mu) + y_2(x,\epsilon,\mu) + y_3(x,\epsilon,\mu) + \cdots + \tag{3.6}$$

It is important to single out that, there is the self-cancelling noise terms appear between various components of the two asymptotic solutions. Then we cancelled the noise terms in the decomposition solution (3.4) and the variational iteration, and we then compared the two approximated solutions see Figure 1.

After cancelling the noise terms in both solutions we ended up with the following solutions

$$Adomian\ y(x,\epsilon,\mu) = A + 6\epsilon\, x - 30\,\mu^2 x^2 - 12\mu\, x^3 - 2\,x^4 + \cdots +$$

$$Variational\ y(x,\epsilon,\mu) = A - 46\,\epsilon\, x - 30\mu^2\, x^2 - 26\,\mu x^3 - 2x^4 + \cdots +$$

The comparison of the graphical representation of the two solution is shown in the below figure 1-2-3.

The graphical representation show that for $0 \le x < 1$ and for very small values of ϵ and , the two solution are equal. For $x \ge 1$ for any values of the small parameters ϵ and μ, the curves of the two solution are equal.

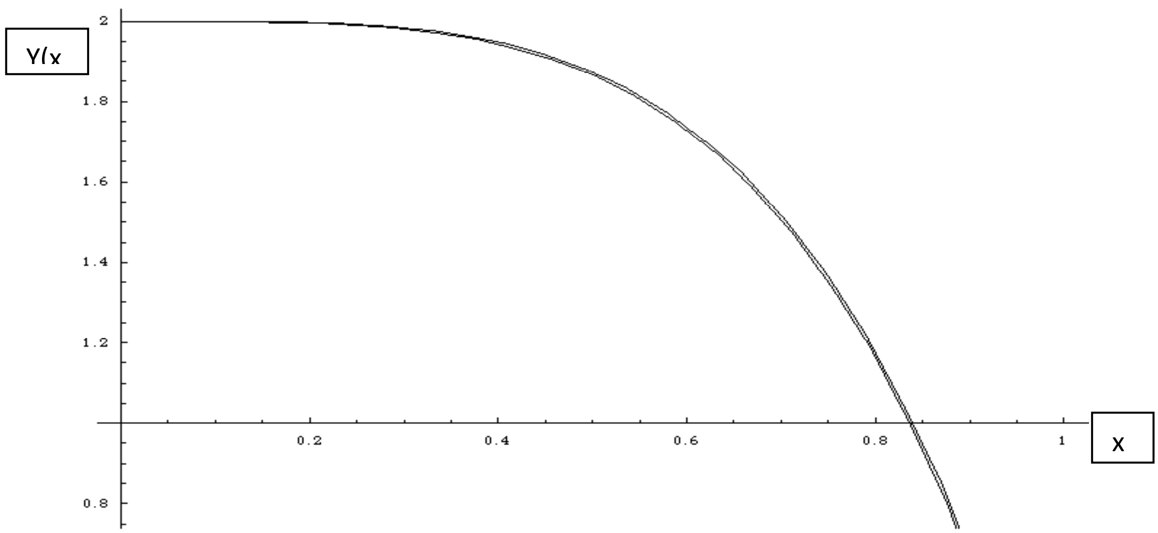

Figure 1: Comparison for $\epsilon = 0.0002$ and $\mu = 0.001$ with small values of x

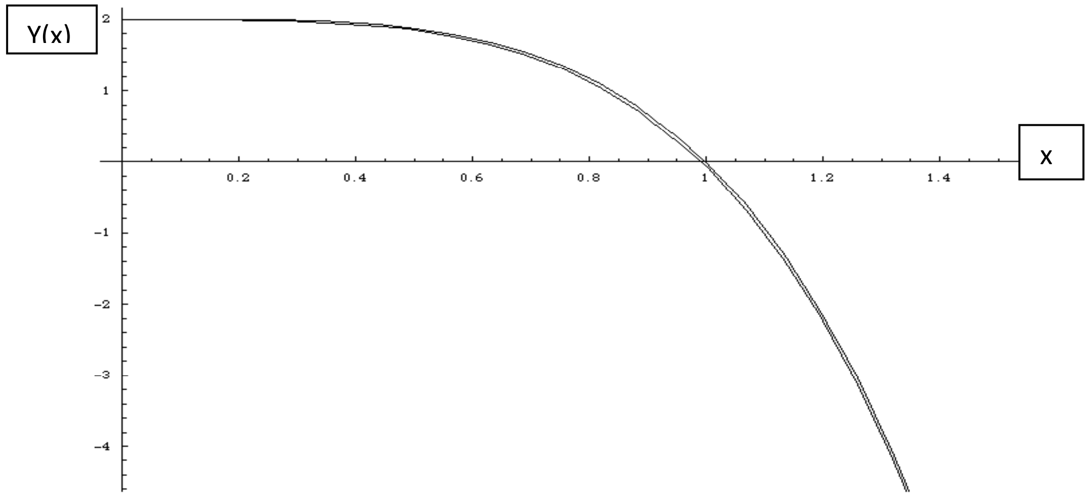

Figure 2: Comparison for $\epsilon = \mu = 0.001$

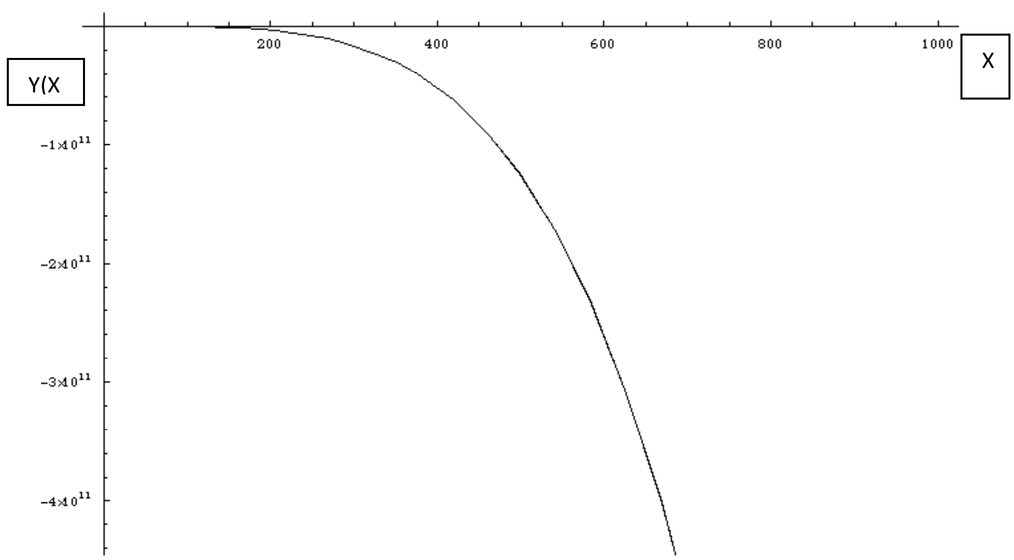

Figure 3: Comparison for $\epsilon = 0.1$ and $\mu = 0.2$ with large value of x

Example 2

Our concern here is considering the following boundary value problem for nonlinear equation of fourth order with parameters of form:

$$\epsilon \frac{d^4 y(x)}{dx^4} + \mu \cos x \frac{d^2 y(x)}{dx^2} + e^x \frac{d^2 y(x)}{dx^2} - \frac{1}{x+1} \frac{dy(x)}{dx} = 1 \tag{3.7}$$

subject to the initial condition introduced earlier.

3.3 Adomian Decomposition Method

Following the discussion presented earlier we introduced the following recursive formula:

$$y_0(x,\epsilon,\mu) = A - \frac{x^2}{2} - x$$

$$y_{n+1}(x,\epsilon,\mu) = \int_0^x \left[(v+1)\epsilon\frac{d^4y_n(v)}{dv^4} + \mu(v+1)\cos(v)\frac{d^2y_n(v)}{dv^2} + (v+1)e^x\frac{d^2y_n(v)}{dv^2}\right]dv \tag{3.8}$$

Then using the zeroth component and equation (3.8) we determined the remaining components as:

$$y_1(x,\epsilon,\mu) = -xe^x$$

$$y_2(x,\epsilon,\mu) = \frac{1}{2}[1 - e^{2x} - 2e^{2x}x - e^{2x}x^2 + 2\epsilon - 2e^x\epsilon - 6e^x x\epsilon - 2e^x x^2\epsilon + 2\mu - e^x(2+4x+x^2)\mu\cos(x) - e^x x(2 + x)\mu\sin(x)]$$

$$y_3(x,\epsilon,\mu) = \frac{1}{135000}\left[170000 - 5000e^{3x}\left(34 + 3x\left(29 + 6x(4+x)\right)\right) - 2615625\epsilon + 945000\,\epsilon^2 + 871776\mu + 1012500\epsilon\mu + 568944\mu^2\right]$$

$$+$$

$$\frac{1}{135000}\left[27e^{2x}\left(-625\left(-155 + 2x\left(155 + x(101 + 18x)\right)\right)\mu\right) - 8\left(4036 + 5x\left(1921 + 5x(278 + 55x)\right)\right)\mu\cos(x) - 8\left(448 + 5x\left(253 + 15x(18 + 5x)\right)\right)\mu\sin(x)\right]$$

$$+$$

$$\frac{109\,e^x}{135000}\left[-625\left(15 + x\left(38 + x(16+x)\right)\right)\epsilon\mu\cos(x) - 2\left(1384 + 5x\left(593 + 10x(34 + 5x)\right)\right)\mu^2\cos(2x) + 625\left(21 + x\left(36 + x(19 + 3x)\right)\right)\epsilon\mu\sin(x) + 2\left(612 + 5x\left(199 + 5x(24 + 5x)\right)\right)\mu^2\sin(2x) - 1250\left(\left(7 + x(3+x)(6+x)\right)\epsilon^2 + \left(2 + x(3+2x)\right)\mu^2 + 73\sinh(x)\right)\right]$$

For this purpose we stopped for N = 3 and the series solution of equation (3.7) obtained via Adomian Decomposition method can be approximated to:

$$y(x,\epsilon,\mu) = y_0(x,\epsilon,\mu) + y_1(x,\epsilon,\mu) + y_2(x,\epsilon,\mu) + y_3(x,\epsilon,\mu) + \cdots + \tag{3.8}$$

3.4 Variational Iteration Method

As discussed earlier we introduced the recursive the following recursive formula:

$$y_0(x,\epsilon,\mu) = A - \frac{x^2}{2} - x$$

$$y_{n+1}(x,\epsilon,\mu) = y_n(x,\epsilon,\mu) - \int_0^x \left[\epsilon(1+v)\frac{d^4y_n(v)}{dv^4} + \mu(1+v)\cos v\frac{d^2y_n(v)}{dv^2} + (1+v)\,e^v\frac{d^2y_n(v)}{dv^2} - \frac{dy_n(v)}{dv} - (v+1)\right]dv$$

So that:

$$y_1(x,\epsilon,\mu) = A - \frac{x^2}{2} - x + xe^x$$

$$y_2(x,\epsilon,\mu) = \frac{1}{2}\left[1 + 2A - 2x - x^2 - e^{2x}(1+x)^2 + 2\epsilon + 2\mu - e^x\left(-6x + 2\left(1 + x(3+x)\right)\epsilon + (2 + x(4+x))\mu\cos(x) + x(2+x)\mu\sin(x)\right)\right]$$

$$y_3(x,\epsilon,\mu) = \frac{67}{54} + A - \frac{1}{2}x(2+x) + \frac{1}{27}e^{3x}\left(34 + 3x\left(29 + 6x(4+x)\right)\right) - 7\epsilon^2 + \frac{1}{8}e^{2x}\left(-20(1+x)^2 + \left(-155 + 2x\left(155 + x(101 + 18x)\right)\right)\epsilon\right) - \frac{15}{8}\epsilon(-13 + 4\mu) - \frac{1}{625}\mu(911 + 2634\mu) + e^x\left(x^3\epsilon^2 + \epsilon(-5 + 7\epsilon) + 2\mu^2 + x^2(\epsilon(-5 + 9\epsilon) + 2\mu^2) + x(7 + 3\epsilon(-5 + 6\epsilon) + 3\mu^2)\right)$$

$$+$$

$$\frac{1}{1250}\left(e^x\mu\left(\left(2e^x\left(4036 + 5x\left(1921 + 5x(278 + 55x)\right)\right)625\left(-5\left(2 + x(4+x)\right) + \left(15 + x\left(38 + (16 + x)\right)\right)\right)\right)\epsilon\right)\right)\cos(x)$$

$$+2\left(1384 + 5x\left(593 + 10x(34 + 5x)\right)\right)\mu\cos(2x) + 2e^x\left(448 + 5x\left(253 + 15x(18 + 5x)\right)\right)\sin(x)$$

$$-625\left(5x(2+x) + \left(21 + x\left(36 + x(19 + 3x)\right)\right)\mu\right)\sin(x) - 2\left(612 + 5x(199 + 5x(24 + 5x)\mu\sin(2x))\right) + 73e^x\epsilon\sinh(x)$$

For this purpose we stopped for N = 3 and the series solution of equation (3.7) obtained via Variational iteration method can be approximated to:

$$y(x,\epsilon,\mu) = y_0(x,\epsilon,\mu) + y_1(x,\epsilon,\mu) + y_2(x,\epsilon,\mu) + y_3(x,\epsilon,\mu) + \cdots + \quad (3.9)$$

The noisy free of both solutions are obtained by canceling the noise terms in both solutions as discussed earlier.

V- CONCLUSIONS

In this work, we applied both the Adomian decomposition and the variational iteration method for solving the initial and boundary value problems of nonlinear equation of fourth order with two parameters .The solution obtained noticeably pointed out the consistency and correctness of the proposed methods. Two examples were given in order to demonstrate the effectiveness of the two techniques for solving the boundary value problems of nonlinear equation of fourth order with two parameters. The suggested method is used directly without using perturbation, linearization or restrictive assumptions.

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