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The Generalized q-Analogue Hermite matrix Polynomials of Two Variables

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Abstract: In this paper, we introduce the q-analogue generalized Hermite matrix polynomials of two variables. Some recurrence relations for these q-polynomials are derived.

Keywords: Hermite Matrix; Polynomials of two Variables; Generating Functions; Recurrence Relations.

1. Introduction

The classical Hermite polynomials have two important properties: (i) they form a family of orthogonal polynomials and (ii) are intimately connected with the commutation properties between the multiplication operator X and the differentiation operator D. In contrast to the discrete q-Hermite polynomials, which generalize both aspects, the continuous q-Hermite polynomials generalize only the first one.

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [6]

$$(a;q)_n = \begin{cases} 1 & , \quad n=0 \\ \prod_{0\le j\le n-1} (1-aq^j), & n=1,2,3,\dots \end{cases}$$

where

$$(q^{-n};q)_k = \begin{cases} 0 & , \quad k>n \\ \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk} & , \quad k\le n \end{cases} \qquad (1.2)$$

$$(0;q)_n = 1$$

also

$$(a;q)_{n+k} = (a;q)_n (aq^n;q)_k , \qquad (1.3)$$

where

$$\lim_{q\to 1^-} \frac{(q^z;q)_k}{(1-q)^k} = (z)_k$$

.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} , \quad 0\le k\le n, \ \ k,n\in N \qquad (1.4)$$

.

The q-derivative with index α is defined by [11]

$$D_\alpha = \frac{f(q^\alpha x)-f(x)}{(q^\alpha -1)x} , \qquad D_1 = D, \qquad (1.5)$$

which for q-derivative of the pair of functions are valid:

$$D(\lambda a(x)+\mu b(x)) = \lambda Da(x) + \mu Db(x) \quad , \qquad (1.6)$$

$$D(a(x).b(x)) = a(qx).Db(x) + Da(x).b(x) \ , \qquad (1.7)$$

$$D\left(\frac{a(x)}{b(x)}\right) = \frac{Da(x).b(x) - a(x).Db(x)}{b(x)b(qx)}$$

. (1.8)

Exton [3] presented the following q-exponential functions:

$$E(\mu,z;q)=\sum_{n=0}^{\infty}\frac{q^{\mu n(n-1)}}{[n]_q!}z^n, \quad (1.9)$$

where $[n]_q!=\frac{(q;q)_n}{(1-q)^n},\quad n\in N_0$ (1.10)

$$\lim_{q\to1}[n]_q!=\lim_{q\to1}\frac{(q;q)_n}{(1-q)^n}=(1)_n=n!\ . \quad (1.11)$$

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $\frac{a}{2}$, we get

$$E\left(\frac{a}{2},\frac{x}{1-q};q\right)=E_q(x,a),$$

where $E_q(x,a)=\sum_{n=0}^{\infty}\frac{q^{a\binom{n}{2}}}{(q;q)_n}x^n,$ (1.12)

which satisfies the functional relation [3]

$$E_q(x,a)-E_q(qx,a)=xE_q(q^ax,a).$$

The above q-function can be rewritten by the formula

$$D_qE_q(x,a)=\frac{1}{1-q}E_q(q^ax,a). \quad (1.13)$$

Also, the q-analogue of the binomial function $(x\pm y)^n$ is given by [9,12]

$$(x\pm y)^n=(x\pm y)_n=x^n(\mp {}^{y}/_{x};q)=x^n\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}(\mp {}^{y}/_{x})^k. \quad (1.14)$$

Hermite Polynomials are defined by means of generating relations [10]

$$exp[2xt-t^2]=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}, \quad (1.15)$$

$$exp[2xt+yt^2]=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}. \quad (1.16)$$

Shrivastava [14] presented and studied the classical Hermite polynomials and its generalizations in the form:

$$exp[2x(t+h)-(y+1)(t+h)^2]=\sum_{n,m=0}^{\infty}H_{n,m}(x,y)\frac{t^nh^m}{n!m!}. \quad (1.17)$$

Jodar and Company [4] introduced the class of Hermite matrix polynomials $H_n(x,A)$ defined by

$$exp\left[xt\sqrt{2A}-t^2I\right]=\sum_{n=0}^{\infty}H_n(x,A)\frac{t^n}{n!}, \quad (1.18)$$

and

$$H_n(x,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k};\quad n\geq 0. \quad (1.19)$$

which appear a finite series solutions of second-order matrix differential equations $y''-xAy'+nAy=0$, for a matrix A in $C^{N\times N}$ whose eigenvalues are all in the right open half-plane.

In [13], Sayyed, Metwally and Batahan introduced a generalization of the Hermite matrix polynomials of the form

$$F(x,t)=exp\left[\lambda\left(xt\sqrt{2A}-t^2I\right)\right]=\sum_{n=0}^{\infty}H_{n,m}^{\lambda}(x,A)\frac{t^n}{n!}. \quad (1.20)$$

Also, Batahan [1] presented a study of the two-variable Hermite matrix polynomials defined by

$$F(x,y,t)=exp\left[xt\sqrt{2A}-yt^2I\right]=\sum_{n=0}^{\infty}H_n(x,y,A)\frac{t^n}{n!}, \quad (1.21)$$

where

$$H_n(x,y,A)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!(n-2k)!}\left(x\sqrt{2A}\right)^{n-2k}y^k, \quad (1.22)$$

Moreover, Kahmmash [5] introduced and studied the Hermite matrix polynomials of two variables defined by

$$exp\left[xt\sqrt{2A}-(y+1)t^2I\right]=\sum_{n=0}^{\infty}H_n(x,y,A)\frac{t^n}{n!},\qquad |t^n|<\infty \quad (1.23)$$

where $H_n(x,y,A)$ is defined by (1.22).

Pathan, Bin Saad and Alsarahi [8] studied on matrix polynomials associated with Hermite matrix polynomials defined by

$$exp\left[x(t+h)\sqrt{2A}-y(t+h)^2I\right]=\sum_{n,m=0}^{\infty}H_{n,m}(x,y;A)\frac{t^nh^m}{n!m!}. \quad (1.24)$$

where

$$H_{n,m}(x,y;A)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{r+s}\frac{n!\,m!\,(2r+2s)!\left(x\sqrt{2A}\right)^{n+m-2r-2s}y^{r+s}}{(r+s)!\,(n-2r)!\,(m-2s)!\,(2r)!\,(2s)!}, \quad (1.25)$$

The following double series transformations that we will occasionally use, are easy to prove

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n-2k), \quad (1.26)$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k). \quad (1.27)$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+2k), \quad (1.28)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k), \quad (1.29)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+mk), \quad (1.30)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}A(k,n-(m-1)k), \quad (1.31)$$

where m,n are a positive integer such that $(n>m)$.

2. The Generalized q-Analogue Hermite Matrix Polynomials of Two-Variable $H_{n,m}(x,y,a;A;q)$.

In this paper, we introduce the generalized q-analogue Hermite matrix polynomial of two variables by the following:

Let A be a matrix such that $A \in C^{N\times N}$ satisfying the condition $\mu \in \sigma(A)$ is not negative integer $\forall \mu$, where $\sigma(A)$ is the set of all eigenvalues of A.

$$H_{n,m}(x,y,a;A;q) = \sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)} \frac{q^{\frac{a}{4}(n+m-2r-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}} \times \left(x\sqrt{2A}\right)^{n+m-2r-2s} y^{r+s}. \quad (2.1)$$

$0<q<1,\ m,n=0,1,2,\ldots.$

Now, we get generating function of the generalized q-analogue Hermite matrix polynomials in the form of the following theorem:

Theorem 2.1. Let A be a positive stable matrix in $C^{N\times N}$ and $0<q<1$, $a\in Z^{+}$, then the following generating function for the generalized q-analogue Hermite matrix polynomials $H_{n,m}(x,y,a;A;q)$ holds true:

$$E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right).E_q\left((-1)^{a+1}q^{a/4}y(t+h)^2I;\frac{a}{2}\right) = \sum_{n,m=0}^{\infty} H_{n,m}(x,y,a;A;q)\,t^n h^m. \quad (2.2)$$

Proof. Let us denote the left hand side of (2.2) by W, then

$$W = E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right).E_q\left((-1)^{a+1}q^{a/4}y(t+h)^2I;\frac{a}{2}\right),$$

appling relation (1.12), we obtain

$$W = \sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}\left(x\sqrt{2A}\right)^n}{(q;q)_n}(t+h)^n \sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{a}{4}r}y^r}{(q;q)_r}(t+h)^{2r}, \quad (2.3)$$

which using relation (1.14), we find

$$W = \sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}\left(x\sqrt{2A}\right)^n}{(q;q)_n}\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q q^{\binom{m}{2}}t^{n-m}h^m \times \sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{a}{4}r}y^r}{(q;q)_r}\sum_{s=0}^{2r}\begin{bmatrix} 2r\\ s\end{bmatrix}_q q^{\binom{s}{2}}t^{2r-s}h^s, \quad (2.4)$$

thus, by using relation (1.4), we get

$$W = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{q^{\frac{a}{4}n^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^n}{(q;q)_{n-m}(q;q)_m}t^{n-m}h^m \times \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}r^2+\binom{2s}{2}}(q;q)_{2r}y^r}{(q;q)_r(q;q)_{2r-2s}(q;q)_{2s}}t^{2r-2s}h^{2s}, \quad (2.5)$$

which on using relation (1.29) and (1.26), gives

$$W = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m}t^n h^m \times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}(r+s)^2+\binom{2s}{2}}(q;q)_{2r+2s}y^{r+s}}{(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}}t^{2r}h^{2s}$$

$$= \sum_{n,m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)} \times \frac{q^{\frac{a}{4}(n-2r+m-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}\left(x\sqrt{2A}\right)^{n+m-2r-2s}y^{r+s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}} t^n h^m.$$

By using definition (2.1), we obtain the required relation (2.2).

Lemma 2.1. The polynomial $H_{n,m}(x,y,a;A;q)$ is a q-analogy of a new Hermite matrix polynomials and the modified Hermite matrix polynomials.

Proof. In (2.1), replacing x and y by $(1-q)x$ and $(1-q)y$ respectively, taking the limit as $q\to 1$ for both sides, we get

$$\lim_{q\to 1} H_{n,m}\left((1-q)x,(1-q)y,a;A;q\right) = \lim_{q\to 1}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)} \frac{q^{\frac{a}{4}(n+m-2r-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}} \times \left((1-q)x\sqrt{2A}\right)^{n+m-2r-2s}\left((1-q)y\right)^{r+s}$$

$$= \sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)}\frac{(2r+2s)!\left(x\sqrt{2A}\right)^{n+m-2r-2s}y^{r+s}}{(n-2r)!\,(m-2s)!\,(r+s)!\,(2r)!\,(2s)!} = H_{n,m}(x,y,a;A), \quad (2.6)$$

where $H_{n,m}(x,y,a;A)$ assumed to be a new Hermite matrix polynomials.

Putting $a=0$ in (2.6), we obtain the known result (1.24).

Also $a=m=0$ and replacing y by $y+1$ in (2.6), we obtain the result (1.22).

3. Recurrence relations

Theorem (3.1). The q-analogue generalized Hermite matrix polynomials of two-index and two-variable $H_{n,m}(x,y,a;A;q)$ satisfy the following relations:

$$\frac{\partial^s}{\partial x^s}H_{n,m}(x,y,a;A;q) = \frac{q^{s^2a/4}\left(\sqrt{2A}\right)^s}{(1-q)^{s-1}}\sum_{k=0}^{s}\frac{q^{\binom{k}{2}}(1-q^2)\ldots(1-q^s)}{(q;q)_{s-k}(q;q)_k}H_{n+k-s,m-k}\left(q^{sa/2}x,y,a;A;q\right), \quad (3.1)$$

and

$$\frac{\partial^s}{\partial y^s}H_{n,m}(x,y,a;A;q) = \frac{(-1)^{s(a+1)}q^{s^2a/4}}{(1-q)^{s-1}}\sum_{k=0}^{2s}\frac{q^{\binom{k}{2}}(1-q^2)\ldots(1-q^{2s})}{(q;q)_{2s-k}(q;q)_k}H_{n+k-2s,m-k}\left(x,q^{sa/2}y,a;A;q\right). \quad (3.2)$$

Proof. Differentiating (2.2) with respect to x yields

$$\sum_{n,m=0}^{\infty}\frac{\partial}{\partial x}H_{n,m}(x,y,a;A;q)\,t^n h^m$$

$= \frac{q^{a/4}\sqrt{2A}(t+h)}{1-q} E_q\left(q^{a/4+a/2} x\sqrt{2A}(t+h); \frac{a}{2}\right).E_q\left((-1)^{a+1} q^{a/4} y(t+h)^2 I; \frac{a}{2}\right),$

which on using relations (1.14), (1.4) and (1.1), gives

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} H_{n,m}(x,y,a;A;q)\, t^n h^m$$

$$= \frac{q^{a/4}\sqrt{2A}}{1-q} \sum_{n,m=0}^{\infty} \sum_{k=0}^{1} q^{\binom{k}{2}} \frac{(q;q)_1}{(q;q)_{1-k}(q;q)_k} H_{n,m} (q^{a/2}x,y,a;A;q)\, t^{n+1-k} h^{m+k}$$

$$= q^{a/4}\sqrt{2A} \sum_{n,m=0}^{\infty} \sum_{k=0}^{1} \frac{q^{\binom{k}{2}}}{(q;q)_{1-k}(q;q)_k} H_{n+k-1,m-k} (q^{a/2}x,y,a;A;q) t^n h^m$$

On comparing the coefficients of $t^n h^m$ on both sides of the above equation, we obtain

$$\frac{\partial}{\partial x} H_{n,m}(x,y,a;A;q) = q^{a/4}\sqrt{2A} \sum_{k=0}^{1} \frac{q^{\binom{k}{2}}}{(q;q)_{1-k}(q;q)_k} H_{n+k-1,m-k}(q^{a/2}x,y,a;A;q)$$

Thus

$$\frac{\partial^2}{\partial x^2} H_{n,m}(x,y,a;A;q) = \frac{q^{2^2 a/4}(\sqrt{2A})^2}{(1-q)} \sum_{k=0}^{2} \frac{q^{\binom{k}{2}}(1-q^2)}{(q;q)_{2-k}(q;q)_k} H_{n+k-2,m-k}\left(q^{2a/2}x,y,a;A;q\right)$$

Hence

$$\frac{\partial^s}{\partial x^s} H_{n,m}(x,y,a;A;q) = \frac{q^{s^2 a/4}(\sqrt{2A})^s}{(1-q)^{s-1}} \sum_{k=0}^{s} \frac{q^{\binom{k}{2}}(1-q^2)\ldots(1-q^s)}{(q;q)_{s-k}(q;q)_k} H_{n+k-s,m-k}(q^{sa/2}x,y,a;A;q)$$

which is the required relation (3.1).

Again, differentiating (2.2) with respect to y yields

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial y} H_{n,m}(x,y,a;A;q)\, t^n h^m = \frac{(-1)^{a+1} q^{a/4}(t+h)^2}{1-q} \times E_q\left(q^{a/4} x\sqrt{2A}(t+h); \frac{a}{2}\right).E_q\left((-1)^{a+1} q^{a/4+a/2} y(t+h)^2 I; \frac{a}{2}\right).$$

By using relations (1.14),(1.4) and (1.1), we find

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial y} H_{n,m}(x,y,a;A;q)\, t^n h^m = (-1)^{a+1} q^{a/4} \sum_{n,m=0}^{\infty} \sum_{k=0}^{2} \frac{q^{\binom{k}{2}}(1-q^2)}{(q;q)_{2-k}(q;q)_k} H_{n+k-2,m-k} (x,q^{a/2}y,a;A;q) t^n h^m$$

On comparing the coefficients of $t^n h^m$ on both sides of the above equation, we get

$$\frac{\partial}{\partial y} H_{n,m}(x,y,a;A;q) = (-1)^{a+1} q^{a/4} \sum_{k=0}^{2} \frac{q^{\binom{k}{2}}(1-q^2)}{(q;q)_{2-k}(q;q)_k} H_{n+k-2,m-k}(x,q^{a/2}y,a;A;q)$$

Thus

$$\frac{\partial^2}{\partial y^2} H_{n,m}(x,y,a;A;q) = \frac{(-1)^{2(a+1)} q^{2^2 a/4}}{(1-q)} \sum_{k=0}^{4} \frac{q^{\binom{k}{2}}(1-q^2)(1-q^3)(1-q^4)}{(q;q)_{4-k}(q;q)_k} H_{n+k-4,m-k}\left(x,q^{2a/2}y,a;A;q\right)$$

Hence

$$\frac{\partial^s}{\partial y^s} H_{n,m}(x,y,a;A;q) = \frac{(-1)^{s(a+1)} q^{s^2 a/4}}{(1-q)^{s-1}} \sum_{k=0}^{2s} \frac{q^{\binom{k}{2}}(1-q^2)\ldots(1-q^{2s})}{(q;q)_{2s-k}(q;q)_k} H_{n+k-2s,m-k}(x,q^{sa/2}y,a;A;q)$$

,
which is the required relation (3.2).

Theorem (3.2).

The polynomials sequence $H_{n,m}(x,y,a;A;q)$ satisfies the next recurrence relation

$$[n+1]H_{n+1,m}(x,y,a;A;q) = 2(-1)^{a+1} q^{a/4} y \sum_{k=0}^{1} \frac{q^{\binom{k}{2}+n-2r}}{(q;q)_{1-k}(q;q)_k} H_{n+k-1,m-k} (qx,q^{a/2}y,a;A;q) + \frac{q^{a/4} x\sqrt{2A}}{1-q} H_{n,m}(q^{a/2}x,y,a;A;q)\,, \tag{3.3}$$

and

$$[m+1]H_{n,m+1}(x,y,a;A;q) = 2(-1)^{a+1} q^{a/4} y \sum_{k=0}^{1} \frac{q^{\binom{k}{2}+m-2s}}{(q;q)_{1-k}(q;q)_k} H_{n+k-1,m-k} (qx,q^{a/2}y,a;A;q) + \frac{q^{a/4} x\sqrt{2A}}{1-q} H_{n,m}(q^{a/2}x,y,a;A;q).$$

(3.4)

Proof. Differentiating (2.2) with respect to t and using (1.13), we find

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial t} H_{n,m}(x,y,a;A;q)\, t^n h^m$$

$$= \frac{2(-1)^{a+1} q^{a/4} y(t+h)}{1-q} \times E_q\left(q^{a/4} x\sqrt{2A}(qt+h); \frac{a}{2}\right).E_q\left((-1)^{a+1} q^{a/4+a/2} y(t+h)^2 I; \frac{a}{2}\right) + \frac{q^{a/4} x\sqrt{2A}}{1-q} E_q\left(q^{a/4+a/2} x\sqrt{2A}(t+h); \frac{a}{2}\right) E_q\left((-1)^{a+1} q^{a/4} y(t+h)^2 I; \frac{a}{2}\right)$$

appling relations (1.14) and (1.4), we obtain

$$\sum_{n,m=0}^{\infty} [n+1]H_{n+1,m}(x,y,a;A;q)\, t^n h^m = 2(-1)^{a+1} q^{a/4} y \times \sum_{n,m=0}^{\infty} \sum_{k=0}^{1} \frac{q^{\binom{k}{2}+n-2r}}{(q;q)_{1-k}(q;q)_k} H_{n,m}\left(x,q^{a/2}y,a;A;q\right) t^{n+1-k} h^{m+k}$$

$$+ \frac{q^{a/4} x\sqrt{2A}}{1-q} \sum_{n,m=0}^{\infty} H_{n,m}(q^{a/2}x,y,a;A;q)\, t^n h^m$$

$$\sum_{n,m=0}^{\infty} [n+1]H_{n+1,m}(x,y,a;A;q)\, t^n h^m = 2(-1)^{a+1} q^{a/4} y$$

$$\times \sum_{n,m=0}^{\infty}\sum_{k=0}^{1}\frac{q^{\binom{k}{2}+n-2r}}{(q;q)_{1-k}(q;q)_k}H_{n+k-1,m-k}\left(x,q^{a/2}y,a;A;q\right)t^nh^m + \frac{q^{a/4}x\sqrt{2A}}{1-q}\sum_{n,m=0}^{\infty}H_{n,m}\left(q^{a/2}x,y,a;A;q\right)t^nh^m$$

Now, on comparing of coefficients of t^nh^m, we get

$$[n+1]H_{n+1,m}(x,y,a;A;q) = 2(-1)^{a+1}q^{a/4}y\sum_{k=0}^{1}\frac{q^{\binom{k}{2}+n-2r}}{(q;q)_{1-k}(q;q)_k}H_{n+k-1,m-k}\left(x,q^{a/2}y,a;A;q\right) + \frac{q^{a/4}x\sqrt{2A}}{1-q}H_{n,m}\left(q^{a/2}x,y,a;A;q\right).$$

Which the required relation (3.3).

In similar way, differentiating (2.2) with respect to h, we get the relation (3.4).

Theorem (3.3). For $H_{n,m}(x,y,a;A;q)$ the following relation holds true:

$$\frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m} = \sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)a}H_{n,m}(x,y,a;A;q) \times \frac{q^{\frac{a}{4}(n-2r+m-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}y^{r+s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}} \quad (3.5)$$

Proof. Using generating function of polynomials $H_{n,m}(x,y,a;A;q)$ and definition expression for function $E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$ and $E_q\left((-1)^aq^{a/4}y(t+h)^2I;\frac{a}{2}\right)$ we have

$$E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right) = E_q\left((-1)^aq^{a/4}y(t+h)^2I;\frac{a}{2}\right)\sum_{n,m=0}^{\infty}H_{n,m}(x,y,a;A;q)\,t^nh^m$$

Using relations (1.12) and (1.4), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{q^{\frac{a}{4}n^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^n}{(q;q)_{n-m}(q;q)_m}t^{n-m}h^m = \sum_{n,m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^{ra}H_{n,m}(x,y,a;A;q)\frac{q^{\frac{a}{4}r^2+\binom{2s}{2}}(q;q)_{2r}y^r}{(q;q)_r(q;q)_{2r-2s}(q;q)_{2s}}t^{n+2r-2s}h^{m+2s}$$

On using relations (1.29) and (1.25), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m}t^nh^m = \sum_{n,m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)a}H_{n,m}(x,y,a;A;q) \times \frac{q^{\frac{a}{4}(n-2r+m-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}y^{r+s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}}t^nh^m$$

Comparing of the coefficients of t^nh^m of the above equation, we obtain the required relation (3.5) .

4. Rodrigue's formula:

Theorem (3.4). Let $q\in(0,1)$, the q-analogue generalized Hermite matrix polynomials $H_{n,m}(qx,y,a;A;q)$ has the following representation:

$$H_{n,m}(qx,y,a;A;q) = \frac{(1-q)^2}{q^a}\frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m}E_q\left[(-1)^{a+1}q^{a/4}y\left(\sqrt{2A}\right)^{-2}\frac{\partial^2}{\partial x^2};\frac{a}{2}\right] \quad (3.6)$$

Proof. Since,

$$\frac{\partial}{\partial x}E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right) = \frac{q^{a/4}}{1-q}\sqrt{2A}(t+h)E_q\left(q^{3a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$

$$\frac{\partial^2}{\partial x^2}E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right) = \frac{q^a}{(1-q)^2}\left(\sqrt{2A}\right)^2(t+h)^2E_q\left(q^{5a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$

$$\therefore \frac{(1-q)^2}{q^a}\left[\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right]^2E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right) = (t+h)^2E_q\left(q^{5a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$

Thus,

$$\frac{(1-q)^2}{q^a}E_q\left[(-1)^{a+1}q^{a/4}y\left(\sqrt{2A}\right)^{-2}\frac{\partial^2}{\partial x^2};\frac{a}{2}\right]E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$
$$= \frac{(1-q)^2}{q^a}\sum_{n=0}^{\infty}(-1)^{n(a+1)}\frac{q^{\frac{a}{2}\binom{n}{2}+n\frac{a}{4}}y^n}{(q;q)_n}\left[\left(\sqrt{2A}\right)^{-1}\frac{\partial}{\partial x}\right]^{2n}E_q\left(q^{a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$
$$= \sum_{n=0}^{\infty}(-1)^{n(a+1)}\frac{q^{\frac{a}{2}\binom{n}{2}+n\frac{a}{4}}y^n}{(q;q)_n}(t+h)^{2n}E_q\left(q^{5a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$
$$= E_q\left((-1)^{(a+1)}q^{a/4}y(t+h)^2;\frac{a}{2}\right)E_q\left(q^{5a/4}x\sqrt{2A}(t+h);\frac{a}{2}\right)$$
$$= \sum_{n,m=0}^{\infty}H_{n,m}(qx,y,a;A;q)\,t^nh^m$$

Hence,

$$\sum_{n,m=0}^{\infty}H_{n,m}(qx,y,a;A;q)\,t^nh^m = \frac{(1-q)^2}{q^a}E_q\left[(-1)^{a+1}q^{a/4}y\left(\sqrt{2A}\right)^{-2}\frac{\partial^2}{\partial x^2};\frac{a}{2}\right]\sum_{n,m=0}^{\infty}\frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m}t^nh^m$$

By comparing the coefficients t^nh^m , we find $H_{n,m}(qx,y,a;A;q)$

$$= \frac{(1-q)^2}{q^a} \frac{q^{\frac{a}{4}(n+m)^2+\binom{m}{2}}\left(x\sqrt{2A}\right)^{n+m}}{(q;q)_n(q;q)_m}$$
$$E_q\left[(-1)^{a+1}q^{a/4}y\left(\sqrt{2A}\right)^{-2}\frac{\partial^2}{\partial x^2};\frac{a}{2}\right]$$

which the required relation (3.6).

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مصفوفة كثيرات حدود هرميت الأساسية المعممة ذات متغيرين

فضل صالح ناصر علي السرحي[1]

الملخص: في هذا البحث قدمنا مصفوفة كثيرات حدود لهرميت الأساسية –أي من النوع كيو– المعممة ذات دليلين ومتغيرين. كما اشتقفتا بعض العلاقات التكرارية لها .

الكلمات المفتاحية: مصفوفة كثيرات حدود هرميت الأساسية المعممة ذات متغيرين و الدوال المولدة والعلاقات التكرارية.