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On the Stability of First Order Ordinary Differential Equation with a Nonlocal Condition

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Abstract: In this paper we study the existence and uniqueness of solution for the first order differential equation, $\frac{dx}{dt} + f\big(t, x(t)\big) = 0, t \in [0, T]$ with the nonlocal condition $x(1) + I^{\gamma}\, x(t)|_{t=t_o} = x_o$., then we prove that the solution is uniformly stable.

Keywords: First order; Ordinary differential equation; Nonlocal condition; Stability.

1. Introduction

In the last decades many authors studied the nonlocal problems with different conditions (for example see [1, 2, 3, 4, 5, 7, 8, 9, 12, 13, 14] and the references therein).

In this work, we study the existence and uniqueness of a solution of the first order ordinary differential equation with the nonlocal condition

$$\begin{cases} \frac{dx}{dt} + f\big(t, x(t)\big) = 0, t \in [0, T] \\ x(1) + I^{\gamma}\, x(t)|_{t=t_o} = x_o, \quad \gamma \in (0,1] \end{cases} \qquad (1)$$

and then we study the stability of the solution.

Preliminaries:

First of all, we give some basic notations and definitions which will be used in this paper.

Let $C(I)$ denotes the class of continuous functions and $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [a, b]$, where $0 \leq a < b < \infty$ and let $\Gamma(.)$ denotes the gamma function.

Definition 1. [15] The fractional-order integral of the function $f \in L^1[a, b]$ of order $\beta \in R^+$ is defined by

$$I_a^{\beta} f(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds.$$

Definition 3. [10] The solution of problem P is uniform stable, if $\forall\, \varepsilon > 0, \exists\, \delta(\varepsilon) > 0$, such that

$$|x_o - \tilde{x}_o| < \delta(\varepsilon) \Rightarrow |x(t) - \tilde{x}(t)| < \varepsilon.$$

Where $\tilde{x}(t)$ is the solution of the problem $\tilde{P}$.

Theorem 1. (Arzela - Ascolis Theorem **)** [11] Let E be a compact metric space and $C(E)$ be the Banach space of real or comlex valyed continuous functions norms by

$$\|f\| = \sup_{t \in E} |f(t)|$$

If $A = \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equi-continuous mapping, then $\bar{A}$ is compact.

Theorem 2. (Lebesgue dominated convergence Theorem**)** [6] Let $\{f_n\}$ be a sequence functions converging to a limit f on A and suppose that

$$|f_n(t)| \leq \varphi(t), \qquad t \in A, n = 1, 2, \ldots$$

Where φ is integrable on A then f is integrable on A and

$$\lim_{n\to\infty} \int_A f_n(t)\, d\mu = \int_A f(t)\, d\mu.$$

2. Integral Representation:

In this section, we study integral representation of the solution of the nonlocal problem (1).

Lemma 1. The solution of the nonlocal problem (1) can be expressed by the integral equation as

$$x(t) = A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\, ds. \qquad (2)$$

$$; A = \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right).$$

Proof. Integrating equation $\frac{dx}{dt} + f(t,x(t)) = 0$, we get

$$x(t) = c - \int_0^t f(s,x(s))\, ds,$$

operating on both sides of the above equation by I^γ, we obtain

$$I^\gamma x(t) = \frac{c\, t^\gamma}{\Gamma(\gamma+1)} - I^{\gamma+1} f(t,x(t))$$

Also from the relation $x(1) + I^\gamma x(t)|_{t=t_o} = x_o$, we have

$$c - \int_0^1 f(s,x(s))ds + \frac{c\, t_o{}^\gamma}{\Gamma(\gamma+1)} - \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds = x_o,$$

then

$$c = \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right)(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds),$$

and

$$x(t) = A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\, ds.$$

where $A = \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right)$.

3. Existence of Solution:

In this section, we discuss the existence of the solution of the nonlocal problem (1).

Consider the problem (1) under the following assumptions:

(i) $f: [0,1] \times R \to R$ is measurable in $t \in [0,1]$ for every $x \in R$,

(ii)) $f: [0,1] \times R \to R$ is continuous in $x \in R$ for every $t \in [0,1]$,

(iii) there exists a function $m \in L^1[0,1]$ such that $|f| \le m$.

Theorem 3. Let the assumption (i)-(ii)-(iii) are satisfied then the nonlocal problem (1) has at least one continuous solution $x \in C[0,1]$.

Proof. Define a subset $Q_r \subset C[0,1]$ by $Q_r = \{x(t) > 0, for\ each\ t \in [0,1], ||u|| \le r\}$, $r = A\, x_o + 3\, ||m||_{L^1}$.

The set Q_r is nonempty, closed and convex.

Let $T: Q_r \to Q_r$ be an operator defined by

$$Tx(t) = A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\, ds.$$

For $x \in Q_r$, let $\{x_n(t)\}$ be a sequence in Q_r converges to $x(t)$, $x_n(t) \to x(t), \forall\, t \in [0,1]$, then

$$\lim_{n\to\infty} Tx_n(t) = Ax_o + A\lim_{n\to\infty}\int_0^1 f(s,x_n(s))ds + A\lim_{n\to\infty}\int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x_n(s))ds - \lim_{n\to\infty}\int_0^t f(s,x_n(s))\, ds$$

Since the assumption (i)-(ii)-(iii) are satisfied then by applying Lebesgue dominated convergence Theorem we get

$$\lim_{n\to\infty} T\, x_n(t) = (Tu)(t)$$

Then T is continuous.

Now, let $u \in Q_r$, then

$$|(Tx)(t)| \le |A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\, ds\,|$$

$$\le A\, x_o + A\int_0^1 |f(s,x(s))|\, ds + A\int_0^{t_o} \frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} |f(s,x(s))|ds + \int_0^t |f(s,x(s))|\, ds$$

$$\le A\, x_o + (A + \frac{A}{\Gamma(\gamma+1)} + 1)\int_0^1 |f(s,x(s))|\, ds$$

$$\le A\, x_o + 3\,||m||_{L^1} = r$$

Then $\{Tx(t)\}$ is uniformly bounded in Q_r.

In what follows we show that T is a completely continuous operator.

For $t_1, t_2 \in (0,1)$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$ we have

$$(Tx)(t_2)-(Tx)(t_1) = -\int_0^{t_2} f(s,x(s))ds + \int_0^{t_1} f(s,x(s))ds$$

$$|(Tx)(t_2)-(Tx)(t_1)| \le \int_{t_1}^{t_2} |f(s,x(s))|\ ds \le \int_{t_1}^{t_2} \mathrm{m}(s)\ ds$$

$$|(Tx)(t_2)-(Tx)(t_1)| \le \epsilon$$

Hence the class of functions $\{Tx(t)\}$ is equi-continuous. By Arzela - Ascolis Theorem $\{Tx(t)\}$ is relatively compact. Since all conditions of Schauder Theorem held, then T has a fixed point in Q_r.

Therefore the integral equation (2) has at least one positive continuous solution $x \in C(0,1)$.

Now,

$$\lim_{t\to 0^+} x(t) = A\lim_{t\to 0^+}\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \lim_{t\to 0^+}\int_0^t f(s,x(s))\,ds$$

$$= A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) = x(0),$$

and

$$\lim_{t\to 1^-} x(t) = A\lim_{t\to 1^-}\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \lim_{t\to 1^-}\int_0^t f(s,x(s))\,ds$$

$$= A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^1 f(s,x(s))\,ds = x(1),$$

Then the integral equation (2) has at least one continuous solution $x \in C[0,1]$.

To complete this proof, differentiating equation (2), we obtain the differential equation of problem (1).

Then, operating on both sides of equation (2) by I^γ, we obtain

$$I^\gamma x(t) = \frac{A\,x_o\,t^\gamma}{\Gamma(\gamma+1)} + \frac{A\ t^\gamma}{\Gamma(\gamma+1)}\int_0^1 f(s,x(s))ds + \frac{A\ t^\gamma}{\Gamma(\gamma+1)}\int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds - \int_0^t \frac{(t-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds$$

let $t = 1$ in equation (2) and $t = t_o$ in the above equation, we get

$$x(1) + I^\gamma x(t)|_{t=t_o} = Ax_o + A\int_0^1 f(s,x(s))ds + A\int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds - A\int_0^1 f(s,x(s))\,ds + \frac{A\,x_o\,t_o{}^\gamma}{\Gamma(\gamma+1)} + \frac{A\ t_o{}^\gamma}{\Gamma(\gamma+1)}\int_0^1 f(s,x(s))ds + \frac{A\ t_o{}^\gamma}{\Gamma(\gamma+1)}\int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds - \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds$$

$$= \left(\frac{t_o{}^\gamma}{\Gamma(\gamma+1)}+1\right)Ax_o + \left(A + \frac{A\,t_o{}^\gamma}{\Gamma(\gamma+1)} - 1\right)\int_0^1 f(s,x(s))ds + \left(A + \frac{A\,t_o{}^\gamma}{\Gamma(\gamma+1)} - 1\right)\int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds$$

$$= \left(\frac{t_o{}^\gamma}{\Gamma(\gamma+1)}+1\right)\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right)x_o + \left(\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right)(1+\frac{A\,t_o{}^\gamma}{\Gamma(\gamma+1)}) - 1\right)\int_0^1 f(s,x(s))ds + \left(\left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)+t_o{}^\gamma}\right)(1+\frac{A\,t_o{}^\gamma}{\Gamma(\gamma+1)}) - 1\right)\int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds = x_o$$

The proof is complete.

4. Uniqueness of the Solution:

For the uniqueness of the solution we have the following theorem:

Theorem 4. Assume that there exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \le k|x-y|\ , \forall\, t \in [0,1], \quad \forall x,y \in C[0,1]$$

If

$$k\left(A + \frac{A\,t_o{}^{\gamma+1}}{\Gamma(\gamma+2)} + 1\right) < 1, \quad (3)$$

then the problem (1) has a unique solution $x \in C[0,1]$.

Proof. Define the operator $H: C[0,1] \to C[0,1]$ by

$$Hx(t) = A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o}\frac{(t_o-s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\,ds. \quad (4)$$

Let $x, y \in C[0,1]$, then

$$Hx(t) - Hy(t) = A\int_0^1 f(s,x(s))ds + A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds - \int_0^t f(s,x(s))\, ds$$
$$-A\int_0^1 f(s,y(s))ds - A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} f(s,y(s))ds + \int_0^t f(s,y(s))\, ds$$
$$= A\int_0^1 (f(s,x(s)) - f(s,y(s)))\, ds + A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} (f(s,x(s)) - f(s,y(s)))\, ds - \int_0^t (f(s,x(s)) - f(s,y(s)))\, ds$$
$$|Hx(t) - Hy(t)| \le k\, A\int_0^1 |x(s) - y(s)|\, ds + k\, A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} |x(s) - y(s)|\, ds - k\int_0^t |x(s) - y(s)|\, ds$$
$$\le k\, A \sup_{t\in[0,1]} |x(t) - y(t)| \int_0^1 ds + k\, A \sup_{t\in[0,1]} |x(t) - y(t)| \int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)}\, ds - k \sup_{t\in[0,1]} |x(t) - y(t)| \int_0^t ds$$
$$\|Hx - Hy\| \le k\, A\|x-y\| + k\, A\|x-y\| \frac{{t_o}^{\gamma+1}}{\Gamma(\gamma+2)} + k\,\|x-y\|$$
$$\le k\left(A + \frac{A\, {t_o}^{\gamma+1}}{\Gamma(\gamma+2)} + 1\right)\|x-y\| = K\|x-y\|$$

but since $K = k\left(A + \frac{A\, {t_o}^{\gamma+1}}{\Gamma(\gamma+2)} + 1\right) < 1$, then we get

$$||Hx - Hy|| \le ||x - y||$$

Therefore the map $H: C[0,1] \to C[0,1]$ is contraction and then equation (2) has a unique fixed point $x \in C[0,1]$.

Stability:

Now we are ready to study the uniform stability of the solution of the nonlocal problem (1).

Theorem 5. Let the assumptions of Theorem 2 be satisfied, then the solution of the problem (2) is uniformly stable.

Proof. Let $x(t)$ be the solution of

$$x(t) = A\left(x_o + \int_0^1 f(s,x(s))ds + \int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds\right) - \int_0^t f(s,x(s))\, ds.$$

and let $\tilde{x}(t)$ be the solution of equation (2) such that $\tilde{x}(1) + I^\gamma\, \tilde{x}(t)|_{t=t_o} = \tilde{x}_o$.

Then

$$x(t) - \tilde{x}(t) = A\,(x_o - \tilde{x}_o) + A\int_0^1 f(s,x(s))ds - A\int_0^1 f(s,\tilde{x}(s))ds$$
$$+A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} f(s,x(s))ds - A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} f(s,\tilde{x}(s))ds - \int_0^t f(s,x(s))\, ds + \int_0^t f(s,\tilde{x}(s))\, ds$$
$$|x(t) - \tilde{x}(t)| = A\,|x_o - \tilde{x}_o| + A\int_0^1 |f(s,x(s)) - f(s,\tilde{x}(s))|\, ds + A\int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)} |f(s,x(s)) - f(s,\tilde{x}(s))|\, ds + \int_0^t |f(s,x(s)) - f(s,\tilde{x}(s))|\, ds$$
$$\le A\,|x_o - \tilde{x}_o| + A \sup_{t\in[0,1]} |x(t) - \tilde{x}(t)| \int_0^1 ds + A \sup_{t\in[0,1]} |x(t) - \tilde{x}(t)| \int_0^{t_o} \frac{(t_o - s)^\gamma}{\Gamma(\gamma+1)}\, ds + \sup_{t\in[0,1]} |x(t) - \tilde{x}(t)| \int_0^t ds$$

Therefore,

$$\|x - \tilde{x}\| \le A\,|x_o - \tilde{x}_o| + A\|x - \tilde{x}\| + A\|x - \tilde{x}\| \frac{{t_o}^{\gamma+1}}{\Gamma(\gamma+2)} + \|x - \tilde{x}\|$$

Therefore, $\forall \epsilon > 0, \exists\, \delta > 0,$ such that

$$|x_o - \tilde{x}_o| < \delta \Rightarrow \|x - \tilde{x}\| < \epsilon$$

which proves that the solution of problem (2) is uniformly stable.

5. Conclusion:

This paper studied a first order ordinary differential equation with a fractional-order integral condition by finding the existence, uniqueness and uniform stability of the solution.

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حول الاستقرار للمعادلة التفاضلية العادية من الدرجة الاولى بشرط غير محلي

ابتسام عمر بن طاهر

الملخص: في هذا البحث نحن ندرس وجود و وحدانية الحل للمعادلة التفاضلية من الدرجة الأولى $dx/dt + f(t,x(t))=0, t\in[0,T]$ مع الشرط غير المحلي $x_o = x(t)|_{t=t_o} + I^{\gamma} x(1)$ ثم نثبت الاستقرار المنتظم للحل. حيث I^{γ} هو التكامل الكسري من الرتبة. γ
والمسألة تحقق الشروط التالية : $f(t,x) \rightarrow t$ قابلة للقياس لكل $x\in R$, $x \rightarrow f(t,x)$ متصلة لكل $t\in [0,1]$, يوجد دالة $m\in L^1 [0,1]$ بحيث ان $|f|\leq m$

كلمات مفتاحية: معادلة تفاضلية عادية من الدرجة الاولى , شرط غير محلي , استقرار.