Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/050205

Discrete (Dynamic) Cumulative Residual Entropy in Bivariate case

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Received: 4 Apr. 2015, Revised: 30 Mar. 2016, Accepted: 2 Apr. 2016
Published online: 1 Jul. 2016

Abstract: Cumulative residual entropy (CRE) is a new measure of uncertainty for continuous distributions which has been introduced by Rao et al. [27] and its discrete version has been defined by Baratpour and Bami [4]. The present paper addresses the question of extending the definition of CRE and its dynamic version to bivariate setup in discrete case and study its properties. We show that the proposed measure is invariance under increasing one-to-one transformation and has additive property. Also, a lower bound for discrete bivariate CRE based on Shannon entropy is obtained. Further more, we introduce scalar and vector bivariate dynamic CRE and their connections with well-known reliability measures such as the discrete bivariate mean residual life time. Finally, the bivariate version of the hazard rate, mean residual life and cumulative residual entropy are obtained for bivariate geometric distribution.

Keywords: Entropy, Cumulative residual entropy, Bivariate hazard rate, Bivariate mean residual life, Bivariate cumulative residual life, Bivariate geometric distribution.

1 Introduction and Preliminaries

The Shannon entropy, which is the well known measure of uncertainty and associated with a discrete integer-valued life distribution, is introduced by Shannon [31]. This measure is given by,

$$H(T) = -\sum_{t=0}^{\infty} p(t)\log p(t),$$

where $p(t)$ is a probability mass function and the base of logarithm is e (i.e. natural logarithm) and also, $0\log 0 = 0$.

Rao et al. [27] defined an alternative measure of uncertainty called cumulative residual entropy (CRE). This measure is based on the cumulative distribution function F and is defined in the univariate case for non-negative continuous random variable X as,

$$\varepsilon(X) = -\int_0^{\infty} \overline{F}(x)\log\overline{F}(x)dx,$$

where $\overline{F}(x) = 1-F(x)$. They also provided some applications of it in reliability engineering and computer vision. Rao [26] developed some mathematical properties of CRE and gave an alternative formula for it. Many interesting properties of CRE are given in a recent paper by Di Crescenzo and Longobardi [7].

Recently, Asadi and Zohrevand [3] proposed a dynamic form of CRE for continuous distributions and obtain some of its properties. They also showed how CRE and dynamic forms of CRE (DCRE) are connected with well known reliability measures such as mean residual lifetime. They introduced dynamic CRE with the form,

$$\varepsilon(X,t) = -\int_t^{\infty} \frac{\overline{F}(x)}{\overline{F}(t)}\log\frac{\overline{F}(x)}{\overline{F}(t)}dx.$$

Nanda and Paul [17,18] defined some orderings and ageing properties in terms of the generalized residual entropy function. Navarro et al. [19] obtained some new results on these functions. They also defined and studied the dynamic

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cumulative past entropy function. For more discussion on the properties and generalization of (dynamic) CRE one may refer to Abbasnejad et al. [1], Kumar and Taneja [12], Navarro et al. [20], Sunoj and Linu [32], Khorashadizadeh et al. [11], Psarrakos and Navarro [22], Navarro et al. [21] and Chamany and Baratpour [5], among others.

In studying the reliability aspects of multi-component system with each component having a lifetime depending on the lifetimes of the next, multivariate life distributions are employed. Reliability characteristics in the univariate case extend to the corresponding multivariate version. Even though a lot of interest has been evoked on the entropy of residual life in the univariate case, only few works seem to have been done in higher dimensions.

The works of Ahmed and Gokhale [2], Zografos [33], Darbellay and Vajda [6], Rajesh and Nair [23], Nadarajah and Zografos [15], Ebrahimi, et al. [8], Rajesh et al. [24] and Sathar et al. [29,30] focus attention on extending information measures in higher dimensions.

Rejesh et al. [25] have proposed the bivariate dynamic CRE for absolutely continuous distributions, through the relationship,

$$\varepsilon(X;t_1,t_2)=-\int_{t_2}^{\infty}\int_{t_1}^{\infty}\frac{\overline{F}(x_1,x_2)}{\overline{F}(t_1,t_2)}\log\frac{\overline{F}(x_1,x_2)}{\overline{F}(t_1,t_2)}dx_1dx_2.$$

They have studied some properties of $\varepsilon(X;t_1,t_2)$ and looked into the problem of characterizing certain bivariate models using the functional form of the $\varepsilon(X;t_1,t_2)$. Further, they defined new classes of life distributions based on this measure. Also, Kundu and Kundu [13] have studied the bivariate dynamic cumulative past entropy.

As we know, it is sometimes impossible or inconvenient to measure the life length of a device, on a continuous scale. In practice, we come across situations, where lifetime of a device is considered to be a discrete random variable. Discrete failure data arise in several common situations such as below,

– Reports on field failures are collected weekly, monthly, and the observations are the number of failures, without specification of the failure times.
– A piece of equipment operates in cycles and the experimenter observes the number of cycles successfully completed prior to failure. For example the life length of many devices in industry such as switches and mechanical devices depend on the number of times that the devices are turned on or off.
– An experimenter often discretizes or groups continuous data.

So, studying the discrete life time random variables is one of the essential subject in reliability literature. Following this idea, the discrete version of the CRE (d-CRE) have been defined by Baratpour and Bami [4] for non-negative integer valued random variable T, as

$$d\varepsilon(T)=-\sum_{t=0}^{\infty}\overline{F}(t)\log\overline{F}(t). \tag{1}$$

They have studied some properties of the $d\varepsilon(T)$ and showed that it is connected with some well-known measures such as discrete mean residual lifetime.

Let $\mathbf{T}=(T_1,T_2)$ be bivariate discrete non-negative integer random variable with joint probability mass function $p(t_1,t_2)$, joint survival function $\overline{F}(t_1,t_2)$, marginal probability mass functions $p_i(t_i); i=1,2$ and marginal survival functions $\overline{F}_i(t_i)$, $i=1,2$.
Nair and Asha [16] defined the bivariate hazard rate function by

$$h(\mathbf{T})=h(\mathrm{T}_1,\mathrm{T}_2)=(h_1(\mathbf{T}),h_2(\mathbf{T})), \tag{2}$$

where

$$h_1(\mathbf{T})=h_1(t_1,t_2)=P(T_1=t_1|\mathbf{T}\geq\mathbf{t})=1-\frac{\overline{F}(t_1+1,t_2)}{\overline{F}(t_1,t_2)},$$

$$h_2(\mathbf{T})=h_2(t_1,t_2)=P(T_2=t_2|\mathbf{T}\geq\mathbf{t})=1-\frac{\overline{F}(t_1,t_2+1)}{\overline{F}(t_1,t_2)}.$$

They have derived various classes of increasing failure rate (IFR) and decreasing failure rate (DFR) models. The failure rate $h(\mathbf{T})$ determines the distribution of $\mathbf{T}$ uniquely through the formula

$$\overline{F}(t_1,t_2)=\prod_{r=0}^{t_1-1}[1-h_1(t_1-r-1,t_2)]\times\prod_{r=0}^{t_2-1}[1-h_2(0,t_2-r-1)].$$

The discrete bivariate mean residual life functions have been defined by Roy [28] as below,

$$m_1(t_1,t_2)=E(T_1-t_1|T_1\geq t_1,T_2\geq t_2), \tag{3}$$

$$m_2(t_1,t_2) = E(T_2 - t_2 | T_1 \geq t_1, T_2 \geq t_2).$$

Where we showed that the $m_i(t_1,t_2), i = 1,2$ can be expressed by survival function as follow,

$$m_1(t_1,t_2) = \frac{\sum_{i=t_1}^{\infty}\sum_{j=t_2}^{\infty}(i-t_1)p(i,j)}{\overline{F}(t_1-1,t_2-1)} = \frac{1}{\overline{F}(t_1-1,t_2-1)}\sum_{i=t_1}^{\infty}\overline{F}(i-1,t_2-1),$$

and similarly,

$$m_2(t_1,t_2) = \frac{1}{\overline{F}(t_1-1,t_2-1)}\sum_{j=t_2}^{\infty}\overline{F}(t_1-1,j-1).$$

In this paper we define the scalar and vector version of bivariate discrete dynamic cumulative residual entropy and show that it is invariant under increasing one-to-one transformations. Then, it is shown that the proposed measure has additive property. Also, its relations with Shannon entropy and some well-known reliability measures are obtained.

2 Main Results

In this section we look into the problem of extending (1) to the bivariate setup. A natural extension of (1) is given by the following definition.

Definition 1. Let $\mathbf{T} = (T_1,T_2)$ be a non-negative integer valued bivariate random vector with survival function $\overline{F}(t_1,t_2)$. We define the bivariate d-CRE through the relationship,

$$d\varepsilon(T_1,T_2) = -\sum_{t_2=0}^{\infty}\sum_{t_1=0}^{\infty}\overline{F}(t_1,t_2)\log\overline{F}(t_1,t_2). \tag{4}$$

In the next theorem we show that the bivariate d-CRE defined in (4) is invariant under non-singular increasing transformations which is different in continuous case, where Rajesh et al. [25] have established that the bivariate CRE in continuous lifetime distributions is not influenced by non-singular increasing transformations.

Theorem 1. If $K_j = \Phi_j(T_j), j = 1,2$, are increasing and one-to-one transformations, with corresponding survival function $\overline{G}(k_1,k_2)$ and bivariate d-CRE $d\varepsilon(K_1,K_2)$, then

$$d\varepsilon(K_1,K_2) = d\varepsilon(T_1,T_2).$$

Proof: Via transformation $K_j = \Phi_j(T_j), j = 1,2$ we have,

$$\begin{aligned}\overline{G}(k_1,k_2) &= P(T_1 > \Phi_1^{-1}(k_1), T_2 > \Phi_2^{-1}(k_2)) \\ &= \overline{F}(\Phi_1^{-1}(k_1),\Phi_2^{-1}(k_2)).\end{aligned}$$

Now, we have,

$$\begin{aligned}d\varepsilon(K_1,K_2) &= -\sum_{k_2=\Phi_2(0)}^{\Phi_2(\infty)}\sum_{k_1=\Phi_1(0)}^{\Phi_1(\infty)}\overline{G}(k_1,k_2)\log\overline{G}(k_1,k_2) \\ &= -\sum_{t_2=0}^{\infty}\sum_{t_1=0}^{\infty}\overline{F}(t_1,t_2)\log\overline{F}(t_1,t_2) \\ &= d\varepsilon(T_1,T_2),\end{aligned}$$

which shows that the d-CRE (4) is invariant under the increasing transformation.□

Ebrahimi et al. [8] has shown that bivariate residual entropy has the additive property. In the following theorem, we show that this property holds good for our proposed bivariate d-CRE also.

Theorem 2. If T_1 and T_2 are independent, then

$$d\varepsilon(T_1,T_2) = \mu_1 d\varepsilon(T_2) + \mu_2 d\varepsilon(T_1),$$

where $\mu_i = E(T_i), i = 1,2$.

Proof: Using definition of d-CRE, we have,

$$\begin{aligned} d\varepsilon(T_1,T_2) &= -\sum_{t_2=0}^{\infty}\sum_{t_1=0}^{\infty}\overline{F}(t_1,t_2)\log\overline{F}(t_1,t_2) \\ &= -\sum_{t_2=0}^{\infty}\sum_{t_1=0}^{\infty}\overline{F}_1(t_1)\overline{F}_2(t_2)\log[\overline{F}_1(t_1)\overline{F}_2(t_2)] \\ &= -\sum_{t_2=0}^{\infty}\overline{F}_2(t_2)\sum_{t_1=0}^{\infty}\overline{F}_1(t_1)\log\overline{F}_1(t_1) \\ &- \sum_{t_1=0}^{\infty}\overline{F}_1(t_1)\sum_{t_2=0}^{\infty}\overline{F}_2(t_2)\log\overline{F}_2(t_2) \\ &= \sum_{t_2=0}^{\infty}\overline{F}_2(t_2)d\varepsilon(T_1)+\sum_{t_1=0}^{\infty}\overline{F}_1(t_1)d\varepsilon(T_2) \\ &= E(T_1)d\varepsilon(T_2)+E(T_2)d\varepsilon(T_1), \end{aligned}$$

which complete the proof.□

In particular if T_1 and T_2 are independent and both are having the same expectation μ, the above relation becomes

$$d\varepsilon(T_1,T_2)=\mu[d\varepsilon(T_2)+d\varepsilon(T_1)].$$

Example 1. Among many bivariate geometric distributions there is a fundamental one defined by Hawkes [10] and Esary and Marshall [9], as follows. Suppose that T_1 and T_2 are bivariate random variables taking values in the set $\{0,1,2,\dots\}$. Then $\mathbf{T}=(T_1,T_2)$ is said to have a bivariate geometric distribution if its survival function be as follow,

$$\overline{F}(t_1,t_2)=p_1^{t_1}p_2^{t_2}\theta^{\max(t_1,t_2)};\quad 0<p_i<1, p_i\theta<1, 0<\theta<1, i=1,2. \tag{5}$$

This distribution has geometric marginal, and it is a discrete analogue of the bivariate exponential distribution introduced by Marshall and Olkin [14]. For this distribution we obtained the bivariate hazard rate function of form (2) as follow,

$$h_1(t_1,t_2)=\begin{cases}1-p_1\theta, & t_1\geq t_2\\ 1-p_1, & t_1<t_2\end{cases} \tag{6}$$

$$h_2(t_1,t_2)=\begin{cases}1-p_2\theta, & t_2\geq t_1\\ 1-p_2, & t_2<t_1\end{cases} \tag{7}$$

We see that despite the univariate, in bivariate case the hazard rate function of bivariate geometric distribution of form (5) is not constant.

Also, the bivariate mean residual life of form (3) in this distribution is given by,

$$m_1(t_1,t_2)=\frac{{p_1}^{1-t_1}{p_2}^{-t_2}\theta^{1-\max(t_1,t_2)}}{1-p_1\theta},$$

$$m_2(t_1,t_2)=\frac{{p_1}^{-t_1}{p_2}^{1-t_2}\theta^{1-\max(t_1,t_2)}}{1-p_2\theta}.$$

Furthermore, we obtain the bivariate d-CRE as follows,

$$d\varepsilon(T_1,T_2)=\ln(p_1)A_1+\ln(p_2)A_2+\ln(\theta)A_3,$$

where,

$$A_1=\frac{2\,\theta^3p_2{p_1}^2-\theta^2p_1-3\,\theta p_1-\theta^2p_2p_1+2\,p_2^{-1}-(\theta p_1p_2)^{-1}+2}{(\theta\,p_1-1)^2\,(\theta\,p_2-1)\,(\theta\,p_2p_1-1)^2},$$

$$A_2=-\frac{1-2\,\theta\,p_2p_1}{(\theta\,p_1-1)\,\theta\,p_2p_1\,(\theta\,p_2p_1-1)^2}-\sum_{t_2=0}^{\infty}(t_2\theta\,p_2-1)^{t_2-1},$$

$$\begin{aligned} A_3 &= -\frac{-1-2\,\theta^5{p_2}^3{p_1}^3+{p_1}^2{p_2}^2\,(1+p_2+p_1)\,\theta^4+4\,\theta^3{p_2}^2{p_1}^2}{p_1p_2\theta\,(\theta\,p_1-1)^2\,(\theta\,p_2-1)^2\,(\theta\,p_2p_1-1)^2} \\ &+ \frac{-3\,p_1\,(p_2+p_1+4/3)\,p_2\theta^2+((2\,p_2+2)\,p_1+2\,p_2)\,\theta}{p_1p_2\theta\,(\theta\,p_1-1)^2\,(\theta\,p_2-1)^2\,(\theta\,p_2p_1-1)^2}, \end{aligned}$$

which is not a simple form.

Next theorem describes the connection between marginal Shannon entropies and $d\varepsilon(T_1,T_2)$ and also gives a lower bound for discrete bivariate CRE.

Theorem 3. For a nonnegative integer valued bivariate random vector $\mathbf{T}=(T_1,T_2)$,

$$d\varepsilon(T_1,T_2)\geq \max\{K_1e^{H(T_1)},K_2e^{H(T_2)}\},$$

where

$$K_1=\exp\left\{E_1\left(\sum_{t_2=0}^{\infty}\overline{F}(T_1,t_2)|\log\overline{F}(T_1,t_2)|\right)\right\},$$

and

$$K_2=\exp\left\{E_2\left(\sum_{t_1=0}^{\infty}\overline{F}(t_1,T_2)|\log\overline{F}(t_1,T_2)|\right)\right\}.$$

Proof: Using log-sum inequality we have:

$$\sum_{t_1=0}^{\infty}\left(p_1(t_1)\log\frac{p_1(t_1)}{\sum_{t_2=0}^{\infty}\overline{F}(t_1,t_2)|\log\overline{F}(t_1,t_2)|}\right)$$
$$\geq\left(\sum_{t_1=0}^{\infty}p_1(t_1)\right)\log\frac{\sum_{t_1=0}^{\infty}p_1(t_1)}{\sum_{t_2=0}^{\infty}\sum_{t_1=0}^{\infty}\overline{F}(t_1,t_2)|\log\overline{F}(t_1,t_2)|}$$
$$=\log\frac{1}{d\varepsilon(T_1,T_2)}.$$

The left hand side is equal to

$$-H(T_1)-E_1\left(\sum_{t_2=0}^{\infty}\overline{F}(T_1,t_2)|\log\overline{F}(T_1,t_2)|\right),$$

which gives

$$d\varepsilon(T_1,T_2)\geq K_1e^{H(T_1)}.$$

Similarly we have $d\varepsilon(T_1,T_2)\geq K_2e^{H(T_2)}$, and the prove is completed.

3 Bivariate dynamic cumulative residual entropy

In the same way of other extentions, we can define the discrete dynamic CRE of form (1) as,

$$d\varepsilon(T,t)=-\sum_{i=t}^{\infty}\frac{\overline{F}(i)}{\overline{F}(t-1)}\log\frac{\overline{F}(i)}{\overline{F}(t-1)},\tag{8}$$

where $d\varepsilon(T,0)=d\varepsilon(T)$. In terms of discrete mean residual life $m(t)=E(T-t|T\geq t)=\frac{\sum_{i=t}^{\infty}\overline{F}(i)}{\overline{F}(t-1)}$ we can write (8) as,

$$d\varepsilon(T,t)=m(t)\log\overline{F}(t-1)-\frac{1}{\overline{F}(t-1)}\sum_{i=t}^{\infty}\overline{F}(i)\log\overline{F}(i).$$

If $\mathbf{T}=(T_1,T_2)$ represents the lifetimes of two components in a system where both the components survived up to times t_1 and t_2, respectively, then, the measure of uncertainty associated with the residual lifetimes of the system, called bivariate dynamic CRE, is given by

$$d\varepsilon(t_1,t_2)=-\sum_{i=t_1}^{\infty}\sum_{j=t_2}^{\infty}\frac{\overline{F}(i,j)}{\overline{F}(t_1-1,t_2-1)}\log\frac{\overline{F}(i,j)}{\overline{F}(t_1-1,t_2-1)}.\tag{9}$$

If T_1 and T_2 are independent then we have:

$$d\varepsilon(t_1,t_2)=m_1(t_1)d\varepsilon(T_2,t_2)+m_2(t_2)d\varepsilon(T_1,t_1),\tag{10}$$

where $m_i(t_i) = \frac{1}{\overline{F}_i(t_i-1)} \sum_{j=t_i}^{\infty} \overline{F}_i(j)$ are the marginal discrete mean residual life of the components $T_i; i = 1,2$.

In particular if T_1 and T_2 are independent and both are having the same discrete mean residual life $m(t)$, the above relation becomes

$$d\varepsilon(t_1,t_2) = m(t)[d\varepsilon(T_2,t_2) + d\varepsilon(T_1,t_1)].$$

Corollary 1. Similar to Theorem 1. it is easy to show that $d\varepsilon(t_1,t_2)$ is invariant under non-singular increasing transformations.

4 Conditional discrete dynamic CRE

Now, we investigate the behavior of the dCRE for conditional distributions. Consider the random variables $K_j = [T_j | T_1 \geq t_1, T_2 \geq t_2]$. The distribution of K_j corresponds the conditional distribution of T_j provided that T_i has survived up to time t_i, $i = 1, 2$, and has reliability function $\frac{\overline{F}(k_1,t_2)}{\overline{F}(t_1-1,t_2-1)}$ for $k_1 \geq t_1$ and $\frac{\overline{F}(t_1,k_2)}{\overline{F}(t_1-1,t_2-1)}$ for $k_2 \geq t_2$. The discrete dynamic CRE for the random variables K_j, $j = 1, 2$ are equal to

$$d\varepsilon_1(t_1,t_2) = -\sum_{i=t_1}^{\infty} \frac{\overline{F}(i,t_2)}{\overline{F}(t_1-1,t_2-1)} \log \frac{\overline{F}(i,t_2)}{\overline{F}(t_1-1,t_2-1)}, \tag{11}$$

and

$$d\varepsilon_2(t_1,t_2) = -\sum_{j=t_2}^{\infty} \frac{\overline{F}(t_1,j)}{\overline{F}(t_1-1,t_2-1)} \log \frac{\overline{F}(t_1,j)}{\overline{F}(t_1-1,t_2-1)}. \tag{12}$$

Analogous to definitions of bivariate discrete hazard rate and bivariate discrete mean residual life as a two component vector, we give another definition of vector discrete bivariate dynamic CRE.

Definition 2. For a non-negative integer valued bivariate random vector $\mathbf{T} = (T_1, T_2)$, vector bivariate dynamic CRE is defined as

$$\mathbf{d}\varepsilon(t_1,t_2) = (d\varepsilon_1(t_1,t_2), d\varepsilon_2(t_1,t_2)), \tag{13}$$

where $d\varepsilon_1(t_1,t_2)$ and $d\varepsilon_2(t_1,t_2)$ are given by (11) and (12), respectively.

$d\varepsilon_1(t_1,t_2)$ measures expected uncertainty contained in random variable T_1 about the predictability of the residual lifetime of the component, after t_1, subject to the revision that T_2 has survived up to time t_2. This interpretation can be said for $d\varepsilon_2(t_1,t_2)$, similarly.

Corollary 2. If T_1 and T_2 are independent, we have $K_j = [T_j | T_1 \geq t_1, T_2 \geq t_2] = [T_j | T_j \geq t_j; j = 1,2]$, then $d\varepsilon_i(t_1,t_2) = d\varepsilon(T_i,t_i); i = 1,2$.

Also, $d\varepsilon_1(t_1,t_2)$ and $d\varepsilon_2(t_1,t_2)$ (given by Eq's (11) and (12)) can be written in terms of discrete bivariate mean residual life (3) as,

$$d\varepsilon_1(t_1,t_2) = m_1(t_1,t_2) \log \overline{F}(t_1-1,t_2-1) - \sum_{i=t_1}^{\infty} \frac{\overline{F}(i,t_2)}{\overline{F}(t_1-1,t_2-1)} \log \overline{F}(i,t_2),$$

and

$$d\varepsilon_2(t_1,t_2) = m_2(t_1,t_2) \log \overline{F}(t_1-1,t_2-1) - \sum_{j=t_2}^{\infty} \frac{\overline{F}(t_1,j)}{\overline{F}(t_1-1,t_2-1)} \log \overline{F}(t_1,j).$$

5 Conclusion

According to importance of the discrete random variables and also the bivariate set up in reliability studies, we proposed a discrete version of (dynamic) cumulative residual entropy and some of its properties are discovered which some of them are different form continuous case. Also a lower bound for bivariate CRE and its connections with other reliability measures has obtained. Finally, an example of bivariate geometric distribution is presented.

Acknowledgement

The authors would like to express their thanks to referees and an associate editor for important and constructive comments, which improved the paper.

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