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# Global Error Minimization method for nonlinear oscillators with highly nonlinearity

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**Abstract:** In this paper, we used Global Error Minimization (GEM)method for nonlinear oscillators. This method convert the nonlinear oscillators into an equivalent optimization problem to obtain an analytical solution of the problem. Approximate solution obtained by GEM method is compared with the solution of He's variational approach. We observe from the results that this method is very simple, easy to apply, and gives a very good accuracy by using first-order approximation and simplest trial functions. Comparison made with other known results show that new method provides a mathematical tool to the determination of limit cycles of more complex nonlinear oscillators. This method is applied on nonlinear differential equations. It has demonstrated the accuracy and efficiency of this method by solving some example. Example is given to illustrate the effectiveness and convenience of the method.

**Keywords:** Nonlinear oscillator; Global Erroe Minimization method, Analytical approximate solutions .

## 1 Introduction

There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [6]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt-Poincare perturbation method, where by the solution is analytically expanded in the power series of a small parameter [2]. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt-Poincare techniques [3,4,5], the homotopy perturbation method [6,7,8,9,10,11,12] or linear delta expansion [13,14,15] are only some examples of them. A recent detailed review of asymptotic methods for strongly nonlinear oscillators can be found in [1,35,36,37,38]. The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [2,16,17,18,19,20,21,22,23,24]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required.

In this paper, we used new variational approach proposed by He [25] to develop a method called GEM (Global Error Minimization) method. In this method, the nonlinear oscillator is converted to an equivalent minimization problem. We combine the general idea of global error minimization in the AVK method [26] and He's variational approach [25] for solving the nonlinear ODE's. The idea of error minimization is a natural process. Therefore, we believe that GEM method provides a natural way to obtain a solution.
Suppose nonlinear oscillator

$$u'' + F\left(u', u, t\right) = 0 \tag{1}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

## 2 Preliminaries

**Definition 1.***Consider the nonlinear system* $(1)$*; we define the following functional for the oscillator equation, called the global error functional [26]. Suppose*

$$E\left(u', u, t\right) = \int_0^T \left|\left|u'' + F\left(u', u, t\right)\right|\right|^2 dt,\; T = \frac{2\pi}{\omega} \tag{2}$$

*$\omega$ is the primary natural frequency and E is a continuous functional.*

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**Definition 2.** *We convert the nonlinear ODE in equation* (1) *and* (2) *to the following minimization problem:*
*Minimize*

$$E\left(u',u,t\right)$$

*such that*

$$u(0)=A, u'(0)=0. \tag{3}$$

**Lemma 1.** *If h is a nonlinear continuous function on* $[0,T]$ *and non-negative* $(h\geq 0)$, *then the necessary and sufficient condition for* $\int_0^T h\,du=0$ *is* $h\equiv 0$ *on* $[0,T]$ *[26].*

*Proof.* See [26]. □

**Theorem 1.** *The necessary and sufficient condition for u to be a solution of the nonlinear ODE [1] with initial condition* $u(0)$ *and* $u'(0)=0$ *is* $E\left(u',u,t\right)=0$ *the minimization problem* 3.

*Proof.* See methods[26]. □

## 3 Outline of the procedure

The solution of equation (1) can be expressed in the form of Fourier series [27]:

$$u=a_0+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right). \tag{4}$$

Here $a_0$, $a_n$ and $b_n$ are constants. These unknown constants could not be determined for the case of infinite Fourier series. However, we can approximate equation (4) by a finite series [28,29]:

$$\widetilde{u}=a_0+\sum_{n=1}^{m}\left(a_n\cos n\omega t+b_n\sin n\omega t\right). \tag{5}$$

Various methods have been developed for determining the unknown constants used in equation (5), [1,28,30,31,32]. In this paper, a natural and efficient method will be developed for determining these unknowns.

The nonlinear problem [1] is first converted to the minimization problem with the unknown constants of equation (5). Consider the case where $E\left(u',u,t\right)=0$; then, with respect to Theorem 1, $\widetilde{u}$ happens to be the exact solution. Generally such a case will not arise for nonlinear problems. However, if $E\left(u',u,t\right)\cong 0$, we find an excellent analytical approximated of the original nonlinear (1). It is worth noting that we know the desired answer of our minimization problem in advance, which is zero. Therefore, we have a valuable measure for comparing the accuracy of the approximated solutions. Note that $E\left(u',u,t\right)$ is the global error and any reduction in this functional, by choosing a better trial solution, would greatly improve the approximation of the analytical solution.

## 4 Applications

In order to assess the advantages and the accuracy of new method, we will execute our examples, we use Maple package 11.

*Example 1.* Now we apply GEM [34] to the following nonlinear oscillator:

$$u''+u+au^3+bu^5+cu^7=0, \tag{6}$$

with initial conditions given by (3) .

We begin the procedure with the simplest trial solution:

$$\widetilde{u}_1\left(t\right)=b\cos\omega t. \tag{7}$$

Next, we convert equation (6) to the minimization problem:
Minimize

$$E\left(u',u,t\right)=\int_0^T\left|\left|\widetilde{u}_1''+\widetilde{u}_1+a\widetilde{u}_1^3+b\widetilde{u}_1^5+c\widetilde{u}_1^7\right|\right|^2dt,\ T=\frac{2\pi}{\omega},$$

such that

$$\widetilde{u}_1\left(0\right)=A,\ \widetilde{u}_1'\left(0\right)=0. \tag{8}$$

The constraints of the minimization problem are readily satisfied by choosing $b=A$. Therefore, by replacing $\widetilde{u}_1(t)=A\cos\omega t$ in (8) and performing the integration, we obtain:
Minimize

$$E\left(u_1',u_1,t\right)=\frac{A^2\pi\left(q_1+q_2\right)}{1024\omega}, \tag{9}$$

where

$$q_1=1024\left(1+\omega^4\right)-2048\omega^2+429c^2A^{12} \tag{10}$$
$$+1280bA^4+1120cA^6+1536aA^2 \tag{11}$$
$$q_2=504b^2A^8+640a^2A^4-1536a\omega^2A^2+1120abA^6 \tag{12}$$
$$+1008acA^8-1280b\omega^2A^4+924bcA^{10}-1120c\omega^2A^6 \tag{13}$$

The solution of equation (9) could be found through the condition,
Minimize

$$\frac{\partial E\left(\widetilde{u}_1',\widetilde{u}_1,t\right)}{\partial\omega}=0.$$

$$\omega=\frac{\sqrt{192+144aA^2+120bA^4+105cA^6-3Q}}{24}, \tag{14}$$

where $Q=\sqrt{Q_1+Q_2}$ with

$$Q_1=9984a^2A^2+17280abA^6+15456acA^8 \tag{15}$$
$$+24576aA^2+7648b^2A^8 \tag{16}$$
$$Q_2=13888bcA^{10}+20480bA^4+6373c^2A^{12} \tag{17}$$
$$+17920cA^6+16384. \tag{18}$$

The comparison of the approximate solution with $\omega$ is given in (10) and exact solution [33].

**Fig. 1:** a = 10, b = c = 1, A = 0.01

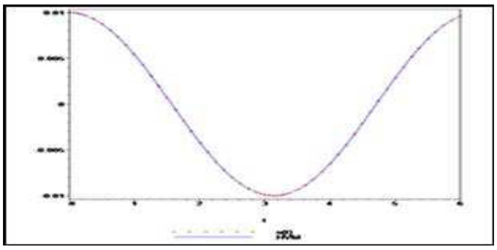

**Fig. 2:** a = 10, b = c = 1, A = 1

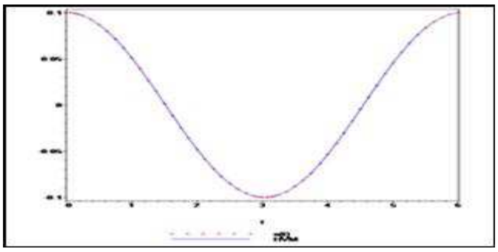

**Fig. 3:** a = 10, b = c = 1, A = 0.1

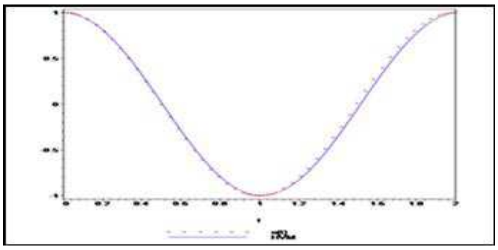

**Fig. 4:** a =0.1, b = c = 1, A = 1

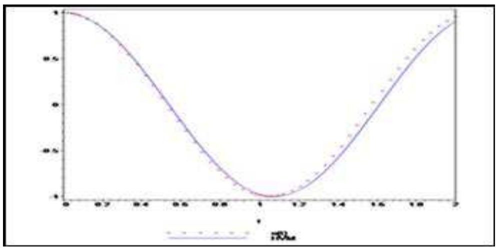

Fig 1,2,3,4 are comparison of the approximate solution $u = A\cos\omega t$, where $\omega$ is defined by equation (10). Dashed line approximate solution; continuous line; He's variational method (HVM, [33]).

For the case $b = c = 0$, equation (6) turns to be the well-known Duffing equation, and its frequency-amplitude relationship obtained by the homotopy perturbation method, the variational iteration method [25], is $\omega = \sqrt{1+\frac{3}{4}aA^2}$.

## 5 Conclusion

In this research, we have used Global Error Minimization (GEM) method for nonlinear oscillators. This method converts the nonlinear oscillators into an equivalent optimization problem. Approximate solution obtained by GEM method is compared with the solution of He's variationl approach [33]. All solutions are almost identical see fig 1,2,3,4. We observe from the results that this method is very simple, easy to apply. This method gives a very good accuracy by using first-order approximation and simplest trial functions. Comparison made with other known results show that new (GEM) method provides a simple way of determination the limit cycles of more complex nonlinear oscillators.

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