

Robustness Study for the Parameter of Power Function Distribution

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Abstract: Sequential testing procedure is developed for shape parameter of power function distribution. Sequential Probability Ratio Test (SPRT) is developed for testing the hypothesis regarding the (shape) parameter of power function distribution. The robustness of SPRT is studied. The expressions for the operating characteristic (OC) and average sample number (ASN) functions are derived and the results presented through tables and graphs.

Keywords: Power Function Distribution · Robustness · OC and ASN functions · SPRT.

1 Introduction

Power function distribution is mostly used in depiction of population data where sample size is predictable. This distribution is also counted as a life testing model to fit the failure data. Power function distribution is a specific case of Pareto distribution. Meniconi and Barry (1995)[5] discussed the application of Power function distribution. They proved that the Power function distribution is the best distribution to check the reliability of any electrical component. In reliability/survival testing this distribution measures hazard rate in specified time interval.

The pioneering work of sequential test is proposed by Wald (1947)[12], who developed Sequential Probability Ratio Test (SPRT) for testing a simple null hypothesis against a simple alternative hypothesis. As a measure of the performance of SPRT, Wald obtained the expression for the Operating Characteristic (OC) and Average Sample Number (ASN) functions. The robustness of the SPRT, dealing with the testing of the parameters under consider has undergone a change has been studied by various authors, while dealing with different probabilistic models useful in reliability/survival analysis. Epstein and Sobel (1955)[2] applied the SPRT for testing the simple null hypothesis against simple alternative hypothesis regarding the parameter of exponential distribution. Harter and Moore (1976)[3] conducted Monte Carlo study to investigate the robustness of SPRT for exponential distribution. Montagne and Singpurwalla (1985)[6] generalized the results of Harter and Moore (1976)[3] from Weibull to a class of distribution having an increasing failure rate. They obtained inequalities for OC and ASN functions in order to demonstrate the robustness. For sequential tests and their robustness for the parameters of some families of continuous distributions, one may refer to Phatarfod (1971)[9], Chaturvedi, Kumar and Surinder (2000)[1], Surinder and Naresh (2009)[10] Parameshwar, Nagaraj and Gudaganavar (2010)[8] and Surinder and Mukesh (2013)[11].

In this paper, we have developed the Sequential Testing Procedure for the parameter of power function distribution. In section 2, we describe the basic introduction of power function distribution. In section 3, we develop the SPRT and obtained the expressions for the OC and ASN functions for parameter. In section 4, we have also studied the robustness of the SPRT when the parameter under consideration has undergone a change. Finally, we illustrate implementation of SPRT for power function distribution, in Section 5.

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2 Set Up of the Problem

Let the random variable (rv) X follows the Power function distribution presented by the probability density function (pdf)

$$f(x) = \frac{C(x-a)^{(C-1)}}{(b-a)^C}; \quad a < x \leq b \quad (1)$$

where C is the shape parameter $C > 0$ and a, b are the boundary points. Taking $a = 0$ the pdf converted in to

$$f(x) = \frac{Cx^{(C-1)}}{b^C}; \quad 0 < x \leq b \quad (2)$$

Given a sequence of observation $X_1, X_2, X_3, \dots, X_n$ from (2.2), suppose one wish to test the simple null hypothesis $H_0 = C = C_0$ against simple alternative hypothesis $H_1 = C = C_1$. The expression for the OC and ASN functions are obtained and their behaviour is studied through graphs and tables.

3 Testing the Hypothesis for C

The SPRT for testing simple null hypothesis against simple alternative hypothesis, corresponding ratio is

$$Z_i = \log\left[\frac{f(x_i, C_1)}{f(x_i, C_0)}\right] = \log\left(\frac{C_1}{C_0}\right) + (C_1 - C_0)\log x_i + (C_0 - C_1)\log b \quad (3)$$

Now we choose two numbers A and B such that $0 < B < 1 < A$. At the n^{th} stage accept H_0 if $\sum Z_i \leq (\log B)$, reject H_0 if $\sum Z_i \geq (\log A)$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. $\alpha \varepsilon(0, 1)$ and $\beta \varepsilon(0, 1)$ are *Type Ist* and *Type IInd* errors respectively, where $A = \left(\frac{1-\beta}{\alpha}\right)$ and $B = \left(\frac{\beta}{1-\alpha}\right)$.

The OC function of SPRT is given by

$$L(C) = \frac{(A^h - 1)}{(A^h - B^h)} \quad (4)$$

where h is the non zero solution of the given equation $E(e^{Z_i}) = 1$
or

$$\int_0^b \left[\frac{f(x_i, C_1)}{f(x_i, C_0)}\right]^h f(x_i, C) dx_i = 1 \quad (5)$$

After solving we get,

$$h = \frac{C\left[\left(\frac{C_1}{C_0}\right)^h - 1\right]}{(C_1 - C_0)}$$

On substituting $d = (C_1 - C_0), a = \left(\frac{C_1}{C_0}\right)$ we get

$$Ca^h = hd + C \quad (6)$$

Taking logarithm in both sides in (3.4),

$$h \log(a) + \log(C) = \log(hd + C)$$

$$h \log(a) = \log\left(1 + \frac{hd}{C}\right)$$

$$h \log a = \frac{hd}{C} - \frac{h^2 d^2}{2C^2} + \frac{h^3 d^3}{3C^3}$$

Taking approximation upto 3^{rd} degree of h , we get quadratic equation

$$2d^3 h^2 - 3Cd^2 h + 6C^2 d - 6C^3 \log a = 0 \quad (7)$$

The roots are

$$h = \frac{3Cd^2 \pm \sqrt{48C^3d^3 \log a + 39C^2d^4}}{4d^3}$$

ASN for power function distribution

$$E(N/C) = \frac{L(C)\log B + [1 - L(C)]\log A}{E(Z_i)} \tag{8}$$

Now calculate $E(Z_i)$ from (3.1), where $E(Z_i) \neq 0$,

$$\begin{aligned} E(Z_i) &= \log\left(\frac{C_1}{C_0}\right) + (C_1 - C_0) \int_0^b \log x_i f(x) dx_i + (C_0 - C_1) \log b \\ &= \log\left(\frac{C_1}{C_0}\right) + (C_1 - C_0) \int_0^b \log x_i \frac{C x_i^{(C-1)}}{b^C} dx_i + (C_0 - C_1) \log b \\ E(Z_i) &= \log\left(\frac{C_1}{C_0}\right) - \frac{(C_1 - C_0)}{C} \end{aligned} \tag{9}$$

4 Robustness of SPRT for the Scale Parameter of Power Function Distribution

Let us suppose that the parameter C has undergone a change then the pdf in (2.2) become $f(x_i, C, b_1)$. To study the robustness of SPRT developed in section 4 with respect to OC function, consider h as the solution of the equation

$$\int_0^{b_1} \left[\frac{f(x_i, C_1, b)}{f(x_i, C_0, b)} \right]^h f(x_i, C, b_1) dx_i = 1 \tag{10}$$

After solving we get

$$\left(\frac{b_1}{b}\right)^{h(C_1 - C_0)} C \left(\frac{C_1}{C_0}\right)^h = h(C_1 - C_0) + C$$

Let $\phi = \left(\frac{b_1}{b}\right), a = \left(\frac{C_1}{C_0}\right), d = (C_1 - C_0)$

The equation is written as

$$C(a^h)(\phi^{hd}) = hd + C$$

$$(a^h)(\phi^{hd}) = \left[\frac{hd}{C} + 1\right] \tag{11}$$

Taking logarithm in both sides in (4.2) and we get results

$$\frac{h^2 d^3}{3C^3} - \frac{hd^2}{2C^2} + \frac{d}{C} - \log(a) - d \log(\phi) = 0 \tag{12}$$

ASN function for SPRT can be obtained as follows

$$\begin{aligned} E(N/C) &= \frac{L(C)\log B + [1 - L(C)]\log A}{E_{b_1}(Z_i)} \\ E_{b_1}(Z_i) &= \int_0^{b_1} Z f(x_i, C, b_1) dx_i \end{aligned} \tag{13}$$

$$\begin{aligned} E_{b_1}(Z_i) &= \log\left(\frac{C_1}{C_0}\right) + (C_0 - C_1) \log b + (C_1 - C_0) \int_0^{b_1} \log x_i f(x_i, C, b_1) dx_i \\ &= \log\left(\frac{C_1}{C_0}\right) + (C_0 - C_1) \log b + (C_1 - C_0) \left[\log b_1 - \frac{1}{C}\right] \\ E_{b_1}(Z_i) &= \log(a) - d \log\left(\frac{1}{\phi}\right) + \frac{d}{C} \end{aligned} \tag{14}$$

where $\phi = \left(\frac{b_1}{b}\right), a = \left(\frac{C_1}{C_0}\right), d = (C_1 - C_0)$

5 Implementation of SPRT for power function distribution

The nature of SPRT in case of power function distribution is described as, let $X_1, X_2, X_3, \dots, X_n$ be identically independent distributed r.v's from power function function distribution (2.2), where $C > 0$, we wish to test the simple null hypothesis $H_0 = C = C_0$ against simple alternative hypothesis $H_1 = C = C_1 (> C_0)$ having pre-assigned $0 < \alpha, \beta < 1$. which are (A and B) defined in section 3 and Z_i defined as

$$Z_i = \log\left[\frac{f(x_i, C_1)}{f(x_i, C_0)}\right] = \log\left(\frac{C_1}{C_0}\right) + (C_1 - C_0)\log x_i + (C_0 - C_1)\log b \tag{15}$$

Let $n(\geq), i = 1, 2, \dots$. The SPRT summarized and simplified to the following. Let $Y(n) = \sum X_i$ and $N = \text{first integer } (n \geq 1)$ for which inequality $Y(n) \leq c_1 + dn$ or $Y(n) \geq c_2 + dn$ holds with constants

$$c_1 = \frac{\ln(B)}{(C_1 - C_0)}, c_2 = \frac{\ln(A)}{(C_1 - C_0)}, d = \frac{\ln\left[\left(\frac{C_1}{C_0}\right)b^{(C_0 - C_1)}\right]}{(C_1 - C_0)}$$

At the stopping stage of sampling, if $Y(N) \leq c_1 + dN$ we accept H_0 and if $Y(N) \geq c_2 + dN$ we reject H_0 for different values of N , where A and B are the fixed constants. **Fig.3** shows acceptance and rejection region for null hypothesis, when $H_0 = C_0 = 20, H_1 = C_1 = 25, b = 1, \alpha = \beta = 0.05$. The calculated values constants are $c_1 = -0.9108, c_2 = 0.59713$ and $d = 0.01938$. from these constants if $Y(N) \leq 0.019N - 0.9108$ we accept H_0 and if $Y(N) \geq 0.019N + 0.59714$ we accept H_1 . At the intermediate stage, we continue sampling.

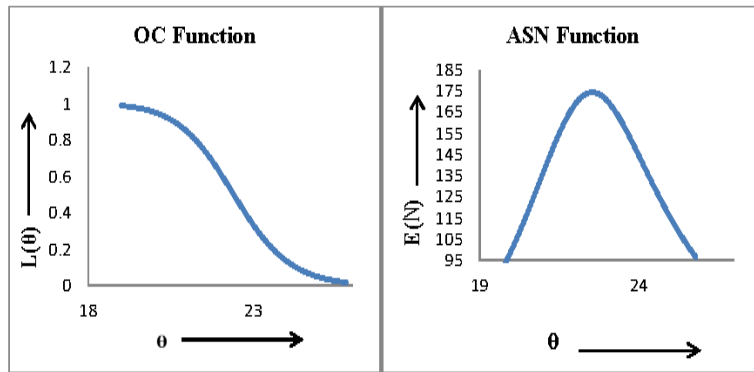


Fig. 1: OC function and ASN function for testing when $C_1 = 20, C_2 = 25, b = 1, \alpha = \beta = 0.05$.

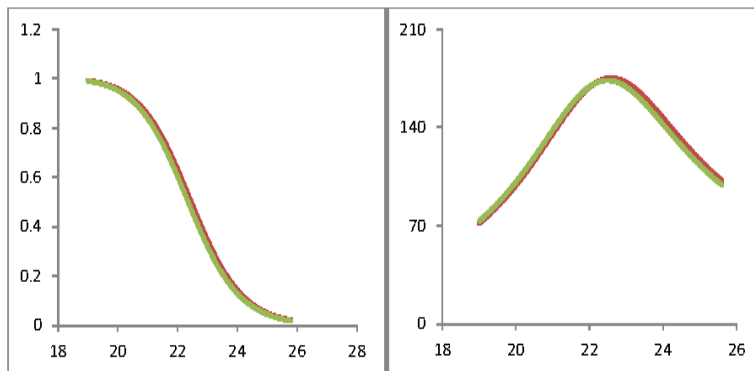


Fig. 2: Robustness of OC and ASN functions for testing $C_0 = 20, C_1 = 25, b = 1, \alpha = \beta = 0.05, p = \phi$.

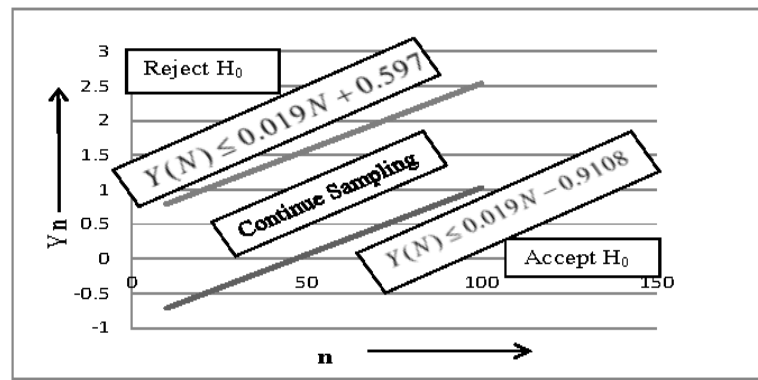


Fig. 3: Acceptance and rejection region for testing $C_0 = 20, C_1 = 25, b = 1, \alpha = \beta = 0.05, p = \phi$.

Table: 1 OC and ASN functions for testing $C_0 = 20, C_1 = 25, b = 1$ and $\alpha = \beta = 0.5, p = \phi$.

θ	$\phi = 1.0000$		$\phi = 0.9999$		$\phi = 1.00011$	
	$L(\theta)$	$E(N)$	$L(\theta)$	$E(N)$	$L(\theta)$	$E(N)$
19.0	0.991363	72.31351	0.992172	71.53846	0.990406	73.17874
19.2	0.987318	76.99261	0.988400	76.14191	0.986043	77.94159
19.4	0.982016	82.06598	0.983439	81.13527	0.980344	83.10313
19.6	0.975152	87.55464	0.977001	86.54096	0.972990	88.68267
19.8	0.966368	93.47254	0.968737	92.37510	0.963604	94.69146
20.0	0.955244	99.82246	0.958246	98.64377	0.951753	101.1284
20.2	0.941309	106.5910	0.945068	105.3381	0.936951	107.9745
20.4	0.924045	113.7422	0.928695	112.4284	0.918671	115.1869
20.6	0.902911	121.2111	0.908590	119.8574	0.896371	122.6915
20.8	0.877375	128.8965	0.884214	127.5332	0.869529	130.3769
21.0	0.846965	136.6559	0.855074	135.3235	0.837704	138.0891
21.2	0.811339	144.3026	0.820789	143.0518	0.800600	145.6304
21.4	0.770356	151.6088	0.781158	150.4991	0.758145	152.7644
21.6	0.724155	158.3153	0.736244	157.4116	0.710572	159.2276
21.8	0.673220	164.1503	0.686438	163.5172	0.658462	164.7502
22.0	0.618398	168.8549	0.632498	168.5496	0.602763	169.0833
22.2	0.560876	172.2127	0.575530	172.2782	0.544740	172.0282
22.4	0.502091	174.0768	0.516919	174.5352	0.485879	173.4642
22.6	0.443597	174.3928	0.458205	175.2418	0.427737	173.3580
22.8	0.386905	173.1983	0.400925	174.4119	0.371786	171.7775
23.0	0.333337	170.6221	0.346460	172.1547	0.319273	168.8698
23.2	0.283912	166.8587	0.295914	168.6504	0.271124	164.8426
23.4	0.239300	162.1437	0.250047	164.1287	0.227908	159.9362
23.6	0.199819	156.7255	0.209262	158.8380	0.189855	154.3971
23.8	0.165483	150.8422	0.173644	153.0222	0.156907	148.4561
24.0	0.136072	144.7059	0.143025	146.9019	0.128790	142.3152
24.2	0.111204	138.4934	0.117056	140.6642	0.105093	136.1401
24.4	0.090408	132.3438	0.095283	134.4583	0.085330	130.0591
24.6	0.073178	126.3606	0.077204	128.3968	0.068991	124.1660
24.8	0.059011	120.6156	0.062312	122.5594	0.055583	118.5246
25.0	0.047437	115.1547	0.050128	116.9981	0.044647	113.1745
25.2	0.038032	110.0033	0.040215	111.7430	0.035771	108.1363
25.4	0.030422	105.1714	0.032186	106.8078	0.028597	103.4164
25.6	0.024288	100.6578	0.025708	102.1935	0.022819	99.01144
25.8	0.019357	96.45376	0.020498	97.89312	0.018179	94.91104

6 Conclusion

The values of OC and ASN functions for the cases $b < b_1, b = b_1, b > b_1$ are plotted in Fig.2 respectively. From the Fig.2 we observe that for $b < b_1$ and $b > b_1$, the OC curve is not shifted to any other side of the curve. From the Fig.2 it is clear that SPRT is robust, as the deviation in OC and ASN functions are insignificant. Thus we conclude that for the present model, the SPRT for testing the hypothesis regarding C , is highly robust for changes in the value of b .

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