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Fixed Point Results of Contractive Mappings with Altering Distance Functions in Ordered b-Metric Spaces

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Abstract: We explore the existence of a fixed point as well as the uniqueness of a mapping in an ordered b-metric space using a generalized $(\breve{\psi},\hat{\eta})$-weak contraction. In addition, some results are posed on a coincidence point and a coupled coincidence point of two mappings under the same contraction condition. These findings generalize and build on a few recent studies in the literature. At the end, we provided some examples to back up our findings.

Keywords: $(\breve{\psi},\hat{\eta})$-weak contraction, fixed points, coincidence and coupled coincidence points, ordered b-metric space.

1 Introduction

In a wide range of pure and applied mathematics problems, fixed points of mappings that satisfy contractive conditions in extended metric spaces are extremely useful. First, Ran and Reuings [31] described the existence of fixed points in this direction for certain maps in ordered metric space and exhibited matrix linear equations applications. Following that, Nieto et al. [28, 29] expanded the result of [31] to nondecreasing mappings and used their findings to obtain differential equations solutions. Agarwal et al. [4] and O'Regan et al. [30] examine the influence of generalized contractions in ordered spaces at the same time. Bhaskar and Lakshmikantham [11] first developed coupled fixed point theory for some maps, then used the results to find a unique solution to periodic boundary value problems. Following that, Lakshmikantham and Ćirić [25], which were the extensions of [11] involving monotone property to a function in the space, pioneered the idea of coupled coincidence, common fixed point results. [15, 16, 17, 19, 21, 24, 26, 35, 36, 37, 38] provide additional information on coupled fixed point effects in various spaces under various contractive conditions.

A b-metric space is one of several generalizations of a standard metric space proposed by Bakhtin in his work [9], and widely used by Czerwik in his work [13, 14]. Following that, a lot of progress was made in acquiring the results of fixed points to single valued as well as multi-valued operators in the space, as evidenced by [1, 2, 3, 5, 6, 7, 8, 10, 18, 20, 22, 23, 27, 32, 33, 34, 39].

We demonstrate some fixed points results for mappings in ordered b-metric space that satisfy a generalized weak contraction in this paper. The results from [10, 11, 12, 19, 21, 25, 34] are expanded here as well as some examples noted to support the findings at the end of our work.

2 Preliminaries

Definition 21[*14*] *A b-metric is a map* $\eth:\mathscr{E}\times\mathscr{E}\to[0,+\infty)$ *that satisfies the properties below in* $\mathscr{E}$ *for all* $\varepsilon,\wp,\zeta$ *and some* $\mathrm{s}\geq 1$,

(a).$\eth(\varepsilon,\wp)=0$ *if and if* $\varepsilon=\wp$.
(b).$\eth(\varepsilon,\wp)=\eth(\wp,\varepsilon)$.
(c).$\eth(\varepsilon,\wp)\leq \mathrm{s}(\eth(\varepsilon,\zeta)+\eth(\zeta,\wp))$.

A b-metric space is specified as $(\mathscr{E},\eth,\mathrm{s})$.

Definition 22[*10, 14*] *In a b-metric space,*

(1).if $\eth(\varepsilon_n,\varepsilon)\to 0$ *as* $n\to+\infty$, $\{\varepsilon_n\}$ *is said to be convergent to* ε.

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(2).if $\eth(\varepsilon_n,\varepsilon_m)\to 0$ is the same as $n,m\to+\infty$, then $\{\varepsilon_n\}$ is a Cauchy sequence.
(3).if $(\mathscr{E},\eth,s)$ is a complete b-metric space, then any Cauchy sequence is convergent.

Definition 23[14, 34, 38] *If $\mathscr{E}$ is a partial ordered set with respect to an ordered relation $\preceq$ and $\eth$ is a metric on it, then $(\mathscr{E},\eth,\preceq)$ is a partially ordered metric space. $(\mathscr{E},\eth,\preceq)$ is a complete partially ordered b-metric space, despite the fact that $\eth$ is complete.*

Definition 24[34, 38] *If $\hbar(\varepsilon)\preceq\hbar(\wp)$ for all $\varepsilon,\wp\in\mathscr{E}$with $\varepsilon\preceq\wp$, the map is called a monotone non-decreasing.*

Definition 25[12] *Let $\hbar,\mathscr{I}:\mathscr{A}\to\mathscr{A}$ be two mappings, and $\mathscr{A}\neq\emptyset\subseteq\mathscr{E}$ be one. If $\hbar\varepsilon=\mathscr{I}\varepsilon=\varepsilon$ ($\hbar\varepsilon=\mathscr{I}\varepsilon$) for $\varepsilon\in\mathscr{A}$, then ε is a common fixed point (coincidence point) of $\hbar$, $\mathscr{I}$.*

Definition 26[12] *If $\hbar\mathscr{I}\varepsilon=\mathscr{I}\hbar\varepsilon$ for all $\varepsilon\in\mathscr{A}$, then $\hbar$ and $\mathscr{I}$ are commuting.*

Definition 27[12, 34] *The two maps $\hbar,\mathscr{I}$ are compatible if $\lim\limits_{n\to+\infty} d(\mathscr{I}\hbar\varepsilon_n,\hbar\mathscr{I}\varepsilon_n)=0$ for each sequence $\{\varepsilon_n\}\subseteq\mathscr{E}$ so that $\lim\limits_{n\to+\infty}\hbar\varepsilon_n=\lim\limits_{n\to+\infty}\mathscr{I}\varepsilon_n=\mu$, for some $\mu\in\mathscr{A}$.*

Definition 28[12, 34] *If $\hbar\varepsilon=\mathscr{I}\varepsilon$ for $\varepsilon\in\mathscr{A}$, then $\hbar\mathscr{I}\varepsilon=\mathscr{I}\hbar\varepsilon$, the mappings $\hbar$ and $\mathscr{I}$are weakly compatible.*

Definition 29[34] *If $\hbar\varepsilon\preceq\hbar\wp$ implies $\mathscr{I}\varepsilon\preceq\mathscr{I}\wp$ for any $\varepsilon,\wp\in\mathscr{E}$, then a map $\mathscr{I}$ is a monotone $\hbar$-non-decreasing.*

Definition 210[11] *Let $\mathscr{I}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ and $\hbar:\mathscr{E}\to\mathscr{E}$ are two mappings,*

(a).a point $(\varepsilon,\wp)\in\mathscr{E}\times\mathscr{E}$ is coupled coincidence point of $\mathscr{I}$, $\hbar$ if $\mathscr{I}(\varepsilon,\wp)=\hbar\varepsilon$ and, $\mathscr{I}(\wp,\varepsilon)=\hbar\wp$. In particular, if $\hbar$ is an identity map, then $(\varepsilon,\wp)$ is a coupled fixed point of $\mathscr{I}$.
(b).an element $\varepsilon\in\mathscr{E}$ is a common fixed point of $\mathscr{I}$, $\hbar$ if $\mathscr{I}(\varepsilon,\varepsilon)=\hbar\varepsilon=\varepsilon$.
(c).if $\mathscr{I}(\hbar\varepsilon,\hbar\wp)=\hbar(\mathscr{I}\varepsilon,\mathscr{I}\wp)$ for all $\varepsilon,\wp\in\mathscr{E}$, then $\mathscr{I}$ and $\hbar$ are commuting each other.
(d).If any two elements in a set $\mathscr{A}\subseteq\mathscr{E}$ are comparable, the set is well ordered.

Definition 211*A self map $\breve{\psi}$ on $[0,+\infty)$ that meets the conditions below is known as an altering distance function:*

(a).$\breve{\psi}$ is a non-decreasing and continuous function.
(b).$\breve{\psi}(\ell)=0$ iff $\ell=0$.

As seen above, the symbol $\hat{\Phi}$ represents the set of all altering distance functions.

Similarly,
$\hat{\Psi}:\{\hat{\eta}|\hat{\eta}$ *is a lower semi* $-$ *continuous self mapping on* $[0,+\infty)$ and, $\hat{\eta}(\ell)=0$ iff $\ell=0\}$.

Lemma 212[27] *Let $\hbar:\mathscr{E}\to\mathscr{E}$ be a mapping, and $\mathscr{E}\neq\emptyset$. Then $\mathscr{M}\subseteq\mathscr{E}$ occurs, resulting in $\hbar\mathscr{M}=\hbar\mathscr{E}$, where $\hbar:\mathscr{M}\to\mathscr{E}$ is one-to-one.*

Lemma 213[2] *Let $\{\varepsilon_n\}$ and $\{\wp_n\}$ be two sequences and b-convergent to ε and $\wp$ in a b-metric space $(\mathscr{E},\eth,s,\preceq)$, where $s>1$. Then*

$$\frac{1}{s^2}\eth(\varepsilon,\wp)\le\lim_{n\to+\infty}\inf\eth(\varepsilon_n,\wp_n)\le\lim_{n\to+\infty}\sup\eth(\varepsilon_n,\wp_n)\\ \le s^2\eth(\varepsilon,\wp).$$

In particular, if $\varepsilon=\wp$, then $\lim\limits_{n\to+\infty}\eth(\varepsilon_n,\wp_n)=0$. In addition, for every $\tau\in\mathscr{E}$, we get

$$\frac{1}{s}\eth(\varepsilon,\tau)\le\lim_{n\to+\infty}\inf\eth(\varepsilon_n,\tau)\le\lim_{n\to+\infty}\sup\eth(\varepsilon_n,\tau)\le sd(\varepsilon,\tau).$$

3 Main Results

Let's get started with the theorem below.

Theorem 31*Suppose $(\mathscr{E},\eth,s,\preceq)$ is a complete partially ordered b-metric space with $s\ge 1$. A map $\mathscr{I}:\mathscr{E}\to\mathscr{E}$ is non-decreasing and continuous with respect to $\preceq$. If $\varepsilon_0\in\mathscr{E}$ is such that $\varepsilon_0\preceq\mathscr{I}\varepsilon_0$ and the following contraction condition is fulfilled, then $\mathscr{I}$ has a fixed point in $\mathscr{E}$.*

$$\breve{\psi}(s\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp))\le\breve{\psi}(\mathscr{P}(\varepsilon,\wp))-\hat{\eta}(\mathscr{P}(\varepsilon,\wp)) \tag{1}$$

for $\breve{\psi}\in\hat{\Phi},\hat{\eta}\in\hat{\Psi}$ and for any $\varepsilon,\wp\in\mathscr{E}$ so that $\varepsilon\preceq\wp$ and

$$\mathscr{P}(\varepsilon,\wp)=\max\{\frac{\eth(\wp,\mathscr{I}\wp)\,[1+\eth(\varepsilon,\mathscr{I}\varepsilon)]}{1+\eth(\varepsilon,\wp)},\\ \frac{\eth(\varepsilon,\mathscr{I}\varepsilon)\;\eth(\wp,\mathscr{I}\wp)}{1+\eth(\varepsilon,\wp)},\\ \frac{\eth(\varepsilon,\mathscr{I}\varepsilon)\;\eth(\wp,\mathscr{I}\wp)}{1+\eth(\mathscr{I}\varepsilon,\mathscr{I}\wp)},\eth(\varepsilon,\wp)\}. \tag{2}$$

*Proof.*For some $\varepsilon_0\in\mathscr{E}$ with $\mathscr{I}\varepsilon_0=\varepsilon_0$, then the result is trivial. Assuming that $\varepsilon_0\prec\mathscr{I}\varepsilon_0$, we describe a sequence $\{\varepsilon_n\}\subset\mathscr{E}$ by $\varepsilon_{n+1}=\mathscr{I}\varepsilon_n$ for all $n\ge 0$. However, we can deduce the following as $\mathscr{I}$ is non-decreasing,

$$\varepsilon_0\prec\mathscr{I}\varepsilon_0=\varepsilon_1\preceq\mathscr{I}\varepsilon_1=\varepsilon_2\preceq\ldots\\ \preceq\mathscr{I}\varepsilon_{n-1}=\varepsilon_n\preceq\mathscr{I}\varepsilon_n=\varepsilon_{n+1}\preceq\ldots\ldots \tag{3}$$

If $\varepsilon_{n_0}=\varepsilon_{n_0+1}$ for $n_0\in\mathbb{N}$, then ε_{n_0} is a fixed point of $\mathscr{I}$ from (3). Otherwise, for all $n\ge 1$, $\varepsilon_n\neq\varepsilon_{n+1}$. For $n\ge 1$, let $D_n=\eth(\varepsilon_{n+1},\varepsilon_n)$ be used. We know that for every $n\ge 1$, $\varepsilon_{n-1}\prec\varepsilon_n$ and, the equation (1) becomes

$$\breve{\psi}(D_n)=\breve{\psi}(\eth(\varepsilon_n,\varepsilon_{n+1}))=\breve{\psi}(\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n))\\ \le\breve{\psi}(s\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n))\\ \le\breve{\psi}(\mathscr{P}(\varepsilon_{n-1},\varepsilon_n))\\ -\hat{\eta}(\mathscr{P}(\varepsilon_{n-1},\varepsilon_n)). \tag{4}$$

From (4), we get

$$\eth(\varepsilon_n,\varepsilon_{n+1})=\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n)\le\frac{1}{\mathfrak{z}}\mathscr{P}(\varepsilon_{n-1},\varepsilon_n), \tag{5}$$

where

$$\mathscr{P}(\varepsilon_{n-1},\varepsilon_n) = \max\{\frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\,[1+\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})]}{1+\eth(\varepsilon_{n-1},\varepsilon_n)}, \frac{\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})\,\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)}{1+\eth(\varepsilon_{n-1},\varepsilon_n)}, \frac{\eth(\varepsilon_{n-1},\mathscr{I}\varepsilon_{n-1})\,\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)}{1+\eth(\mathscr{I}\varepsilon_{n-1},\mathscr{I}\varepsilon_n)}, \eth(\varepsilon_{n-1},\varepsilon_n)\} \le \max\{\eth(\varepsilon_n,\varepsilon_{n+1}),\eth(\varepsilon_{n-1},\varepsilon_n)\} \le \max\{D_n,D_{n-1}\}. \tag{6}$$

If $\max\{D_n,D_{n-1}\} = D_n$ for certain $n \ge 1$, equation (5) is then accompanied by

$$\eth(\varepsilon_n,\varepsilon_{n+1}) \le \frac{1}{\jmath}\eth(\varepsilon_n,\varepsilon_{n+1}),$$

this is a contradiction. Thus, $\max\{D_n,D_{n-1}\} = D_{n-1}$ for $n \ge 1$. Hence, equation (5) becomes

$$\eth(\varepsilon_n,\varepsilon_{n+1}) \le \frac{1}{\jmath}\eth(\varepsilon_n,\varepsilon_{n-1}),$$

Since $\frac{1}{\jmath} \in (0,1)$ then $\{\varepsilon_n\}$ is a Cauchy sequence from [1,6,8,18]. Also, the completeness of $\mathscr{E}$ gives that $\varepsilon_n \to \mu \in \mathscr{E}$.

We may also deduce the following from $\mathscr{I}$'s continuity:

$$\mathscr{I}\mu = \mathscr{I}(\lim_{n\to+\infty}\varepsilon_n) = \lim_{n\to+\infty}\mathscr{I}\varepsilon_n = \lim_{n\to+\infty}\varepsilon_{n+1} = \mu. \tag{7}$$

As a result, $\mathscr{I}$ in $\mathscr{E}$ has a fixed point μ.

The continuity assumption on $\mathscr{I}$ is extracted from Theorem 31 and used to derive the following theorem.

Theorem 32*In Theorem 31, if $\mathscr{E}$ satisfies below condition, then $\mathscr{I}$ has a fixed point.*

$$\text{If a non-decreasing sequence } \{\varepsilon_n\} \subseteq \mathscr{E} \text{ and } \varepsilon_n \to \sigma \text{ then } \varepsilon_n \le \sigma, \text{ for each } n \in \mathbb{N}, \text{ i.e., } \sigma = \sup \varepsilon_n. \tag{8}$$

*Proof.*We have an increasing sequence $\{\varepsilon_n\} \subseteq \mathscr{E}$ that eventually converges to some $\sigma \in \mathscr{E}$ as a result of Theorem 31. But by the hypotheses for all n, $\varepsilon_n \preceq \sigma$, which means that $\sigma = \sup \varepsilon_n$.

We can now assert that σ is a fixed point of $\mathscr{I}$. Assume $\mathscr{I}\sigma \neq \sigma$ is not true. Let

$$\mathscr{P}(\varepsilon_n,\sigma) = \max\{\frac{\eth(\sigma,\mathscr{I}\sigma)\,[1+\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)]}{1+\eth(\varepsilon_n,\sigma)}, \frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\,\eth(\sigma,\mathscr{I}\sigma)}{1+\eth(\varepsilon_n,\sigma)}, \frac{\eth(\varepsilon_n,\mathscr{I}\varepsilon_n)\,\eth(\sigma,\mathscr{I}\sigma)}{1+\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma)}, \eth(\varepsilon_n,\sigma)\}, \tag{9}$$

then taking limit as $n \to +\infty$ in the equation (9) and making use of $\lim_{n\to+\infty}\varepsilon_n = \sigma$, we get

$$\lim_{n\to+\infty}\mathscr{P}(\varepsilon_n,\sigma) = \max\{\eth(\sigma,\mathscr{I}\sigma),0\} = \eth(\sigma,\mathscr{I}\sigma), \tag{10}$$

Since, $\varepsilon_n \preceq \sigma$ for each n, then we obtain the following from equations (1) and (9)

$$\breve{\psi}(\eth(\varepsilon_{n+1},\mathscr{I}\sigma)) = \breve{\psi}(\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma)) \le \breve{\psi}(s\eth(\mathscr{I}\varepsilon_n,\mathscr{I}\sigma)) \le \breve{\psi}(\mathscr{P}(\varepsilon_n,\sigma)) - \hat{\eta}(\mathscr{P}(\varepsilon_n,\sigma)). \tag{11}$$

Take limit as $n \to +\infty$ in (11) and from equation (10) as well as the properties of $\breve{\psi}$, $\hat{\eta}$, we have

$$\breve{\psi}(\eth(\sigma,\mathscr{I}\sigma)) \le \breve{\psi}(\eth(\sigma,\mathscr{I}\sigma)) - \hat{\eta}(\eth(\sigma,\mathscr{I}\sigma)) < \breve{\psi}(\eth(\sigma,\mathscr{I}\sigma)). \tag{12}$$

This is a contradiction to $\mathscr{I}\sigma \neq \sigma$. Hence, $\mathscr{I}\sigma = \sigma$.

In the above theorems, the fixed point is unique if $\mathscr{E}$ meets the following condition.

$$\text{There is an } \sigma \text{ in } \mathscr{E} \text{ that is comparable to } \varepsilon \text{ and } \wp \text{ for each } \varepsilon,\wp \in \mathscr{E}. \tag{13}$$

Theorem 33*If $\mathscr{E}$ assumes the condition (13) in Theorem 31 & 32, then $\mathscr{I}$ has a unique fixed point in $\mathscr{E}$.*

*Proof.*Theorems 31 & 32 show that the set of fixed points of $\mathscr{I}$ is nonempty. Assume $\varepsilon^* \neq \wp^*$ are fixed points of $\mathscr{I}$ to ensure uniqueness. Following that,

$$\breve{\psi}(\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)) \le \breve{\psi}(s\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)) \le \breve{\psi}(\mathscr{P}(\varepsilon^*,\wp^*)) - \hat{\eta}(\mathscr{P}(\varepsilon^*,\wp^*)) \tag{14}$$

where

$$\mathscr{P}(\varepsilon^*,\wp^*) = \max\{\frac{\eth(\wp^*,\mathscr{I}\wp^*)\,[1+\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)]}{1+\eth(\varepsilon^*,\wp^*)}, \frac{\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)\,\eth(\wp^*,\mathscr{I}\wp^*)}{1+\eth(\varepsilon^*,\wp^*)}, \frac{\eth(\varepsilon^*,\mathscr{I}\varepsilon^*)\,\eth(\wp^*,\mathscr{I}\wp^*)}{1+\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)}, \eth(\varepsilon^*,\wp^*)\}. \tag{15}$$

Therefore, from equations (14) and (15), we have

$$\breve{\psi}(\eth(\varepsilon^*,\wp^*)) = \breve{\psi}(\eth(\mathscr{I}\varepsilon^*,\mathscr{I}\wp^*)) \le \breve{\psi}(\eth(\varepsilon^*,\wp^*)) - \hat{\eta}(\eth(\varepsilon^*,\wp^*)) < \breve{\psi}(\eth(\varepsilon^*,\wp^*)), \tag{16}$$

this contradicts to $\varepsilon^* \neq \wp^*$. Hence, $\varepsilon^* = \wp^*$.

Now, we have the below corollary from Theorems 31 to 33.

Corollary 34*Let $(\mathscr{E},\eth,\preceq)$ be a partially ordered b-metric space. Suppose the mappings $\mathscr{I},\hbar : \mathscr{E} \to \mathscr{E}$ are continuous such that*

(C_1).

$$\breve{\psi}(s\eth(\mathscr{J}\varepsilon,\mathscr{J}\wp)) \le \breve{\psi}(\mathscr{P}_\hbar(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp)) \quad (17)$$

for every $\varepsilon,\wp\in\mathscr{E}$ *with* $\hbar\varepsilon\preceq\hbar\wp$, $s>1$, $\breve{\psi}\in\hat{\Phi}$, $\hat{\eta}\in\hat{\Psi}$ *and, where*

$$\mathscr{P}_\hbar(\varepsilon,\wp) = \max\{\frac{\eth(\hbar\wp,\mathscr{J}\wp)[1+\eth(\hbar\varepsilon,\mathscr{J}\varepsilon)]}{1+\eth(\hbar\varepsilon,\hbar\wp)}, \frac{\eth(\hbar\varepsilon,\mathscr{J}\varepsilon)\,\eth(\hbar\wp,\mathscr{J}\wp)}{1+\eth(\hbar\varepsilon,\hbar\wp)}, \frac{\eth(\hbar\varepsilon,\mathscr{J}\varepsilon)\,\eth(\hbar\wp,\mathscr{J}\wp)}{1+\eth(\mathscr{J}\varepsilon,\mathscr{J}\wp)}, \eth(\hbar\varepsilon,\hbar\wp)\}. \quad (18)$$

(C_2).$\mathscr{J}\mathscr{E}\subset\hbar\mathscr{E}$ *and* $\hbar\mathscr{E}\subseteq\mathscr{E}$ *is complete,*
(C_3).$\mathscr{J}$ *is monotone* $\hbar$*-non-decreasing and*
(C_4).$\mathscr{J}$ *and* $\hbar$ *are compatible.*

If for some $\varepsilon_0\in\mathscr{E}$ *such that* $\hbar\varepsilon_0\preceq\mathscr{J}\varepsilon_0$*, then there is a coincidence point in* $\mathscr{E}$ *for a pair of mappings* $(\mathscr{J},\hbar)$.

Proof. According to lemma 212, there is a subset $\mathscr{M}$ of $\mathscr{E}$ so that $\hbar\mathscr{M}\subset\mathscr{E}$ is a complete subspace, and $\hbar:\mathscr{E}\to\mathscr{E}$ is one-to-one. Following [27]'s Corollary 2.1, there is a sequence $\{\hbar\varepsilon_n\}\subset\hbar\mathscr{M}$ for some $\varepsilon_0\in\mathscr{M}$ so that $\hbar\varepsilon_{n+1}=\mathscr{J}\varepsilon_n=\kappa(\hbar\varepsilon_n)$, $(n\ge 0)$ and, where $\kappa:\hbar\mathscr{M}\to\hbar\mathscr{M}$ is a mapping so that $\kappa(\hbar\varepsilon)=\mathscr{J}\varepsilon$, $\varepsilon\in\mathscr{M}$.

Thus from equation (17), we get

$$\breve{\psi}(s\eth(\kappa(\hbar\varepsilon),\kappa(\hbar\wp))) \le \breve{\psi}(\mathscr{P}_\hbar(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp)), \quad (19)$$

for every $\varepsilon,\wp\in\mathscr{E}$ with $\hbar\varepsilon\preceq\hbar\wp$ and, where

$$\mathscr{P}_\hbar(\varepsilon,\wp) = \max\{\frac{\eth(\hbar\wp,\kappa(\hbar\wp))[1+\eth(\hbar\varepsilon,\kappa(\hbar\varepsilon))]}{1+\eth(\hbar\varepsilon,\hbar\wp)}, \frac{\eth(\hbar\varepsilon,\kappa(\hbar\varepsilon))\,\eth(\hbar\wp,\kappa(\hbar\wp))}{1+\eth(\hbar\varepsilon,\hbar\wp)}, \frac{\eth(\hbar\varepsilon,\kappa(\hbar\varepsilon))\,\eth(\hbar\wp,\kappa(\hbar\wp))}{1+\eth(\kappa(\hbar\varepsilon),\kappa(\hbar\wp))}, \eth(\hbar\varepsilon,\hbar\wp)\}. \quad (20)$$

We can deduce from Theorem 31 that $\{\hbar\varepsilon_n\}\subset\hbar\mathscr{M}$ is a b-Cauchy sequence that converging on $\upsilon\in\hbar\mathscr{M}$.

We get from the condition (C_4) that,

$$\lim_{n\to+\infty}\eth(\kappa(\mathscr{J}\varepsilon_n),\mathscr{J}(\hbar\varepsilon_n)) = 0.$$

We have from a b-metrics triangular inequality that

$$\eth(\mathscr{J}\upsilon,\hbar\upsilon) \le s\eth(\mathscr{J}\upsilon,\mathscr{J}(\hbar\varepsilon_n)) + s^2\eth(\mathscr{J}(\hbar\varepsilon_n),\kappa(\mathscr{J}\varepsilon_n)) + s^2\eth(\kappa(\mathscr{J}\varepsilon_n),\hbar\upsilon). \quad (21)$$

As $n\to+\infty$ in (21), $\eth(\mathscr{J}\upsilon,\hbar\upsilon)=0$ this indicates that υ is a coincidence point of $\mathscr{J}$, $\hbar$.

The following result can get from Corollary 34 by weakening its hypotheses.

Corollary 35 *If* $\mathscr{E}$ *satisfies the following condition in Corollary 34,*

for very nondecreasing sequence $\{\hbar\varepsilon_n\}\subseteq\mathscr{E}$ *so that* $\hbar\varepsilon_n\to\hbar\sigma$*, then* $\hbar\varepsilon_n\le\hbar\sigma$ $(n\ge 0)$*, i.e.,* $\hbar\sigma=\sup\hbar\varepsilon_n$. (22)

then, if $\hbar\mu\preceq\hbar(\hbar\mu)$ *for some coincidence point* μ*, a coincidence point exists for the weakly compatible mappings* $(\mathscr{J},\hbar)$*. Moreover,* $(\mathscr{J},\hbar)$ *has only one common fixed point iff the set of common fixed points is well ordered.*

Proof. A pair of maps $(\mathscr{J},\hbar)$ has a coincidence point, according to Theorem 33 and Corollary 34.

Next, assume $(\mathscr{J},\hbar)$ is only weakly compatible. Let $\upsilon\in\mathscr{E}$ be a point with $\upsilon=\mathscr{J}\mu=\hbar\mu$. Thence, $\mathscr{J}\upsilon=\mathscr{J}(\hbar\mu)=\hbar(\mathscr{J}\mu)=\hbar\upsilon$.

Therefore,

$$\begin{aligned}\mathscr{P}_\hbar(\mu,\upsilon) &= \max\{\frac{\eth(\hbar\upsilon,\mathscr{J}\upsilon)[1+\eth(\hbar\mu,\mathscr{J}\mu)]}{1+\eth(\hbar\mu,\hbar\upsilon)}, \frac{\eth(\hbar\mu,\mathscr{J}\mu)\,\eth(\hbar\upsilon,\mathscr{J}\upsilon)}{1+\eth(\hbar\mu,\hbar\upsilon)}, \frac{\eth(\hbar\mu,\mathscr{J}\mu)\,\eth(\hbar\upsilon,\mathscr{J}\upsilon)}{1+\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)}, \eth(\hbar\mu,\hbar\upsilon)\}\\ &= \max\{0,\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)\}\\ &= \eth(\mathscr{J}\mu,\mathscr{J}\upsilon).\end{aligned} \quad (23)$$

Thus from equation (17), we get

$$\begin{aligned}\breve{\psi}(\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)) &\le \breve{\psi}(\mathscr{P}_\hbar(\mu,\upsilon)) - \hat{\eta}(\mathscr{P}_\hbar(\mu,\upsilon))\\ &\le \breve{\psi}(\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)) - \hat{\eta}(\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)).\end{aligned} \quad (24)$$

By the property of $\hat{\eta}$, we get $\eth(\mathscr{J}\mu,\mathscr{J}\upsilon)=0$ implies that $\mathscr{J}\upsilon=\hbar\upsilon=\upsilon$.

Finally, we can deduce from Theorem 33 that $(\mathscr{J},\hbar)$ only has one common fixed point iff the common fixed points of $(\mathscr{J},\hbar)$ is well ordered.

Remark 36 *Theorems 31 to 33 are the extension of Theorems 2.1,.2.2 & 2.3 of [12].*

Remark 37 *Corollaries 34 & 35 are the generalizations of Corollaries 2.1 & 2.2 of [27] respectively.*

Definition 38 *Consider the partially ordered b-metric space,* $(\mathscr{E},\eth,\preceq)$*. A map* $\mathscr{J}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ *is known to be a generalized* $(\breve{\psi},\hat{\eta})$*-contractive map with regards to* $\hbar:\mathscr{E}\to\mathscr{E}$*, if*

$$\breve{\psi}(s^k\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\zeta,\Im))) \le \breve{\psi}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)) - \hat{\eta}(\mathscr{P}_\hbar(\varepsilon,\wp,\zeta,\Im)), \quad (25)$$

for all $\varepsilon, \wp, \zeta, \Im \in \mathscr{E}$ *with* $\hslash\varepsilon \preceq \hslash\zeta$ *and* $\hslash\wp \succeq \hslash\Im$, $k > 2$, $s > 1$, $\breve{\psi} \in \hat{\Phi}$, $\hat{\eta} \in \hat{\Psi}$ *and where*

$$\begin{aligned}\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im) = \max\{&\frac{\eth(\hslash\zeta,\mathscr{J}(\zeta,\Im))\,[1+\eth(\hslash\varepsilon,\mathscr{J}(\varepsilon,\wp))]}{1+\eth(\hslash\varepsilon,\hslash\zeta)},\\ &\frac{\eth(\hslash\varepsilon,\mathscr{J}(\varepsilon,\wp))\,\eth(\hslash\zeta,\mathscr{J}(\zeta,\Im))}{1+\eth(\hslash\varepsilon,\hslash\zeta)},\\ &\frac{\eth(\hslash\varepsilon,\mathscr{J}(\varepsilon,\wp))\,\eth(\hslash\zeta,\mathscr{J}(\zeta,\Im))}{1+\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\zeta,\Im))},\\ &\eth(\hslash\varepsilon,\hslash\zeta)\}.\end{aligned}$$

Theorem 39*Suppose* $(\mathscr{E},\eth,\preceq)$ *be a complete partially ordered b-metric space. A map* $\mathscr{J}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ *satisfies the condition* (25) *and,* $\mathscr{J}$*,* $\hslash$ *are continuous,* $\mathscr{J}$ *has mixed* $\hslash$*-monotone property and also commutes with* $\hslash$*. Assume, if some* $(\varepsilon_0,\wp_0) \in \mathscr{E}\times\mathscr{E}$ *so that* $\hslash\varepsilon_0 \preceq \mathscr{J}(\varepsilon_0,\wp_0)$*,* $\hslash\wp_0 \succeq \mathscr{J}(\wp_0,\varepsilon_0)$ *and* $\mathscr{J}(\mathscr{E}\times\mathscr{E}) \subseteq \hslash(\mathscr{E})$*, then* $\mathscr{J}$ *and* $\hslash$ *in* $\mathscr{E}$ *have a coupled coincidence point.*

*Proof.*From [7] of Theorem 2.2, there will be two sequences $\{\varepsilon_n\},\{\wp_n\} \subset \mathscr{E}$ so that

$$\hslash\varepsilon_{n+1} = \mathscr{J}(\varepsilon_n,\wp_n),\quad \hslash\wp_{n+1} = \mathscr{J}(\wp_n,\varepsilon_n), n \geq 0.$$

In particular, the sequences $\{\hslash\varepsilon_n\}$, $\{\hslash\wp_n\}$ are non-decreasing and non-increasing in $\mathscr{E}$. Put $\varepsilon=\varepsilon_n, \wp=\wp_n, \zeta=\varepsilon_{n+1}, \Im=\wp_{n+1}$ in (25), we get

$$\begin{aligned}\breve{\psi}(s^k\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2})) &= \breve{\psi}(s^k\eth(\mathscr{J}(\varepsilon_n,\wp_n),\mathscr{J}(\varepsilon_{n+1},\wp_{n+1})))\\ &\leq \breve{\psi}(\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1}))\\ &\quad -\hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1})),\end{aligned} \tag{26}$$

where

$$\begin{aligned}\mathscr{P}_{\hslash}(\varepsilon_n,\wp_n,\varepsilon_{n+1},\wp_{n+1}) \leq \max\{&\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\\ &\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2})\}\end{aligned} \tag{27}$$

As a result of (26), we get

$$\begin{aligned}&\breve{\psi}(s^k\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}))\\ &\leq \breve{\psi}(\max\{\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2})\})\\ &\quad -\hat{\eta}(\max\{\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2})\}).\end{aligned} \tag{28}$$

Likewise by taking $\varepsilon=\wp_{n+1}, \wp=\varepsilon_{n+1}, \zeta=\varepsilon_n, \Im=\varepsilon_n$ in (25), we get

$$\begin{aligned}&\breve{\psi}(s^k\eth(\hslash\wp_{n+1},\hslash\wp_{n+2}))\\ &\leq \breve{\psi}(\max\{\eth(\hslash\wp_n,\hslash\wp_{n+1}),\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\})\\ &-\hat{\eta}(\max\{\eth(\hslash\wp_n,\hslash\wp_{n+1}),\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\}).\end{aligned} \tag{29}$$

We know that $\max\{\breve{\psi}(l_1),\breve{\psi}(l_2)\} = \breve{\psi}\{\max\{l_1,l_2\}\}$ for $l_1,l_2 \in [0,+\infty)$. Then we add (28) and (29) together to get,

$$\begin{aligned}&\breve{\psi}(s^k\Gamma_n)\\ &\leq \breve{\psi}(\max\{\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}),\\ &\qquad \eth(\hslash\wp_n,\hslash\wp_{n+1}),\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\})\\ &\quad -\hat{\eta}(\max\{\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}),\\ &\qquad \eth(\hslash\wp_n,\hslash\wp_{n+1}),\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\})\end{aligned} \tag{30}$$

where

$$\Gamma_n = \max\{\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}),\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\}. \tag{31}$$

Let us denote,

$$\begin{aligned}\varkappa_n = \max\{&\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}),\eth(\hslash\wp_n,\hslash\wp_{n+1}),\\ &\eth(\hslash\wp_{n+1},\hslash\wp_{n+2})\}.\end{aligned} \tag{32}$$

Hence from equations (28)-(31), we obtain

$$s^k\Gamma_n \leq \varkappa_n. \tag{33}$$

Now to claim that

$$\Gamma_n \leq \lambda\Gamma_{n-1}, \tag{34}$$

for $n \geq 1$ and $\lambda = \frac{1}{s^k} \in [0,1)$.

Suppose that if $\varkappa_n = \Gamma_n$ then from (33), we get $s^k\Gamma_n \leq \Gamma_n$ this leads to $\Gamma_n = 0$ since $s > 1$ and thus (34) holds.

Suppose $\varkappa_n = \max\{\eth(\hslash\varepsilon_n,\hslash\varepsilon_{n+1}),\eth(\hslash\wp_n,\hslash\wp_{n+1})\}$, i.e., $\varkappa_n = \Gamma_{n-1}$ then (33) follows (34).

Now from (33), we obtain that $\Gamma_n \leq \lambda^n\delta_0$ and hence,

$$\eth(\hslash\varepsilon_{n+1},\hslash\varepsilon_{n+2}) \leq \lambda^n\Gamma_0 \text{ and } \eth(\hslash\wp_{n+1},\hslash\wp_{n+2}) \leq \lambda^n\Gamma_0, \tag{35}$$

which shows that $\{\hslash\varepsilon_n\}$, $\{\hslash\wp_n\}$ in $\mathscr{E}$ are Cauchy sequences by Lemma 3.1 of [22]. Therefore, we can conclude from Theorem 2.2 of [5] that in $\mathscr{E}$, $\mathscr{J}$ and $\hslash$ have a coincidence point.

Corollary 310*Suppose* $(\mathscr{E},\eth,\preceq)$ *be a complete partially ordered b-metric space. A continuous map* $\mathscr{J}:\mathscr{E}\times\mathscr{E}\to\mathscr{E}$ *has mixed monotone property is satisfying the below contraction conditions for all* $\varepsilon,\wp,\zeta,\Im \in \mathscr{E}$ *such that* $\varepsilon \preceq \zeta$ *and* $\wp \succeq \Im$*,* $k > 2$*,* $s > 1$*,* $\breve{\psi} \in \hat{\Phi}$ *and* $\hat{\eta} \in \hat{\Psi}$*:*

(i).

$$\begin{aligned}\breve{\psi}(s^k\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\zeta,\Im))) \leq \breve{\psi}(&\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im))\\ &-\hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im)),\end{aligned}$$

(ii).

$$\begin{aligned}\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\zeta,\Im)) \leq &\frac{1}{s^k}\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im)\\ &-\frac{1}{s^k}\hat{\eta}(\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im)).\end{aligned}$$

where

$$\begin{aligned}\mathscr{P}_{\hslash}(\varepsilon,\wp,\zeta,\Im) = \max\{&\frac{\eth(\zeta,\mathscr{J}(\zeta,\Im))\,[1+\eth(\varepsilon,\mathscr{J}(\varepsilon,\wp))]}{1+\eth(\varepsilon,\zeta)},\\ &\frac{\eth(\varepsilon,\mathscr{J}(\varepsilon,\wp))\,\eth(\zeta,\mathscr{J}(\zeta,\Im))}{1+\eth(\varepsilon,\zeta)},\\ &\frac{\eth(\varepsilon,\mathscr{J}(\varepsilon,\wp))\,\eth(\zeta,\mathscr{J}(\zeta,\Im))}{1+\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\zeta,\Im))},\eth(\varepsilon,\zeta)\}.\end{aligned}$$

If there exists $(\varepsilon_0,\wp_0) \in \mathscr{E}\times\mathscr{E}$ *so that* $\varepsilon_0 \preceq \mathscr{J}(\varepsilon_0,\wp_0)$ *and* $\wp_0 \succeq \mathscr{J}(\wp_0,\varepsilon_0)$*, then* $\mathscr{J}$ *in* $\mathscr{E}$ *has a coupled fixed point.*

Theorem 311 *The unique coupled common fixed point for $\mathcal{J}$ and $\hbar$ exists in Theorem 39, if for every $(\varepsilon,\wp),(\hbar,\ell)\in\mathscr{E}\times\mathscr{E}$ there is some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$ such that $(\mathcal{J}(\Lambda,\Upsilon),\mathcal{J}(\Upsilon,\Lambda))$ is comparable to $(\mathcal{J}(\varepsilon,\wp),\mathcal{J}(\wp,\varepsilon))$ and to $(\mathcal{J}(\hbar,\mathcal{J}),\mathcal{J}(\ell,\hbar))$.*

Proof. The existence of a coupled coincidence point for $\mathcal{J}$, $\hbar$ is guaranteed by the Theorem 39. Let $(\varepsilon,\wp),(\hbar,\ell)\in\mathscr{E}\times\mathscr{E}$ are coupled coincidence points of $\mathcal{J}$, $\hbar$. Now, we assert that $\hbar\varepsilon=\hbar\hbar$ and $\hbar\wp=\hbar\ell$. By hypotheses $(\mathcal{J}(\Lambda,\Upsilon),\mathcal{J}(\Upsilon,\Lambda))$ is comparable to $(\mathcal{J}(\varepsilon,\wp),\mathcal{J}(\wp,\varepsilon))$ and $(\mathcal{J}(\hbar,\mathcal{J}),\mathcal{J}(\ell,\hbar))$ for some $(\Lambda,\Upsilon)\in\mathscr{E}\times\mathscr{E}$.

Now, assume the following

$$(\mathcal{J}(\varepsilon,\wp),\mathcal{J}(\wp,\varepsilon))\le(\mathcal{J}(\Lambda,\Upsilon),\mathcal{J}(\Upsilon,\Lambda))\text{ and }$$
$$(\mathcal{J}(\hbar,\ell),\mathcal{J}(\ell,\hbar))\le(\mathcal{J}(\Lambda,\Upsilon),\mathcal{J}(\Upsilon,\Lambda)).$$

Suppose $\Lambda_0=\Lambda$ and $\Upsilon_0=\Upsilon$ then there is a point $(\Lambda_1,\Upsilon_1)\in\mathscr{E}\times\mathscr{E}$ such that

$$\hbar\Lambda_1=\mathcal{J}(\Lambda_0,\Upsilon_0),\ \ \hbar\Upsilon_1=\mathcal{J}(\Upsilon_0,\Lambda_0)\ \ (n\ge1).$$

As by applying the preceding argument repeatedly, we have the sequences $\{\hbar\Lambda_n\}$ and $\{\hbar\Upsilon_n\}$ in $\mathscr{E}$ with

$$\hbar\Lambda_{n+1}=\mathcal{J}(\Lambda_n,\Upsilon_n),\ \ \hbar\Upsilon_{n+1}=\mathcal{J}(\Upsilon_n,\Lambda_n)\ \ (n\ge0).$$

Define the sequences in the same way $\{\hbar\varepsilon_n\}$, $\{\hbar\wp_n\}$ and, $\{\hbar\hbar_n\}$, $\{\hbar\ell_n\}$ in $\mathscr{E}$ by setting $\varepsilon_0=\varepsilon$, $\wp_0=\wp$ and $\hbar_0=\hbar$, $\ell_0=\ell$. Further, we have that

$$\begin{aligned}&\hbar\varepsilon_n\to\mathcal{J}(\varepsilon,\wp),\ \hbar\wp_n\to\mathcal{J}(\wp,\varepsilon),\\&\hbar\hbar_n\to\mathcal{J}(\hbar,\ell),\ \hbar\ell_n\to\mathcal{J}(\ell,\hbar)(n\ge1).\end{aligned}\tag{36}$$

Thus by induction, we get

$$(\hbar\varepsilon_n,\hbar\wp_n)\le(\hbar\Lambda_n,\hbar\Upsilon_n)\ \text{ for every } n.\tag{37}$$

As a consequence of (25), we have

$$\begin{aligned}\breve{\psi}(\eth(\hbar\varepsilon,\hbar\Lambda_{n+1}))&\le\breve{\psi}(s^k\eth(\hbar\varepsilon,\hbar\Lambda_{n+1}))\\&=\breve{\psi}(s^k\eth(\mathcal{J}(\varepsilon,\wp),\mathcal{J}(\Lambda_n,\Upsilon_n)))\\&\le\breve{\psi}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\Lambda_n,\Upsilon_n))\\&\quad-\hat{\eta}(\mathscr{P}_{\hbar}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)),\end{aligned}\tag{38}$$

where

$$\begin{aligned}&\mathscr{P}_{\hbar}(\varepsilon,\wp,\Lambda_n,\Upsilon_n)\\&\quad=\max\Big\{\frac{\eth(\hbar\Lambda_n,\mathcal{J}(\Lambda_n,\Upsilon_n))\,[1+\eth(\hbar\varepsilon,\mathcal{J}(\varepsilon,\wp))]}{1+\eth(\hbar\varepsilon,\hbar\Lambda_n)},\\&\quad\frac{\eth(\hbar\varepsilon,\mathcal{J}(\varepsilon,\wp))\,\eth(\hbar\Lambda_n,\mathcal{J}(\Lambda_n,\Upsilon_n))}{1+\eth(\hbar\varepsilon,\hbar\Lambda_n)},\\&\quad\frac{\eth(\hbar\varepsilon,\mathcal{J}(\varepsilon,\wp))\,\eth(\hbar\Lambda_n,\mathcal{J}(\Lambda_n,\Upsilon_n))}{1+\eth(\mathcal{J}(\varepsilon,\wp),\mathcal{J}(\Lambda_n,\Upsilon_n))},\\&\quad\eth(\hbar\varepsilon,\hbar\Lambda_n)\Big\}\\&=\max\{0,\eth(\hbar\varepsilon,\hbar\Lambda_n)\}\\&=\eth(\hbar\varepsilon,\hbar\Lambda_n).\end{aligned}$$

As a result of (38), we now have

$$\breve{\psi}(\eth(\hbar\varepsilon,\hbar\Lambda_{n+1}))\le\breve{\psi}(\eth(\hbar\varepsilon,\hbar\Lambda_n))-\hat{\eta}(\eth(\hbar\varepsilon,\hbar\Lambda_n)).\tag{39}$$

As by the similar argument, we acquire that

$$\breve{\psi}(\eth(\hbar\wp,\hbar\Upsilon_{n+1}))\le\breve{\psi}(\eth(\hbar\wp,\hbar\Upsilon_n))-\hat{\eta}(\eth(\hbar\wp,\hbar\Upsilon_n)).\tag{40}$$

Hence from (39) and (40), we have

$$\begin{aligned}&\breve{\psi}(\max\{\eth(\hbar\varepsilon,\hbar\Lambda_{n+1}),\eth(\hbar\wp,\hbar\Upsilon_{n+1})\})\\&\le\breve{\psi}(\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\})\\&-\hat{\eta}(\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\})\\&<\breve{\psi}(\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\}).\end{aligned}\tag{41}$$

Thus, the property of $\breve{\psi}$ implies,

$$\begin{aligned}&\max\{\eth(\hbar\varepsilon,\hbar\Lambda_{n+1}),\eth(\hbar\wp,\hbar\Upsilon_{n+1})\}\\&\quad<\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\}.\end{aligned}$$

Hence, $\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\}$ is a decreasing sequence of positive reals and bounded below and by a result, we have

$$\lim_{n\to+\infty}\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\}=\Gamma,\ \Gamma\ge0.$$

Therefore as $n\to+\infty$ in equation (41), we get

$$\breve{\psi}(\Gamma)\le\breve{\psi}(\Gamma)-\hat{\eta}(\Gamma),\tag{42}$$

from which we have $\hat{\eta}(\Gamma)=0$, implies that $\Gamma=0$. Therefore,

$$\lim_{n\to+\infty}\max\{\eth(\hbar\varepsilon,\hbar\Lambda_n),\eth(\hbar\wp,\hbar\Upsilon_n)\}=0.$$

Hence, we have that,

$$\lim_{n\to+\infty}\eth(\hbar\varepsilon,\hbar\Lambda_n)=0\text{ and }\lim_{n\to+\infty}\eth(\hbar\wp,\hbar\Upsilon_n)=0.\tag{43}$$

By the similar argument as above, we obtain

$$\lim_{n\to+\infty}\eth(\hbar\hbar,\hbar\Lambda_n)=0\text{ and }\lim_{n\to+\infty}\eth(\hbar\mathcal{J},\hbar\Upsilon_n)=0.\tag{44}$$

Therefore from (43) and (44), we get $\hbar\varepsilon=\hbar\hbar$ and $\hbar\wp=\hbar\mathcal{J}$. Since $\hbar\varepsilon=\mathcal{J}(\varepsilon,\wp)$ and $\hbar\wp=\mathcal{J}(\wp,\varepsilon)$ and, the commutativity property of $\mathcal{J}$, $\hbar$ implies that

$$\begin{aligned}&\hbar(\hbar\varepsilon)=\hbar(\mathcal{J}(\varepsilon,\wp))=\mathcal{J}(\hbar\varepsilon,\hbar\wp)\text{ and}\\&\hbar(\hbar\wp)=\hbar(\mathcal{J}(\wp,\varepsilon))=\mathcal{J}(\hbar\wp,\hbar\varepsilon).\end{aligned}\tag{45}$$

If $\hbar\varepsilon=\Lambda^*$ and $\hbar\wp=\Upsilon^*$ then from (45), we get

$$\hbar(\Lambda)=\mathcal{J}(\Lambda^*,\Upsilon^*)\text{ and }\hbar(\Upsilon^*)=\mathcal{J}(\Upsilon^*,\Lambda^*),\tag{46}$$

which exhibits that (Λ^*,Υ^*) is a coupled coincidence point of $\mathcal{J}$, $\hbar$. Hence, $\hbar(\Lambda^*)=\hbar\hbar$ and $\hbar(\Upsilon^*)=\hbar\mathcal{J}$ which in turn gives that $\hbar(\Lambda)=\Lambda^*$ and $\hbar(\Upsilon^*)=\Upsilon^*$. Therefore from (46), (Λ^*, Υ^*) is a coupled common fixed point of $\mathscr{J}$, $\hbar$.

Let $(\Lambda_1^*, \Upsilon_1^*)$ be another coupled common fixed point of $\mathscr{J}$, $\hbar$. Thence, $\Lambda_1^* = \hbar\Lambda_1^* = \mathscr{J}(\Lambda_1^*, \Upsilon_1^*)$ and $\Upsilon_1^* = \hbar\Upsilon_1^* = \mathscr{J}(\Upsilon_1^*, \Lambda_1^*)$. But $(\Lambda_1^*, \Upsilon_1^*)$ is a coupled common fixed point of $\mathscr{J}$, $\hbar$ then, $\hbar\Lambda_1^* = \hbar\varepsilon = \Lambda$ and $\hbar\Upsilon_1^* = \hbar\wp = \Upsilon^*$. Therefore, $\Lambda_1^* = \hbar\Lambda_1^* = \hbar\Lambda = \Lambda$ and $\Upsilon_1^* = \hbar\Upsilon_1^* = \hbar\Upsilon^* = \Upsilon^*$. Hence the uniqueness.

Theorem 312*In Theorem 311, if $\hbar\varepsilon_0 \preceq \hbar\wp_0$ or $\hbar\varepsilon_0 \succeq \hbar\wp_0$, then an unique common fixed point of $\mathscr{J}$, $\hbar$ can be found.*

*Proof.*Assume $(\varepsilon, \wp) \in \mathscr{E}$ is a unique coupled common fixed point of $\mathscr{J}$, $\hbar$. Then, to demonstrate that $\varepsilon = \wp$. Suppose that $\hbar\varepsilon_0 \preceq \hbar\wp_0$ then we get by induction, $\hbar\varepsilon_n \preceq \hbar\wp_n$, $n \geq 0$. From Lemma 2 of [23], we have

$$\begin{aligned}
\breve{\psi}(s^{k-2}\eth(\varepsilon,\wp)) &= \breve{\psi}(s^k\frac{1}{s^2}\eth(\varepsilon,\wp)) \\
&\leq \lim_{n\to+\infty}\sup\breve{\psi}(s^k\eth(\varepsilon_{n+1},\wp_{n+1})) \\
&= \lim_{n\to+\infty}\sup\breve{\psi}(s^k\eth(\mathscr{J}(\varepsilon_n,\wp_n),\mathscr{J}(\wp_n,\varepsilon_n))) \\
&\leq \lim_{n\to+\infty}\sup\breve{\psi}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) \\
&\quad - \lim_{n\to+\infty}\inf\hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) \\
&\leq \breve{\psi}(\eth(\varepsilon,\wp)) \\
&\quad - \lim_{n\to+\infty}\inf\hat{\eta}(\mathscr{P}_\hbar(\varepsilon_n,\wp_n,\wp_n,\varepsilon_n)) \\
&< \breve{\psi}(\eth(\varepsilon,\wp)),
\end{aligned}$$

a contradiction. Hence, $\varepsilon = \wp$.

The result can also see in the case of $\hbar\varepsilon_0 \succeq \hbar\wp_0$.

Remark 313*While $s = 1$ and the result of [21], the condition*

$$\begin{aligned}
&\breve{\psi}(\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\eth,\Im))) \\
&\quad \leq \breve{\psi}(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\}) \\
&\quad - \hat{\eta}(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\})
\end{aligned}$$

is equivalent to,

$$\eth(\mathscr{J}(\varepsilon,\wp),\mathscr{J}(\eth,\Im)) \leq \varphi(\max\{\eth(\hbar\varepsilon,\hbar\eth),\eth(\hbar\wp,\hbar\Im)\}),$$

here $\breve{\psi} \in \breve{\psi}$, $\hat{\eta} \in \hat{\eta}$ and φ is a continuous self map on $[0,+\infty)$ with $\varphi(y) < y$ for every $y > 0$ with $\varphi(y) = 0$ iff $y = 0$. Hence the results found here are generalized and extended the results of [11, 18, 25, 26, 27] and a number of comparable results.

Now, depending on the type of metric, some examples are shown.

Example 314*Let $\mathscr{E} = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}\}$ and $\eth : \mathscr{E}\times\mathscr{E} \to \mathscr{E}$ be a metric defined by*

$$\begin{aligned}
&\eth(\varepsilon,\wp) = \eth(\wp,\varepsilon) = 0, \\
&\quad \text{if } \varepsilon = \wp = \{e_{11},e_{22},e_{33},e_{44},e_{55},e_{66}\} \text{ and } \varepsilon = \wp, \\
&\eth(\varepsilon,\wp) = \eth(\wp,\varepsilon) = 3, \\
&\quad \text{if } \varepsilon = \wp = \{e_{11},e_{22},e_{33},e_{44},e_{55}\} \text{ and } \varepsilon \neq \wp, \\
&\eth(\varepsilon,\wp) = \eth(\wp,\varepsilon) = 12, \\
&\quad \text{if } \varepsilon = \{e_{11},e_{22},e_{33},e_{44}\} \text{ and } \wp = e_{66}, \\
&\eth(\varepsilon,\wp) = \eth(\wp,\varepsilon) = 20, \text{ if } \varepsilon = e_{55} \text{ and } \wp = e_{66}, \\
&\quad \text{with usual order } \leq .
\end{aligned}$$

A self-map $\mathscr{J}$ on $\mathscr{E}$ defined by $\mathscr{J}e_{11} = \mathscr{J}e_{22} = \mathscr{J}e_{33} = \mathscr{J}e_{44} = \mathscr{J}e_{55} = 1, \mathscr{J}e_{66} = 2$ has a fixed point with $\breve{\psi}(y) = \frac{y}{2}$, $\hat{\eta}(y) = \frac{y}{4}$ where $y \in [0,+\infty)$.

*Proof.*When $s = 2$, $(\mathscr{E}, \eth, \leq)$ is a complete partially ordered b-metric space. Let $\varepsilon, \wp \in \mathscr{E}$ such that $\varepsilon < \wp$ then we'll look at the cases below.

Case 1. If $\varepsilon, \wp \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$ then $\eth(\mathscr{J}\varepsilon, \mathscr{J}\wp) = \eth(e_{11}, e_{11}) = 0$. Hence,

$$\breve{\psi}(2\eth(\mathscr{J}\varepsilon,\mathscr{J}\wp)) = 0 \leq \breve{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

Case 2. If $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$ and $\wp = e_{66}$, then $\eth(\mathscr{J}\varepsilon, \mathscr{J}\wp) = \eth(e_{11}, e_{22}) = 3$, $\mathscr{P}(e_{66}, e_{55}) = 20$ and $\mathscr{P}(\varepsilon, e_{66}) = 12$, for $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}\}$. Hence,

$$\breve{\psi}(2\eth(\mathscr{J}\varepsilon,\mathscr{J}\wp)) \leq \frac{\mathscr{P}(\varepsilon,\wp)}{4} = \breve{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).$$

As a result, all of the conditions of Theorem 31 are met, and $\mathscr{J}$ has a fixed point.

Example 315*Let us define a metric $\eth$ with usual order $\leq$ by*

$$\eth(\varepsilon,\wp) = \begin{cases} 0 & , \text{ if } \varepsilon = \wp \\ 1 & , \text{ if } \varepsilon \neq \wp \in \{0,1\} \\ |\varepsilon - \wp| & , \text{ if } \varepsilon,\wp \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\} \\ 6 & , \text{ otherwise.} \end{cases}$$

where $\mathscr{E} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\}$. A self-map $\mathscr{J}$ on $\mathscr{E}$ by $\mathscr{J}0 = 0, \mathscr{J}\frac{1}{n} = \frac{1}{12n}(n \geq 1)$ has a fixed point with $\breve{\psi}(y) = y$, $\hat{\eta}(y) = \frac{4y}{5}$ for $y \in [0,+\infty)$.

Proof.$\eth$ is clearly discontinuous, and $(\mathscr{E}, \eth, \leq)$ is a complete partially ordered b-metric space for $s = \frac{12}{5}$. Now we'll look at the cases for $\varepsilon, \wp \in \mathscr{E}$ with $\varepsilon < \wp$.

Case 1. Suppose $\varepsilon = 0$ and $\wp = \frac{1}{n}$ $(n > 0)$, then $\eth(\mathscr{J}\varepsilon, \mathscr{J}\wp) = \eth(0, \frac{1}{12n}) = \frac{1}{12n}$ and $\mathscr{P}(\varepsilon, \wp) = \frac{1}{n}$ and $\mathscr{P}(\varepsilon, \wp) = \{1, 6\}$. Thus,

$$\begin{aligned}
\breve{\psi}\left(\frac{12}{5}\eth(\mathscr{J}\varepsilon,\mathscr{J}\wp)\right) &\leq \frac{\mathscr{P}(\varepsilon,\wp)}{5} \\
&= \breve{\psi}(\mathscr{P}(\varepsilon,\wp)) - \hat{\eta}(\mathscr{P}(\varepsilon,\wp)).
\end{aligned}$$

Case 2. Let $\varepsilon = \frac{1}{m}$ and $\wp = \frac{1}{n}$ where $m > n \geq 1$, thence

$$\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp) = \eth(\frac{1}{12m}, \frac{1}{12n}) \text{ and}$$
$$\mathscr{P}(\varepsilon, \wp) \geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathscr{P}(\varepsilon, \wp) = 6.$$

Thus,

$$\breve{\psi}\left(\frac{12}{5}\eth(\mathscr{I}\varepsilon, \mathscr{I}\wp)\right) \leq \frac{\mathscr{P}(\varepsilon, \wp)}{5}$$
$$= \breve{\psi}(\mathscr{P}(\varepsilon, \wp)) - \hat{\eta}(\mathscr{P}(\varepsilon, \wp)).$$

Hence, we have the conclusion from Theorem 31 as all assumptions are fulfilled.

Example 316*Define a metric $\eth : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$, where $\mathscr{E} = \{\tilde{\ell}/\tilde{\ell} : [a_1, a_2] \to [a_1, a_2]$ continuous$\}$ by*

$$\eth(\tilde{\ell}_1, \tilde{\ell}_2) = \sup_{y \in [a_1, a_2]} \{|\tilde{\ell}_1(y) - \tilde{\ell}_2(y)|^2\}$$

for any $\tilde{\ell}_1, \tilde{\ell}_2 \in \mathscr{E}$, $0 \leq a_1 < a_2$ with $\tilde{\ell}_1 \preceq \tilde{\ell}_2$ implies $a_1 \leq \tilde{\ell}_1(y) \leq \tilde{\ell}_2(y) \leq a_2, y \in [a_1, a_2]$. A self-map $\mathscr{I}$ on $\mathscr{E}$ defined by $\mathscr{I}\tilde{\ell} = \frac{\tilde{\ell}}{5}, \tilde{\ell} \in \mathscr{E}$ has a unique fixed point with $\breve{\psi}(y) = y$, $\hat{\eta}(y) = \frac{y}{3}$, for any $y \in [0, +\infty]$.

*Proof.*As $\min(\tilde{\ell}_1, \tilde{\ell}_2)(y) = \min\{\tilde{\ell}_1(y), \tilde{\ell}_2(y)\}$ is continuous and all other assumptions of Theorem 33 are fulfilled for $s = 2$. Therefore, $0 \in \mathscr{E}$ is an unique fixed point $\mathscr{I}$.

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The authors declare that they have no competing interests.

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