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# Fixed Point Results of Contractive Mappings with Altering Distance Functions in Ordered $b$ -Metric Spaces

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**Abstract:** We explore the existence of a fixed point as well as the uniqueness of a mapping in an ordered  $b$ -metric space using a generalized  $(\psi, \hat{\eta})$ -weak contraction. In addition, some results are posed on a coincidence point and a coupled coincidence point of two mappings under the same contraction condition. These findings generalize and build on a few recent studies in the literature. At the end, we provided some examples to back up our findings.

**Keywords:**  $(\psi, \hat{\eta})$ -weak contraction, fixed points, coincidence and coupled coincidence points, ordered  $b$ -metric space.

## 1 Introduction

In a wide range of pure and applied mathematics problems, fixed points of mappings that satisfy contractive conditions in extended metric spaces are extremely useful. First, Ran and Reuings [31] described the existence of fixed points in this direction for certain maps in ordered metric space and exhibited matrix linear equations applications. Following that, Nieto et al. [28, 29] expanded the result of [31] to nondecreasing mappings and used their findings to obtain differential equations solutions. Agarwal et al. [4] and O'Regan et al. [30] examine the influence of generalized contractions in ordered spaces at the same time. Bhaskar and Lakshmikantham [11] first developed coupled fixed point theory for some maps, then used the results to find a unique solution to periodic boundary value problems. Following that, Lakshmikantham and Ćirić [25], which were the extensions of [11] involving monotone property to a function in the space, pioneered the idea of coupled coincidence, common fixed point results. [15, 16, 17, 19, 21, 24, 26, 35, 36, 37, 38] provide additional information on coupled fixed point effects in various spaces under various contractive conditions.

A  $b$ -metric space is one of several generalizations of a standard metric space proposed by Bakhtin in his work [9], and widely used by Czerwik in his work [13, 14].

Following that, a lot of progress was made in acquiring the results of fixed points to single valued as well as multi-valued operators in the space, as evidenced by [1, 2, 3, 5, 6, 7, 8, 10, 18, 20, 22, 23, 27, 32, 33, 34, 39].

We demonstrate some fixed points results for mappings in ordered  $b$ -metric space that satisfy a generalized weak contraction in this paper. The results from [10, 11, 12, 19, 21, 25, 34] are expanded here as well as some examples noted to support the findings at the end of our work.

## 2 Preliminaries

**Definition 21**[14] A  $b$ -metric is a map  $\bar{d} : \mathcal{E} \times \mathcal{E} \rightarrow [0, +\infty)$  that satisfies the properties below in  $\mathcal{E}$  for all  $\varepsilon, \wp, \zeta$  and some  $s \geq 1$ ,

- (a).  $\bar{d}(\varepsilon, \wp) = 0$  if and if  $\varepsilon = \wp$ .
- (b).  $\bar{d}(\varepsilon, \wp) = \bar{d}(\wp, \varepsilon)$ .
- (c).  $\bar{d}(\varepsilon, \wp) \leq s(\bar{d}(\varepsilon, \zeta) + \bar{d}(\zeta, \wp))$ .

A  $b$ -metric space is specified as  $(\mathcal{E}, \bar{d}, s)$ .

**Definition 22**[10, 14] In a  $b$ -metric space,

- (1).if  $\bar{d}(\varepsilon_n, \varepsilon) \rightarrow 0$  as  $n \rightarrow +\infty$ ,  $\{\varepsilon_n\}$  is said to be convergent to  $\varepsilon$ .

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- (2).if  $\bar{d}(\epsilon_n, \epsilon_m) \rightarrow 0$  is the same as  $n, m \rightarrow +\infty$ , then  $\{\epsilon_n\}$  is a Cauchy sequence.
- (3).if  $(\mathcal{E}, \bar{d}, s)$  is a complete b-metric space, then any Cauchy sequence is convergent.

**Definition 23**[14, 34, 38] If  $\mathcal{E}$  is a partial ordered set with respect to an ordered relation  $\preceq$  and  $\bar{d}$  is a metric on it, then  $(\mathcal{E}, \bar{d}, \preceq)$  is a partially ordered metric space.  $(\mathcal{E}, \bar{d}, \preceq)$  is a complete partially ordered b-metric space, despite the fact that  $\bar{d}$  is complete.

**Definition 24**[34, 38] If  $\mathcal{h}(\epsilon) \preceq \mathcal{h}(\wp)$  for all  $\epsilon, \wp \in \mathcal{E}$  with  $\epsilon \preceq \wp$ , the map is called a monotone non-decreasing.

**Definition 25**[12] Let  $\mathcal{h}, \mathcal{F} : \mathcal{A} \rightarrow \mathcal{A}$  be two mappings, and  $\mathcal{A} \neq \emptyset \subseteq \mathcal{E}$  be one. If  $\mathcal{h}\epsilon = \mathcal{F}\epsilon = \epsilon$  ( $\mathcal{h}\epsilon = \mathcal{F}\epsilon$ ) for  $\epsilon \in \mathcal{A}$ , then  $\epsilon$  is a common fixed point (coincidence point) of  $\mathcal{h}, \mathcal{F}$ .

**Definition 26**[12] If  $\mathcal{h}\mathcal{F}\epsilon = \mathcal{F}\mathcal{h}\epsilon$  for all  $\epsilon \in \mathcal{A}$ , then  $\mathcal{h}$  and  $\mathcal{F}$  are commuting.

**Definition 27**[12, 34] The two maps  $\mathcal{h}, \mathcal{F}$  are compatible if  $\lim_{n \rightarrow +\infty} d(\mathcal{F}\mathcal{h}\epsilon_n, \mathcal{h}\mathcal{F}\epsilon_n) = 0$  for each sequence  $\{\epsilon_n\} \subseteq \mathcal{E}$  so that  $\lim_{n \rightarrow +\infty} \mathcal{h}\epsilon_n = \lim_{n \rightarrow +\infty} \mathcal{F}\epsilon_n = \mu$ , for some  $\mu \in \mathcal{A}$ .

**Definition 28**[12, 34] If  $\mathcal{h}\epsilon = \mathcal{F}\epsilon$  for  $\epsilon \in \mathcal{A}$ , then  $\mathcal{h}\mathcal{F}\epsilon = \mathcal{F}\mathcal{h}\epsilon$ , the mappings  $\mathcal{h}$  and  $\mathcal{F}$  are weakly compatible.

**Definition 29**[34] If  $\mathcal{h}\epsilon \preceq \mathcal{h}\wp$  implies  $\mathcal{F}\epsilon \preceq \mathcal{F}\wp$  for any  $\epsilon, \wp \in \mathcal{E}$ , then a map  $\mathcal{F}$  is a monotone  $\mathcal{h}$ -non-decreasing.

**Definition 210**[11] Let  $\mathcal{F} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  and  $\mathcal{h} : \mathcal{E} \rightarrow \mathcal{E}$  are two mappings,

- (a).a point  $(\epsilon, \wp) \in \mathcal{E} \times \mathcal{E}$  is coupled coincidence point of  $\mathcal{F}, \mathcal{h}$  if  $\mathcal{F}(\epsilon, \wp) = \mathcal{h}\epsilon$  and,  $\mathcal{F}(\wp, \epsilon) = \mathcal{h}\wp$ . In particular, if  $\mathcal{h}$  is an identity map, then  $(\epsilon, \wp)$  is a coupled fixed point of  $\mathcal{F}$ .
- (b).an element  $\epsilon \in \mathcal{E}$  is a common fixed point of  $\mathcal{F}, \mathcal{h}$  if  $\mathcal{F}(\epsilon, \epsilon) = \mathcal{h}\epsilon = \epsilon$ .
- (c).if  $\mathcal{F}(\mathcal{h}\epsilon, \mathcal{h}\wp) = \mathcal{h}(\mathcal{F}\epsilon, \mathcal{F}\wp)$  for all  $\epsilon, \wp \in \mathcal{E}$ , then  $\mathcal{F}$  and  $\mathcal{h}$  are commuting each other.
- (d).If any two elements in a set  $\mathcal{A} \subseteq \mathcal{E}$  are comparable, the set is well ordered.

**Definition 211**A self map  $\check{\Psi}$  on  $[0, +\infty)$  that meets the conditions below is known as an altering distance function:

- (a). $\check{\Psi}$  is a non-decreasing and continuous function.
- (b). $\check{\Psi}(l) = 0$  iff  $l = 0$ .

As seen above, the symbol  $\hat{\Phi}$  represents the set of all altering distance functions.

Similarly,  
 $\hat{\Psi} : \{\hat{\eta} | \hat{\eta}$  is a lower semi – continuous self mapping on  $[0, +\infty)$  and,  $\hat{\eta}(l) = 0$  iff  $l = 0\}$ .

**Lemma 212**[27] Let  $\mathcal{h} : \mathcal{E} \rightarrow \mathcal{E}$  be a mapping, and  $\mathcal{E} \neq \emptyset$ . Then  $\mathcal{M} \subseteq \mathcal{E}$  occurs, resulting in  $\mathcal{h}\mathcal{M} = \mathcal{h}\mathcal{E}$ , where  $\mathcal{h} : \mathcal{M} \rightarrow \mathcal{E}$  is one-to-one.

**Lemma 213**[2] Let  $\{\epsilon_n\}$  and  $\{\wp_n\}$  be two sequences and b-convergent to  $\epsilon$  and  $\wp$  in a b-metric space  $(\mathcal{E}, \bar{d}, s, \preceq)$ , where  $s > 1$ . Then

$$\frac{1}{s^2} \bar{d}(\epsilon, \wp) \leq \liminf_{n \rightarrow +\infty} \bar{d}(\epsilon_n, \wp_n) \leq \limsup_{n \rightarrow +\infty} \bar{d}(\epsilon_n, \wp_n) \leq s^2 \bar{d}(\epsilon, \wp).$$

In particular, if  $\epsilon = \wp$ , then  $\lim_{n \rightarrow +\infty} \bar{d}(\epsilon_n, \wp_n) = 0$ . In addition, for every  $\tau \in \mathcal{E}$ , we get

$$\frac{1}{s} \bar{d}(\epsilon, \tau) \leq \liminf_{n \rightarrow +\infty} \bar{d}(\epsilon_n, \tau) \leq \limsup_{n \rightarrow +\infty} \bar{d}(\epsilon_n, \tau) \leq s \bar{d}(\epsilon, \tau).$$

### 3 Main Results

Let's get started with the theorem below.

**Theorem 31**Suppose  $(\mathcal{E}, \bar{d}, s, \preceq)$  is a complete partially ordered b-metric space with  $s \geq 1$ . A map  $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{E}$  is non-decreasing and continuous with respect to  $\preceq$ . If  $\epsilon_0 \in \mathcal{E}$  is such that  $\epsilon_0 \preceq \mathcal{F}\epsilon_0$  and the following contraction condition is fulfilled, then  $\mathcal{F}$  has a fixed point in  $\mathcal{E}$ .

$$\check{\Psi}(s\bar{d}(\mathcal{F}\epsilon, \mathcal{F}\wp)) \leq \check{\Psi}(\mathcal{P}(\epsilon, \wp)) - \hat{\eta}(\mathcal{P}(\epsilon, \wp)) \quad (1)$$

for  $\check{\Psi} \in \hat{\Phi}, \hat{\eta} \in \hat{\Psi}$  and for any  $\epsilon, \wp \in \mathcal{E}$  so that  $\epsilon \preceq \wp$  and

$$\mathcal{P}(\epsilon, \wp) = \max\left\{ \frac{\bar{d}(\wp, \mathcal{F}\wp) [1 + \bar{d}(\epsilon, \mathcal{F}\epsilon)]}{1 + \bar{d}(\epsilon, \wp)}, \frac{\bar{d}(\epsilon, \mathcal{F}\epsilon) \bar{d}(\wp, \mathcal{F}\wp)}{1 + \bar{d}(\epsilon, \wp)}, \frac{\bar{d}(\epsilon, \mathcal{F}\epsilon) \bar{d}(\wp, \mathcal{F}\wp)}{1 + \bar{d}(\mathcal{F}\epsilon, \mathcal{F}\wp)}, \bar{d}(\epsilon, \wp) \right\}.$$

*Proof.*For some  $\epsilon_0 \in \mathcal{E}$  with  $\mathcal{F}\epsilon_0 = \epsilon_0$ , then the result is trivial. Assuming that  $\epsilon_0 \prec \mathcal{F}\epsilon_0$ , we describe a sequence  $\{\epsilon_n\} \subset \mathcal{E}$  by  $\epsilon_{n+1} = \mathcal{F}\epsilon_n$  for all  $n \geq 0$ . However, we can deduce the following as  $\mathcal{F}$  is non-decreasing,

$$\epsilon_0 \prec \mathcal{F}\epsilon_0 = \epsilon_1 \preceq \mathcal{F}\epsilon_1 = \epsilon_2 \preceq \dots \preceq \mathcal{F}\epsilon_{n-1} = \epsilon_n \preceq \mathcal{F}\epsilon_n = \epsilon_{n+1} \preceq \dots \quad (3)$$

If  $\epsilon_n = \epsilon_{n+1}$  for  $n_0 \in \mathbb{N}$ , then  $\epsilon_{n_0}$  is a fixed point of  $\mathcal{F}$  from (3). Otherwise, for all  $n \geq 1, \epsilon_n \neq \epsilon_{n+1}$ . For  $n \geq 1$ , let  $D_n = \bar{d}(\epsilon_{n+1}, \epsilon_n)$  be used. We know that for every  $n \geq 1, \epsilon_{n-1} \prec \epsilon_n$  and, the equation (1) becomes

$$\check{\Psi}(D_n) = \check{\Psi}(\bar{d}(\epsilon_n, \epsilon_{n+1})) = \check{\Psi}(\bar{d}(\mathcal{F}\epsilon_{n-1}, \mathcal{F}\epsilon_n)) \leq \check{\Psi}(s\bar{d}(\mathcal{F}\epsilon_{n-1}, \mathcal{F}\epsilon_n)) \leq \check{\Psi}(\mathcal{P}(\epsilon_{n-1}, \epsilon_n)) - \hat{\eta}(\mathcal{P}(\epsilon_{n-1}, \epsilon_n)). \quad (4)$$

From (4), we get

$$\bar{d}(\epsilon_n, \epsilon_{n+1}) = \bar{d}(\mathcal{F}\epsilon_{n-1}, \mathcal{F}\epsilon_n) \leq \frac{1}{s} \mathcal{P}(\epsilon_{n-1}, \epsilon_n), \quad (5)$$

where

$$\begin{aligned} \mathcal{P}(\varepsilon_{n-1}, \varepsilon_n) &= \max\left\{\frac{\bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n)[1 + \bar{\vartheta}(\varepsilon_{n-1}, \mathcal{F}\varepsilon_{n-1})]}{1 + \bar{\vartheta}(\varepsilon_{n-1}, \varepsilon_n)}, \right. \\ &\quad \frac{\bar{\vartheta}(\varepsilon_{n-1}, \mathcal{F}\varepsilon_{n-1}) \bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n)}{1 + \bar{\vartheta}(\varepsilon_{n-1}, \varepsilon_n)}, \\ &\quad \left. \frac{\bar{\vartheta}(\varepsilon_{n-1}, \mathcal{F}\varepsilon_{n-1}) \bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n)}{1 + \bar{\vartheta}(\mathcal{F}\varepsilon_{n-1}, \mathcal{F}\varepsilon_n)}, \bar{\vartheta}(\varepsilon_{n-1}, \varepsilon_n)\right\} \\ &\leq \max\{\bar{\vartheta}(\varepsilon_n, \varepsilon_{n+1}), \bar{\vartheta}(\varepsilon_{n-1}, \varepsilon_n)\} \\ &\leq \max\{D_n, D_{n-1}\}. \end{aligned} \tag{6}$$

If  $\max\{D_n, D_{n-1}\} = D_n$  for certain  $n \geq 1$ , equation (5) is then accompanied by

$$\bar{\vartheta}(\varepsilon_n, \varepsilon_{n+1}) \leq \frac{1}{\beta} \bar{\vartheta}(\varepsilon_n, \varepsilon_{n+1}),$$

this is a contradiction. Thus,  $\max\{D_n, D_{n-1}\} = D_{n-1}$  for  $n \geq 1$ . Hence, equation (5) becomes

$$\bar{\vartheta}(\varepsilon_n, \varepsilon_{n+1}) \leq \frac{1}{\beta} \bar{\vartheta}(\varepsilon_n, \varepsilon_{n-1}),$$

Since  $\frac{1}{\beta} \in (0, 1)$  then  $\{\varepsilon_n\}$  is a Cauchy sequence from [1, 6, 8, 18]. Also, the completeness of  $\mathcal{E}$  gives that  $\varepsilon_n \rightarrow \mu \in \mathcal{E}$ .

We may also deduce the following from  $\mathcal{F}$ 's continuity:

$$\mathcal{F}\mu = \mathcal{F}\left(\lim_{n \rightarrow +\infty} \varepsilon_n\right) = \lim_{n \rightarrow +\infty} \mathcal{F}\varepsilon_n = \lim_{n \rightarrow +\infty} \varepsilon_{n+1} = \mu. \tag{7}$$

As a result,  $\mathcal{F}$  in  $\mathcal{E}$  has a fixed point  $\mu$ .

The continuity assumption on  $\mathcal{F}$  is extracted from Theorem 31 and used to derive the following theorem.

**Theorem 32** *In Theorem 31, if  $\mathcal{E}$  satisfies below condition, then  $\mathcal{F}$  has a fixed point.*

*If a non-decreasing sequence  $\{\varepsilon_n\} \subseteq \mathcal{E}$  and  $\varepsilon_n \rightarrow \sigma$  then  $\varepsilon_n \leq \sigma$ , for each  $n \in \mathbb{N}$ , i.e.,  $\sigma = \sup \varepsilon_n$ .* (8)

*Proof.* We have an increasing sequence  $\{\varepsilon_n\} \subseteq \mathcal{E}$  that eventually converges to some  $\sigma \in \mathcal{E}$  as a result of Theorem 31. But by the hypotheses for all  $n$ ,  $\varepsilon_n \leq \sigma$ , which means that  $\sigma = \sup \varepsilon_n$ .

We can now assert that  $\sigma$  is a fixed point of  $\mathcal{F}$ . Assume  $\mathcal{F}\sigma \neq \sigma$  is not true. Let

$$\begin{aligned} \mathcal{P}(\varepsilon_n, \sigma) &= \max\left\{\frac{\bar{\vartheta}(\sigma, \mathcal{F}\sigma)[1 + \bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n)]}{1 + \bar{\vartheta}(\varepsilon_n, \sigma)}, \right. \\ &\quad \frac{\bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n) \bar{\vartheta}(\sigma, \mathcal{F}\sigma)}{1 + \bar{\vartheta}(\varepsilon_n, \sigma)}, \\ &\quad \left. \frac{\bar{\vartheta}(\varepsilon_n, \mathcal{F}\varepsilon_n) \bar{\vartheta}(\sigma, \mathcal{F}\sigma)}{1 + \bar{\vartheta}(\mathcal{F}\varepsilon_n, \mathcal{F}\sigma)}, \bar{\vartheta}(\varepsilon_n, \sigma)\right\}, \end{aligned} \tag{9}$$

then taking limit as  $n \rightarrow +\infty$  in the equation (9) and making use of  $\lim_{n \rightarrow +\infty} \varepsilon_n = \sigma$ , we get

$$\lim_{n \rightarrow +\infty} \mathcal{P}(\varepsilon_n, \sigma) = \max\{\bar{\vartheta}(\sigma, \mathcal{F}\sigma), 0\} = \bar{\vartheta}(\sigma, \mathcal{F}\sigma), \tag{10}$$

Since,  $\varepsilon_n \leq \sigma$  for each  $n$ , then we obtain the following from equations (1) and (9)

$$\begin{aligned} \check{\Psi}(\bar{\vartheta}(\varepsilon_{n+1}, \mathcal{F}\sigma)) &= \check{\Psi}(\bar{\vartheta}(\mathcal{F}\varepsilon_n, \mathcal{F}\sigma)) \leq \check{\Psi}(s\bar{\vartheta}(\mathcal{F}\varepsilon_n, \mathcal{F}\sigma)) \\ &\leq \check{\Psi}(\mathcal{P}(\varepsilon_n, \sigma)) - \hat{\eta}(\mathcal{P}(\varepsilon_n, \sigma)). \end{aligned} \tag{11}$$

Take limit as  $n \rightarrow +\infty$  in (11) and from equation (10) as well as the properties of  $\check{\Psi}$ ,  $\hat{\eta}$ , we have

$$\begin{aligned} \check{\Psi}(\bar{\vartheta}(\sigma, \mathcal{F}\sigma)) &\leq \check{\Psi}(\bar{\vartheta}(\sigma, \mathcal{F}\sigma)) - \hat{\eta}(\bar{\vartheta}(\sigma, \mathcal{F}\sigma)) \\ &< \check{\Psi}(\bar{\vartheta}(\sigma, \mathcal{F}\sigma)). \end{aligned} \tag{12}$$

This is a contradiction to  $\mathcal{F}\sigma \neq \sigma$ . Hence,  $\mathcal{F}\sigma = \sigma$ .

In the above theorems, the fixed point is unique if  $\mathcal{E}$  meets the following condition.

There is an  $\sigma$  in  $\mathcal{E}$  that is comparable to  $\varepsilon$  and  $\rho$  for each  $\varepsilon, \rho \in \mathcal{E}$ . (13)

**Theorem 33** *If  $\mathcal{E}$  assumes the condition (13) in Theorem 31 & 32, then  $\mathcal{F}$  has a unique fixed point in  $\mathcal{E}$ .*

*Proof.* Theorems 31 & 32 show that the set of fixed points of  $\mathcal{F}$  is nonempty. Assume  $\varepsilon^* \neq \rho^*$  are fixed points of  $\mathcal{F}$  to ensure uniqueness. Following that,

$$\begin{aligned} \check{\Psi}(\bar{\vartheta}(\mathcal{F}\varepsilon^*, \mathcal{F}\rho^*)) &\leq \check{\Psi}(s\bar{\vartheta}(\mathcal{F}\varepsilon^*, \mathcal{F}\rho^*)) \\ &\leq \check{\Psi}(\mathcal{P}(\varepsilon^*, \rho^*)) - \hat{\eta}(\mathcal{P}(\varepsilon^*, \rho^*)) \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathcal{P}(\varepsilon^*, \rho^*) &= \max\left\{\frac{\bar{\vartheta}(\rho^*, \mathcal{F}\rho^*)[1 + \bar{\vartheta}(\varepsilon^*, \mathcal{F}\varepsilon^*)]}{1 + \bar{\vartheta}(\varepsilon^*, \rho^*)}, \right. \\ &\quad \frac{\bar{\vartheta}(\varepsilon^*, \mathcal{F}\varepsilon^*) \bar{\vartheta}(\rho^*, \mathcal{F}\rho^*)}{1 + \bar{\vartheta}(\varepsilon^*, \rho^*)}, \\ &\quad \left. \frac{\bar{\vartheta}(\varepsilon^*, \mathcal{F}\varepsilon^*) \bar{\vartheta}(\rho^*, \mathcal{F}\rho^*)}{1 + \bar{\vartheta}(\mathcal{F}\varepsilon^*, \mathcal{F}\rho^*)}, \bar{\vartheta}(\varepsilon^*, \rho^*)\right\}. \end{aligned} \tag{15}$$

Therefore, from equations (14) and (15), we have

$$\begin{aligned} \check{\Psi}(\bar{\vartheta}(\varepsilon^*, \rho^*)) &= \check{\Psi}(\bar{\vartheta}(\mathcal{F}\varepsilon^*, \mathcal{F}\rho^*)) \\ &\leq \check{\Psi}(\bar{\vartheta}(\varepsilon^*, \rho^*)) - \hat{\eta}(\bar{\vartheta}(\varepsilon^*, \rho^*)) \\ &< \check{\Psi}(\bar{\vartheta}(\varepsilon^*, \rho^*)), \end{aligned} \tag{16}$$

this contradicts to  $\varepsilon^* \neq \rho^*$ . Hence,  $\varepsilon^* = \rho^*$ .

Now, we have the below corollary from Theorems 31 to 33.

**Corollary 34** *Let  $(\mathcal{E}, \bar{\vartheta}, \leq)$  be a partially ordered b-metric space. Suppose the mappings  $\mathcal{F}, \mathcal{h} : \mathcal{E} \rightarrow \mathcal{E}$  are continuous such that*

(C<sub>1</sub>).

$$\Psi(s\bar{d}(\mathcal{F}\varepsilon, \mathcal{F}\rho)) \leq \Psi(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho)) - \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho)) \tag{17}$$

for every  $\varepsilon, \rho \in \mathcal{E}$  with  $\mathcal{H}\varepsilon \preceq \mathcal{H}\rho, s > 1, \Psi \in \hat{\Phi}, \hat{\eta} \in \hat{\Psi}$  and, where

$$\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho) = \max\left\{ \frac{\bar{d}(\mathcal{H}\rho, \mathcal{F}\rho)[1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{F}\varepsilon)]}{1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho)}, \frac{\bar{d}(\mathcal{H}\varepsilon, \mathcal{F}\varepsilon) \bar{d}(\mathcal{H}\rho, \mathcal{F}\rho)}{1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho)}, \frac{\bar{d}(\mathcal{H}\varepsilon, \mathcal{F}\varepsilon) \bar{d}(\mathcal{H}\rho, \mathcal{F}\rho)}{1 + \bar{d}(\mathcal{F}\varepsilon, \mathcal{F}\rho)}, \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho) \right\}. \tag{18}$$

(C<sub>2</sub>).  $\mathcal{F}\mathcal{E} \subset \mathcal{H}\mathcal{E}$  and  $\mathcal{H}\mathcal{E} \subseteq \mathcal{E}$  is complete,

(C<sub>3</sub>).  $\mathcal{F}$  is monotone  $\mathcal{H}$ -non-decreasing and

(C<sub>4</sub>).  $\mathcal{F}$  and  $\mathcal{H}$  are compatible.

If for some  $\varepsilon_0 \in \mathcal{E}$  such that  $\mathcal{H}\varepsilon_0 \preceq \mathcal{F}\varepsilon_0$ , then there is a coincidence point in  $\mathcal{E}$  for a pair of mappings  $(\mathcal{F}, \mathcal{H})$ .

*Proof.* According to lemma 212, there is a subset  $\mathcal{M}$  of  $\mathcal{E}$  so that  $\mathcal{H}\mathcal{M} \subset \mathcal{E}$  is a complete subspace, and  $\mathcal{H} : \mathcal{E} \rightarrow \mathcal{E}$  is one-to-one. Following [27]’s Corollary 2.1, there is a sequence  $\{\mathcal{H}\varepsilon_n\} \subset \mathcal{H}\mathcal{M}$  for some  $\varepsilon_0 \in \mathcal{M}$  so that  $\mathcal{H}\varepsilon_{n+1} = \mathcal{F}\varepsilon_n = \mathcal{H}(\mathcal{H}\varepsilon_n)$ , ( $n \geq 0$ ) and, where  $\mathcal{H} : \mathcal{H}\mathcal{M} \rightarrow \mathcal{H}\mathcal{M}$  is a mapping so that  $\mathcal{H}(\mathcal{H}\varepsilon) = \mathcal{F}\varepsilon, \varepsilon \in \mathcal{M}$ .

Thus from equation (17), we get

$$\Psi(s\bar{d}(\mathcal{H}(\mathcal{H}\varepsilon), \mathcal{H}(\mathcal{H}\rho))) \leq \Psi(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho)) - \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho)), \tag{19}$$

for every  $\varepsilon, \rho \in \mathcal{E}$  with  $\mathcal{H}\varepsilon \preceq \mathcal{H}\rho$  and, where

$$\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho) = \max\left\{ \frac{\bar{d}(\mathcal{H}\rho, \mathcal{H}(\mathcal{H}\rho))[1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}(\mathcal{H}\varepsilon))]}{1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho)}, \frac{\bar{d}(\mathcal{H}\varepsilon, \mathcal{H}(\mathcal{H}\varepsilon)) \bar{d}(\mathcal{H}\rho, \mathcal{H}(\mathcal{H}\rho))}{1 + \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho)}, \frac{\bar{d}(\mathcal{H}\varepsilon, \mathcal{H}(\mathcal{H}\varepsilon)) \bar{d}(\mathcal{H}\rho, \mathcal{H}(\mathcal{H}\rho))}{1 + \bar{d}(\mathcal{H}(\mathcal{H}\varepsilon), \mathcal{H}(\mathcal{H}\rho))}, \bar{d}(\mathcal{H}\varepsilon, \mathcal{H}\rho) \right\}. \tag{20}$$

We can deduce from Theorem 31 that  $\{\mathcal{H}\varepsilon_n\} \subset \mathcal{H}\mathcal{M}$  is a  $b$ -Cauchy sequence that converging on  $v \in \mathcal{H}\mathcal{M}$ .

We get from the condition (C<sub>4</sub>) that,

$$\lim_{n \rightarrow +\infty} \bar{d}(\mathcal{H}(\mathcal{F}\varepsilon_n), \mathcal{F}(\mathcal{H}\varepsilon_n)) = 0.$$

We have from a  $b$ -metrics triangular inequality that

$$\bar{d}(\mathcal{F}v, \mathcal{H}v) \leq s\bar{d}(\mathcal{F}v, \mathcal{F}(\mathcal{H}\varepsilon_n)) + s^2\bar{d}(\mathcal{F}(\mathcal{H}\varepsilon_n), \mathcal{H}(\mathcal{F}\varepsilon_n)) + s^2\bar{d}(\mathcal{H}(\mathcal{F}\varepsilon_n), \mathcal{H}v). \tag{21}$$

As  $n \rightarrow +\infty$  in (21),  $\bar{d}(\mathcal{F}v, \mathcal{H}v) = 0$  this indicates that  $v$  is a coincidence point of  $\mathcal{F}, \mathcal{H}$ .

The following result can get from Corollary 34 by weakening its hypotheses.

**Corollary 35** If  $\mathcal{E}$  satisfies the following condition in Corollary 34,

for very nondecreasing sequence  $\{\mathcal{H}\varepsilon_n\} \subseteq \mathcal{E}$  so that  $\mathcal{H}\varepsilon_n \rightarrow \mathcal{H}\sigma$ , then  $\mathcal{H}\varepsilon_n \preceq \mathcal{H}\sigma$  ( $n \geq 0$ ), i.e.,  $\mathcal{H}\sigma = \sup \mathcal{H}\varepsilon_n$ . (22)

then, if  $\mathcal{H}\mu \preceq \mathcal{H}(\mathcal{H}\mu)$  for some coincidence point  $\mu$ , a coincidence point exists for the weakly compatible mappings  $(\mathcal{F}, \mathcal{H})$ . Moreover,  $(\mathcal{F}, \mathcal{H})$  has only one common fixed point iff the set of common fixed points is well ordered.

*Proof.* A pair of maps  $(\mathcal{F}, \mathcal{H})$  has a coincidence point, according to Theorem 33 and Corollary 34.

Next, assume  $(\mathcal{F}, \mathcal{H})$  is only weakly compatible. Let  $v \in \mathcal{E}$  be a point with  $v = \mathcal{F}\mu = \mathcal{H}\mu$ . Thence,  $\mathcal{F}v = \mathcal{F}(\mathcal{H}\mu) = \mathcal{H}(\mathcal{F}\mu) = \mathcal{H}v$ .

Therefore,

$$\begin{aligned} \mathcal{P}_{\mathcal{H}}(\mu, v) &= \max\left\{ \frac{\bar{d}(\mathcal{H}v, \mathcal{F}v)[1 + \bar{d}(\mathcal{H}\mu, \mathcal{F}\mu)]}{1 + \bar{d}(\mathcal{H}\mu, \mathcal{H}v)}, \frac{\bar{d}(\mathcal{H}\mu, \mathcal{F}\mu) \bar{d}(\mathcal{H}v, \mathcal{F}v)}{1 + \bar{d}(\mathcal{H}\mu, \mathcal{H}v)}, \frac{\bar{d}(\mathcal{H}\mu, \mathcal{F}\mu) \bar{d}(\mathcal{H}v, \mathcal{F}v)}{1 + \bar{d}(\mathcal{F}\mu, \mathcal{F}v)}, \bar{d}(\mathcal{H}\mu, \mathcal{H}v) \right\} \\ &= \max\{0, \bar{d}(\mathcal{F}\mu, \mathcal{F}v)\} \\ &= \bar{d}(\mathcal{F}\mu, \mathcal{F}v). \end{aligned} \tag{23}$$

Thus from equation (17), we get

$$\begin{aligned} \Psi(\bar{d}(\mathcal{F}\mu, \mathcal{F}v)) &\leq \Psi(\mathcal{P}_{\mathcal{H}}(\mu, v)) - \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\mu, v)) \\ &\leq \Psi(\bar{d}(\mathcal{F}\mu, \mathcal{F}v)) - \hat{\eta}(\bar{d}(\mathcal{F}\mu, \mathcal{F}v)). \end{aligned} \tag{24}$$

By the property of  $\hat{\eta}$ , we get  $\bar{d}(\mathcal{F}\mu, \mathcal{F}v) = 0$  implies that  $\mathcal{F}v = \mathcal{H}v = v$ .

Finally, we can deduce from Theorem 33 that  $(\mathcal{F}, \mathcal{H})$  only has one common fixed point iff the common fixed points of  $(\mathcal{F}, \mathcal{H})$  is well ordered.

**Remark 36** Theorems 31 to 33 are the extension of Theorems 2.1, 2.2 & 2.3 of [12].

**Remark 37** Corollaries 34 & 35 are the generalizations of Corollaries 2.1 & 2.2 of [27] respectively.

**Definition 38** Consider the partially ordered  $b$ -metric space,  $(\mathcal{E}, \bar{d}, \preceq)$ . A map  $\mathcal{F} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  is known to be a generalized  $(\Psi, \hat{\eta})$ -contractive map with regards to  $\mathcal{H} : \mathcal{E} \rightarrow \mathcal{E}$ , if

$$\Psi(s^k \bar{d}(\mathcal{F}(\varepsilon, \rho), \mathcal{F}(\zeta, \mathfrak{S}))) \leq \Psi(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho, \zeta, \mathfrak{S})) - \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\varepsilon, \rho, \zeta, \mathfrak{S})), \tag{25}$$

for all  $\varepsilon, \rho, \zeta, \mathfrak{S} \in \mathcal{E}$  with  $\mathfrak{h}\varepsilon \preceq \mathfrak{h}\zeta$  and  $\mathfrak{h}\rho \succeq \mathfrak{h}\mathfrak{S}$ ,  $k > 2$ ,  $s > 1$ ,  $\check{\Psi} \in \check{\Phi}$ ,  $\hat{\eta} \in \check{\Psi}$  and where

$$\mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S}) = \max\left\{\frac{\check{\partial}(\mathfrak{h}\zeta, \mathcal{F}(\zeta, \mathfrak{S}))[1 + \check{\partial}(\mathfrak{h}\varepsilon, \mathcal{F}(\varepsilon, \rho))]}{1 + \check{\partial}(\mathfrak{h}\varepsilon, \mathfrak{h}\zeta)}, \frac{\check{\partial}(\mathfrak{h}\varepsilon, \mathcal{F}(\varepsilon, \rho)) \check{\partial}(\mathfrak{h}\zeta, \mathcal{F}(\zeta, \mathfrak{S}))}{1 + \check{\partial}(\mathfrak{h}\varepsilon, \mathfrak{h}\zeta)}, \frac{\check{\partial}(\mathfrak{h}\varepsilon, \mathcal{F}(\varepsilon, \rho)) \check{\partial}(\mathfrak{h}\zeta, \mathcal{F}(\zeta, \mathfrak{S}))}{1 + \check{\partial}(\mathcal{F}(\varepsilon, \rho), \mathcal{F}(\zeta, \mathfrak{S}))}, \check{\partial}(\mathfrak{h}\varepsilon, \mathfrak{h}\zeta)\right\}.$$

**Theorem 39** Suppose  $(\mathcal{E}, \check{\partial}, \preceq)$  be a complete partially ordered  $b$ -metric space. A map  $\mathcal{F} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  satisfies the condition (25) and,  $\mathcal{F}$ ,  $\mathfrak{h}$  are continuous,  $\mathcal{F}$  has mixed  $\mathfrak{h}$ -monotone property and also commutes with  $\mathfrak{h}$ . Assume, if some  $(\varepsilon_0, \rho_0) \in \mathcal{E} \times \mathcal{E}$  so that  $\mathfrak{h}\varepsilon_0 \preceq \mathcal{F}(\varepsilon_0, \rho_0)$ ,  $\mathfrak{h}\rho_0 \succeq \mathcal{F}(\rho_0, \varepsilon_0)$  and  $\mathcal{F}(\mathcal{E} \times \mathcal{E}) \subseteq \mathfrak{h}(\mathcal{E})$ , then  $\mathcal{F}$  and  $\mathfrak{h}$  in  $\mathcal{E}$  have a coupled coincidence point.

*Proof.* From [7] of Theorem 2.2, there will be two sequences  $\{\varepsilon_n\}, \{\rho_n\} \subset \mathcal{E}$  so that

$$\mathfrak{h}\varepsilon_{n+1} = \mathcal{F}(\varepsilon_n, \rho_n), \quad \mathfrak{h}\rho_{n+1} = \mathcal{F}(\rho_n, \varepsilon_n), n \geq 0.$$

In particular, the sequences  $\{\mathfrak{h}\varepsilon_n\}, \{\mathfrak{h}\rho_n\}$  are non-decreasing and non-increasing in  $\mathcal{E}$ . Put  $\varepsilon = \varepsilon_n, \rho = \rho_n, \zeta = \varepsilon_{n+1}, \mathfrak{S} = \rho_{n+1}$  in (25), we get

$$\begin{aligned} \check{\Psi}(s^k \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2})) &= \check{\Psi}(s^k \check{\partial}(\mathcal{F}(\varepsilon_n, \rho_n), \mathcal{F}(\varepsilon_{n+1}, \rho_{n+1}))) \\ &\leq \check{\Psi}(\mathcal{P}_{\mathfrak{h}}(\varepsilon_n, \rho_n, \varepsilon_{n+1}, \rho_{n+1})) \\ &\quad - \hat{\eta}(\mathcal{P}_{\mathfrak{h}}(\varepsilon_n, \rho_n, \varepsilon_{n+1}, \rho_{n+1})), \end{aligned} \tag{26}$$

where

$$\mathcal{P}_{\mathfrak{h}}(\varepsilon_n, \rho_n, \varepsilon_{n+1}, \rho_{n+1}) \leq \max\left\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2})\right\} \tag{27}$$

As a result of (26), we get

$$\begin{aligned} \check{\Psi}(s^k \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2})) &\leq \check{\Psi}(\max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2})\}) \\ &\quad - \hat{\eta}(\max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2})\}). \end{aligned} \tag{28}$$

Likewise by taking  $\varepsilon = \rho_{n+1}, \rho = \varepsilon_{n+1}, \zeta = \varepsilon_n, \mathfrak{S} = \varepsilon_n$  in (25), we get

$$\begin{aligned} \check{\Psi}(s^k \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})) &\leq \check{\Psi}(\max\{\check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}) \\ &\quad - \hat{\eta}(\max\{\check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}). \end{aligned} \tag{29}$$

We know that  $\max\{\check{\Psi}(l_1), \check{\Psi}(l_2)\} = \check{\Psi}\{\max\{l_1, l_2\}\}$  for  $l_1, l_2 \in [0, +\infty)$ . Then we add (28) and (29) together to get,

$$\begin{aligned} \check{\Psi}(s^k \Gamma_n) &\leq \check{\Psi}(\max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2}), \\ &\quad \check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}) \\ &\quad - \hat{\eta}(\max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2}), \\ &\quad \check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}) \end{aligned} \tag{30}$$

where

$$\Gamma_n = \max\{\check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}. \tag{31}$$

Let us denote,

$$\varkappa_n = \max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2}), \check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1}), \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2})\}. \tag{32}$$

Hence from equations (28)-(31), we obtain

$$s^k \Gamma_n \leq \varkappa_n. \tag{33}$$

Now to claim that

$$\Gamma_n \leq \lambda \Gamma_{n-1}, \tag{34}$$

for  $n \geq 1$  and  $\lambda = \frac{1}{s^k} \in [0, 1)$ .

Suppose that if  $\varkappa_n = \Gamma_n$  then from (33), we get  $s^k \Gamma_n \leq \Gamma_n$  this leads to  $\Gamma_n = 0$  since  $s > 1$  and thus (34) holds.

Suppose

$$\varkappa_n = \max\{\check{\partial}(\mathfrak{h}\varepsilon_n, \mathfrak{h}\varepsilon_{n+1}), \check{\partial}(\mathfrak{h}\rho_n, \mathfrak{h}\rho_{n+1})\}, \quad \text{i.e.,} \\ \varkappa_n = \Gamma_{n-1} \text{ then (33) follows (34).}$$

Now from (33), we obtain that  $\Gamma_n \leq \lambda^n \delta_0$  and hence,

$$\check{\partial}(\mathfrak{h}\varepsilon_{n+1}, \mathfrak{h}\varepsilon_{n+2}) \leq \lambda^n \Gamma_0 \text{ and } \check{\partial}(\mathfrak{h}\rho_{n+1}, \mathfrak{h}\rho_{n+2}) \leq \lambda^n \Gamma_0, \tag{35}$$

which shows that  $\{\mathfrak{h}\varepsilon_n\}, \{\mathfrak{h}\rho_n\}$  in  $\mathcal{E}$  are Cauchy sequences by Lemma 3.1 of [22]. Therefore, we can conclude from Theorem 2.2 of [5] that in  $\mathcal{E}$ ,  $\mathcal{F}$  and  $\mathfrak{h}$  have a coincidence point.

**Corollary 310** Suppose  $(\mathcal{E}, \check{\partial}, \preceq)$  be a complete partially ordered  $b$ -metric space. A continuous map  $\mathcal{F} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  has mixed monotone property is satisfying the below contraction conditions for all  $\varepsilon, \rho, \zeta, \mathfrak{S} \in \mathcal{E}$  such that  $\varepsilon \preceq \zeta$  and  $\rho \succeq \mathfrak{S}$ ,  $k > 2$ ,  $s > 1$ ,  $\check{\Psi} \in \check{\Phi}$  and  $\hat{\eta} \in \check{\Psi}$ :

(i).

$$\check{\Psi}(s^k \check{\partial}(\mathcal{F}(\varepsilon, \rho), \mathcal{F}(\zeta, \mathfrak{S}))) \leq \check{\Psi}(\mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S})) - \hat{\eta}(\mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S})),$$

(ii).

$$\begin{aligned} \check{\partial}(\mathcal{F}(\varepsilon, \rho), \mathcal{F}(\zeta, \mathfrak{S})) &\leq \frac{1}{s^k} \mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S}) \\ &\quad - \frac{1}{s^k} \hat{\eta}(\mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S})). \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_{\mathfrak{h}}(\varepsilon, \rho, \zeta, \mathfrak{S}) &= \max\left\{\frac{\check{\partial}(\zeta, \mathcal{F}(\zeta, \mathfrak{S}))[1 + \check{\partial}(\varepsilon, \mathcal{F}(\varepsilon, \rho))]}{1 + \check{\partial}(\varepsilon, \zeta)}, \frac{\check{\partial}(\varepsilon, \mathcal{F}(\varepsilon, \rho)) \check{\partial}(\zeta, \mathcal{F}(\zeta, \mathfrak{S}))}{1 + \check{\partial}(\varepsilon, \zeta)}, \frac{\check{\partial}(\varepsilon, \mathcal{F}(\varepsilon, \rho)) \check{\partial}(\zeta, \mathcal{F}(\zeta, \mathfrak{S}))}{1 + \check{\partial}(\mathcal{F}(\varepsilon, \rho), \mathcal{F}(\zeta, \mathfrak{S}))}, \check{\partial}(\varepsilon, \zeta)\right\}. \end{aligned}$$

If there exists  $(\varepsilon_0, \rho_0) \in \mathcal{E} \times \mathcal{E}$  so that  $\varepsilon_0 \preceq \mathcal{F}(\varepsilon_0, \rho_0)$  and  $\rho_0 \succeq \mathcal{F}(\rho_0, \varepsilon_0)$ , then  $\mathcal{F}$  in  $\mathcal{E}$  has a coupled fixed point.

**Theorem 311** *The unique coupled common fixed point for  $\mathcal{F}$  and  $\mathcal{h}$  exists in Theorem 39, if for every  $(\varepsilon, \wp), (\mathcal{k}, \ell) \in \mathcal{E} \times \mathcal{E}$  there is some  $(\Lambda, \Upsilon) \in \mathcal{E} \times \mathcal{E}$  such that  $(\mathcal{F}(\Lambda, \Upsilon), \mathcal{F}(\Upsilon, \Lambda))$  is comparable to  $(\mathcal{F}(\varepsilon, \wp), \mathcal{F}(\wp, \varepsilon))$  and to  $(\mathcal{F}(\mathcal{k}, \mathcal{F}), \mathcal{F}(\mathcal{F}, \mathcal{k}))$ .*

*Proof.* The existence of a coupled coincidence point for  $\mathcal{F}$ ,  $\mathcal{h}$  is guaranteed by the Theorem 39. Let  $(\varepsilon, \wp), (\mathcal{k}, \ell) \in \mathcal{E} \times \mathcal{E}$  are coupled coincidence points of  $\mathcal{F}$ ,  $\mathcal{h}$ . Now, we assert that  $\mathcal{h}\varepsilon = \mathcal{h}\mathcal{k}$  and  $\mathcal{h}\wp = \mathcal{h}\ell$ . By hypotheses  $(\mathcal{F}(\Lambda, \Upsilon), \mathcal{F}(\Upsilon, \Lambda))$  is comparable to  $(\mathcal{F}(\varepsilon, \wp), \mathcal{F}(\wp, \varepsilon))$  and  $(\mathcal{F}(\mathcal{k}, \mathcal{F}), \mathcal{F}(\mathcal{F}, \mathcal{k}))$  for some  $(\Lambda, \Upsilon) \in \mathcal{E} \times \mathcal{E}$ .

Now, assume the following

$$\begin{aligned} (\mathcal{F}(\varepsilon, \wp), \mathcal{F}(\wp, \varepsilon)) &\leq (\mathcal{F}(\Lambda, \Upsilon), \mathcal{F}(\Upsilon, \Lambda)) \text{ and} \\ (\mathcal{F}(\mathcal{k}, \mathcal{F}), \mathcal{F}(\mathcal{F}, \mathcal{k})) &\leq (\mathcal{F}(\Lambda, \Upsilon), \mathcal{F}(\Upsilon, \Lambda)). \end{aligned}$$

Suppose  $\Lambda_0 = \Lambda$  and  $\Upsilon_0 = \Upsilon$  then there is a point  $(\Lambda_1, \Upsilon_1) \in \mathcal{E} \times \mathcal{E}$  such that

$$\mathcal{h}\Lambda_1 = \mathcal{F}(\Lambda_0, \Upsilon_0), \quad \mathcal{h}\Upsilon_1 = \mathcal{F}(\Upsilon_0, \Lambda_0) \quad (n \geq 1).$$

As by applying the preceding argument repeatedly, we have the sequences  $\{\mathcal{h}\Lambda_n\}$  and  $\{\mathcal{h}\Upsilon_n\}$  in  $\mathcal{E}$  with

$$\mathcal{h}\Lambda_{n+1} = \mathcal{F}(\Lambda_n, \Upsilon_n), \quad \mathcal{h}\Upsilon_{n+1} = \mathcal{F}(\Upsilon_n, \Lambda_n) \quad (n \geq 0).$$

Define the sequences in the same way  $\{\mathcal{h}\varepsilon_n\}$ ,  $\{\mathcal{h}\wp_n\}$  and,  $\{\mathcal{h}\mathcal{k}_n\}$ ,  $\{\mathcal{h}\ell_n\}$  in  $\mathcal{E}$  by setting  $\varepsilon_0 = \varepsilon$ ,  $\wp_0 = \wp$  and  $\mathcal{k}_0 = \mathcal{k}$ ,  $\ell_0 = \ell$ . Further, we have that

$$\begin{aligned} \mathcal{h}\varepsilon_n &\rightarrow \mathcal{F}(\varepsilon, \wp), \quad \mathcal{h}\wp_n \rightarrow \mathcal{F}(\wp, \varepsilon), \\ \mathcal{h}\mathcal{k}_n &\rightarrow \mathcal{F}(\mathcal{k}, \mathcal{F}), \quad \mathcal{h}\ell_n \rightarrow \mathcal{F}(\mathcal{F}, \mathcal{k}) \quad (n \geq 1). \end{aligned} \tag{36}$$

Thus by induction, we get

$$(\mathcal{h}\varepsilon_n, \mathcal{h}\wp_n) \leq (\mathcal{h}\Lambda_n, \mathcal{h}\Upsilon_n) \text{ for every } n. \tag{37}$$

As a consequence of (25), we have

$$\begin{aligned} \check{\Psi}(\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_{n+1})) &\leq \check{\Psi}(s^k \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_{n+1})) \\ &= \check{\Psi}(s^k \check{\mathcal{D}}(\mathcal{F}(\varepsilon, \wp), \mathcal{F}(\Lambda_n, \Upsilon_n))) \\ &\leq \check{\Psi}(\mathcal{P}_{\mathcal{h}}(\varepsilon, \wp, \Lambda_n, \Upsilon_n)) \\ &\quad - \hat{\eta}(\mathcal{P}_{\mathcal{h}}(\varepsilon, \wp, \Lambda_n, \Upsilon_n)), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \mathcal{P}_{\mathcal{h}}(\varepsilon, \wp, \Lambda_n, \Upsilon_n) &= \max\left\{ \frac{\check{\mathcal{D}}(\mathcal{h}\Lambda_n, \mathcal{F}(\Lambda_n, \Upsilon_n)) [1 + \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{F}(\varepsilon, \wp))]}{1 + \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n)}, \right. \\ &\quad \frac{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{F}(\varepsilon, \wp)) \check{\mathcal{D}}(\mathcal{h}\Lambda_n, \mathcal{F}(\Lambda_n, \Upsilon_n))}{1 + \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n)}, \\ &\quad \frac{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{F}(\varepsilon, \wp)) \check{\mathcal{D}}(\mathcal{h}\Lambda_n, \mathcal{F}(\Lambda_n, \Upsilon_n))}{1 + \check{\mathcal{D}}(\mathcal{F}(\varepsilon, \wp), \mathcal{F}(\Lambda_n, \Upsilon_n))}, \\ &\quad \left. \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n) \right\} \\ &= \max\{0, \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n)\} \\ &= \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n). \end{aligned}$$

As a result of (38), we now have

$$\check{\Psi}(\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_{n+1})) \leq \check{\Psi}(\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n)) - \hat{\eta}(\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n)). \tag{39}$$

As by the similar argument, we acquire that

$$\check{\Psi}(\check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_{n+1})) \leq \check{\Psi}(\check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)) - \hat{\eta}(\check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)). \tag{40}$$

Hence from (39) and (40), we have

$$\begin{aligned} &\check{\Psi}(\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_{n+1}), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_{n+1})\}) \\ &\leq \check{\Psi}(\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\}) \\ &\quad - \hat{\eta}(\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\}) \\ &< \check{\Psi}(\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\}). \end{aligned} \tag{41}$$

Thus, the property of  $\check{\Psi}$  implies,

$$\begin{aligned} &\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_{n+1}), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_{n+1})\} \\ &< \max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\}. \end{aligned}$$

Hence,  $\max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\}$  is a decreasing sequence of positive reals and bounded below and by a result, we have

$$\lim_{n \rightarrow +\infty} \max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\} = \Gamma, \quad \Gamma \geq 0.$$

Therefore as  $n \rightarrow +\infty$  in equation (41), we get

$$\check{\Psi}(\Gamma) \leq \check{\Psi}(\Gamma) - \hat{\eta}(\Gamma), \tag{42}$$

from which we have  $\hat{\eta}(\Gamma) = 0$ , implies that  $\Gamma = 0$ . Therefore,

$$\lim_{n \rightarrow +\infty} \max\{\check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n), \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n)\} = 0.$$

Hence, we have that,

$$\lim_{n \rightarrow +\infty} \check{\mathcal{D}}(\mathcal{h}\varepsilon, \mathcal{h}\Lambda_n) = 0 \text{ and } \lim_{n \rightarrow +\infty} \check{\mathcal{D}}(\mathcal{h}\wp, \mathcal{h}\Upsilon_n) = 0. \tag{43}$$

By the similar argument as above, we obtain

$$\lim_{n \rightarrow +\infty} \check{\mathcal{D}}(\mathcal{h}\mathcal{k}, \mathcal{h}\Lambda_n) = 0 \text{ and } \lim_{n \rightarrow +\infty} \check{\mathcal{D}}(\mathcal{h}\mathcal{F}, \mathcal{h}\Upsilon_n) = 0. \tag{44}$$

Therefore from (43) and (44), we get  $\mathcal{h}\varepsilon = \mathcal{h}\mathcal{k}$  and  $\mathcal{h}\wp = \mathcal{h}\mathcal{F}$ . Since  $\mathcal{h}\varepsilon = \mathcal{F}(\varepsilon, \wp)$  and  $\mathcal{h}\wp = \mathcal{F}(\wp, \varepsilon)$  and, the commutativity property of  $\mathcal{F}$ ,  $\mathcal{h}$  implies that

$$\begin{aligned} \mathcal{h}(\mathcal{h}\varepsilon) &= \mathcal{h}(\mathcal{F}(\varepsilon, \wp)) = \mathcal{F}(\mathcal{h}\varepsilon, \mathcal{h}\wp) \text{ and} \\ \mathcal{h}(\mathcal{h}\wp) &= \mathcal{h}(\mathcal{F}(\wp, \varepsilon)) = \mathcal{F}(\mathcal{h}\wp, \mathcal{h}\varepsilon). \end{aligned} \tag{45}$$

If  $\mathcal{h}\varepsilon = \Lambda^*$  and  $\mathcal{h}\wp = \Upsilon^*$  then from (45), we get

$$\mathcal{h}(\Lambda) = \mathcal{F}(\Lambda^*, \Upsilon^*) \text{ and } \mathcal{h}(\Upsilon^*) = \mathcal{F}(\Upsilon^*, \Lambda^*), \tag{46}$$

which exhibits that  $(\Lambda^*, \Upsilon^*)$  is a coupled coincidence point of  $\mathcal{F}$ ,  $\mathcal{h}$ . Hence,  $\mathcal{h}(\Lambda^*) = \mathcal{h}\mathcal{k}$  and  $\mathcal{h}(\Upsilon^*) = \mathcal{h}\mathcal{F}$  which in turn gives that  $\mathcal{h}(\Lambda) = \Lambda^*$  and  $\mathcal{h}(\Upsilon^*) = \Upsilon^*$ . Therefore

from (46),  $(\Lambda^*, \Upsilon^*)$  is a coupled common fixed point of  $\mathcal{F}$ ,  $\mathcal{H}$ .

Let  $(\Lambda_1^*, \Upsilon_1^*)$  be another coupled common fixed point of  $\mathcal{F}$ ,  $\mathcal{H}$ . Then,  $\Lambda_1^* = \mathcal{H}\Lambda_1^* = \mathcal{F}(\Lambda_1^*, \Upsilon_1^*)$  and  $\Upsilon_1^* = \mathcal{H}\Upsilon_1^* = \mathcal{F}(\Upsilon_1^*, \Lambda_1^*)$ . But  $(\Lambda_1^*, \Upsilon_1^*)$  is a coupled common fixed point of  $\mathcal{F}$ ,  $\mathcal{H}$  then,  $\mathcal{H}\Lambda_1^* = \mathcal{H}\Lambda = \Lambda$  and  $\mathcal{H}\Upsilon_1^* = \mathcal{H}\Upsilon = \Upsilon$ . Therefore,  $\Lambda_1^* = \mathcal{H}\Lambda_1^* = \mathcal{H}\Lambda = \Lambda$  and  $\Upsilon_1^* = \mathcal{H}\Upsilon_1^* = \mathcal{H}\Upsilon = \Upsilon$ . Hence the uniqueness.

**Theorem 312** In Theorem 311, if  $\mathcal{H}\varepsilon_0 \leq \mathcal{H}\varrho_0$  or  $\mathcal{H}\varepsilon_0 \geq \mathcal{H}\varrho_0$ , then an unique common fixed point of  $\mathcal{F}$ ,  $\mathcal{H}$  can be found.

*Proof.* Assume  $(\varepsilon, \varrho) \in \mathcal{E}$  is a unique coupled common fixed point of  $\mathcal{F}$ ,  $\mathcal{H}$ . Then, to demonstrate that  $\varepsilon = \varrho$ . Suppose that  $\mathcal{H}\varepsilon_0 \leq \mathcal{H}\varrho_0$  then we get by induction,  $\mathcal{H}\varepsilon_n \leq \mathcal{H}\varrho_n, n \geq 0$ . From Lemma 2 of [23], we have

$$\begin{aligned} \check{\Psi}(s^{k-2}\check{\mathfrak{D}}(\varepsilon, \varrho)) &= \check{\Psi}(s^k \frac{1}{s^2}\check{\mathfrak{D}}(\varepsilon, \varrho)) \\ &\leq \limsup_{n \rightarrow +\infty} \check{\Psi}(s^k \check{\mathfrak{D}}(\varepsilon_{n+1}, \varrho_{n+1})) \\ &= \limsup_{n \rightarrow +\infty} \check{\Psi}(s^k \check{\mathfrak{D}}(\mathcal{F}(\varepsilon_n, \varrho_n), \mathcal{F}(\varrho_n, \varepsilon_n))) \\ &\leq \limsup_{n \rightarrow +\infty} \check{\Psi}(\mathcal{P}_{\mathcal{H}}(\varepsilon_n, \varrho_n, \varrho_n, \varepsilon_n)) \\ &\quad - \liminf_{n \rightarrow +\infty} \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\varepsilon_n, \varrho_n, \varrho_n, \varepsilon_n)) \\ &\leq \check{\Psi}(\check{\mathfrak{D}}(\varepsilon, \varrho)) \\ &\quad - \liminf_{n \rightarrow +\infty} \hat{\eta}(\mathcal{P}_{\mathcal{H}}(\varepsilon_n, \varrho_n, \varrho_n, \varepsilon_n)) \\ &< \check{\Psi}(\check{\mathfrak{D}}(\varepsilon, \varrho)), \end{aligned}$$

a contradiction. Hence,  $\varepsilon = \varrho$ .

The result can also see in the case of  $\mathcal{H}\varepsilon_0 \geq \mathcal{H}\varrho_0$ .

**Remark 313** While  $s = 1$  and the result of [21], the condition

$$\begin{aligned} \check{\Psi}(\check{\mathfrak{D}}(\mathcal{F}(\varepsilon, \varrho), \mathcal{F}(\check{\mathfrak{D}}, \mathfrak{D}))) \\ \leq \check{\Psi}(\max\{\check{\mathfrak{D}}(\mathcal{H}\varepsilon, \mathcal{H}\check{\mathfrak{D}}), \check{\mathfrak{D}}(\mathcal{H}\varrho, \mathcal{H}\mathfrak{D})\}) \\ - \hat{\eta}(\max\{\check{\mathfrak{D}}(\mathcal{H}\varepsilon, \mathcal{H}\check{\mathfrak{D}}), \check{\mathfrak{D}}(\mathcal{H}\varrho, \mathcal{H}\mathfrak{D})\}) \end{aligned}$$

is equivalent to,

$$\check{\mathfrak{D}}(\mathcal{F}(\varepsilon, \varrho), \mathcal{F}(\check{\mathfrak{D}}, \mathfrak{D})) \leq \varphi(\max\{\check{\mathfrak{D}}(\mathcal{H}\varepsilon, \mathcal{H}\check{\mathfrak{D}}), \check{\mathfrak{D}}(\mathcal{H}\varrho, \mathcal{H}\mathfrak{D})\}),$$

here  $\check{\Psi} \in \check{\Psi}$ ,  $\hat{\eta} \in \hat{\eta}$  and  $\varphi$  is a continuous self map on  $[0, +\infty)$  with  $\varphi(y) < y$  for every  $y > 0$  with  $\varphi(y) = 0$  iff  $y = 0$ . Hence the results found here are generalized and extended the results of [11, 18, 25, 26, 27] and a number of comparable results.

Now, depending on the type of metric, some examples are shown.

**Example 314** Let  $\mathcal{E} = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}\}$  and  $\check{\mathfrak{D}}: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$  be a metric defined by

$$\begin{aligned} \check{\mathfrak{D}}(\varepsilon, \varrho) &= \check{\mathfrak{D}}(\varrho, \varepsilon) = 0, \\ \text{if } \varepsilon = \varrho &= \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}\} \text{ and } \varepsilon = \varrho, \\ \check{\mathfrak{D}}(\varepsilon, \varrho) &= \check{\mathfrak{D}}(\varrho, \varepsilon) = 3, \\ \text{if } \varepsilon = \varrho &= \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\} \text{ and } \varepsilon \neq \varrho, \\ \check{\mathfrak{D}}(\varepsilon, \varrho) &= \check{\mathfrak{D}}(\varrho, \varepsilon) = 12, \\ \text{if } \varepsilon &= \{e_{11}, e_{22}, e_{33}, e_{44}\} \text{ and } \varrho = e_{66}, \\ \check{\mathfrak{D}}(\varepsilon, \varrho) &= \check{\mathfrak{D}}(\varrho, \varepsilon) = 20, \text{ if } \varepsilon = e_{55} \text{ and } \varrho = e_{66}, \\ &\text{with usual order } \leq. \end{aligned}$$

A self-map  $\mathcal{F}$  on  $\mathcal{E}$  defined by  $\mathcal{F}e_{11} = \mathcal{F}e_{22} = \mathcal{F}e_{33} = \mathcal{F}e_{44} = \mathcal{F}e_{55} = 1, \mathcal{F}e_{66} = 2$  has a fixed point with  $\check{\Psi}(y) = \frac{y}{2}, \hat{\eta}(y) = \frac{y}{4}$  where  $y \in [0, +\infty)$ .

*Proof.* When  $s = 2$ ,  $(\mathcal{E}, \check{\mathfrak{D}}, \leq)$  is a complete partially ordered  $b$ -metric space. Let  $\varepsilon, \varrho \in \mathcal{E}$  such that  $\varepsilon < \varrho$  then we'll look at the cases below.

**Case 1.** If  $\varepsilon, \varrho \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$  then  $\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho) = \check{\mathfrak{D}}(e_{11}, e_{11}) = 0$ . Hence,

$$\check{\Psi}(2\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho)) = 0 \leq \check{\Psi}(\mathcal{P}(\varepsilon, \varrho)) - \hat{\eta}(\mathcal{P}(\varepsilon, \varrho)).$$

**Case 2.** If  $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}\}$  and  $\varrho = e_{66}$ , then  $\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho) = \check{\mathfrak{D}}(e_{11}, e_{22}) = 3, \mathcal{P}(e_{66}, e_{55}) = 20$  and  $\mathcal{P}(\varepsilon, e_{66}) = 12$ , for  $\varepsilon \in \{e_{11}, e_{22}, e_{33}, e_{44}\}$ . Hence,

$$\check{\Psi}(2\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho)) \leq \frac{\mathcal{P}(\varepsilon, \varrho)}{4} = \check{\Psi}(\mathcal{P}(\varepsilon, \varrho)) - \hat{\eta}(\mathcal{P}(\varepsilon, \varrho)).$$

As a result, all of the conditions of Theorem 31 are met, and  $\mathcal{F}$  has a fixed point.

**Example 315** Let us define a metric  $\check{\mathfrak{D}}$  with usual order  $\leq$  by

$$\check{\mathfrak{D}}(\varepsilon, \varrho) = \begin{cases} 0 & , \text{ if } \varepsilon = \varrho \\ 1 & , \text{ if } \varepsilon \neq \varrho \in \{0, 1\} \\ |\varepsilon - \varrho| & , \text{ if } \varepsilon, \varrho \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\} \\ 6 & , \text{ otherwise.} \end{cases}$$

where  $\mathcal{E} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots\}$ . A self-map  $\mathcal{F}$  on  $\mathcal{E}$  by  $\mathcal{F}0 = 0, \mathcal{F}\frac{1}{n} = \frac{1}{12n} (n \geq 1)$  has a fixed point with  $\check{\Psi}(y) = y, \hat{\eta}(y) = \frac{4y}{5}$  for  $y \in [0, +\infty)$ .

*Proof.*  $\check{\mathfrak{D}}$  is clearly discontinuous, and  $(\mathcal{E}, \check{\mathfrak{D}}, \leq)$  is a complete partially ordered  $b$ -metric space for  $s = \frac{12}{5}$ . Now we'll look at the cases for  $\varepsilon, \varrho \in \mathcal{E}$  with  $\varepsilon < \varrho$ .

**Case 1.** Suppose  $\varepsilon = 0$  and  $\varrho = \frac{1}{n} (n > 0)$ , then  $\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho) = \check{\mathfrak{D}}(0, \frac{1}{12n}) = \frac{1}{12n}$  and  $\mathcal{P}(\varepsilon, \varrho) = \frac{1}{n}$  and  $\mathcal{P}(\varepsilon, \varrho) = \{1, 6\}$ . Thus,

$$\begin{aligned} \check{\Psi}\left(\frac{12}{5}\check{\mathfrak{D}}(\mathcal{F}\varepsilon, \mathcal{F}\varrho)\right) &\leq \frac{\mathcal{P}(\varepsilon, \varrho)}{5} \\ &= \check{\Psi}(\mathcal{P}(\varepsilon, \varrho)) - \hat{\eta}(\mathcal{P}(\varepsilon, \varrho)). \end{aligned}$$



**Case 2.** Let  $\varepsilon = \frac{1}{m}$  and  $\rho = \frac{1}{n}$  where  $m > n \geq 1$ , thence

$$\begin{aligned} \bar{\partial}(\mathcal{F}\varepsilon, \mathcal{F}\rho) &= \bar{\partial}\left(\frac{1}{12m}, \frac{1}{12n}\right) \text{ and} \\ \mathcal{P}(\varepsilon, \rho) &\geq \frac{1}{n} - \frac{1}{m} \text{ or } \mathcal{P}(\varepsilon, \rho) = 6. \end{aligned}$$

Thus,

$$\begin{aligned} \bar{\Psi}\left(\frac{12}{5}\bar{\partial}(\mathcal{F}\varepsilon, \mathcal{F}\rho)\right) &\leq \frac{\mathcal{P}(\varepsilon, \rho)}{5} \\ &= \bar{\Psi}(\mathcal{P}(\varepsilon, \rho)) - \hat{\eta}(\mathcal{P}(\varepsilon, \rho)). \end{aligned}$$

Hence, we have the conclusion from Theorem 31 as all assumptions are fulfilled.

**Example 316** Define a metric  $\bar{\partial} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ , where  $\mathcal{E} = \{\bar{\ell}/\bar{\ell}' : [a_1, a_2] \rightarrow [a_1, a_2] \text{ continuous}\}$  by

$$\bar{\partial}(\bar{\ell}_1, \bar{\ell}_2) = \sup_{y \in [a_1, a_2]} \{|\bar{\ell}_1(y) - \bar{\ell}_2(y)|^2\}$$

for any  $\bar{\ell}_1, \bar{\ell}_2 \in \mathcal{E}$ ,  $0 \leq a_1 < a_2$  with  $\bar{\ell}_1 \preceq \bar{\ell}_2$  implies  $a_1 \leq \bar{\ell}_1(y) \leq \bar{\ell}_2(y) \leq a_2, y \in [a_1, a_2]$ . A self-map  $\mathcal{F}$  on  $\mathcal{E}$  defined by  $\mathcal{F}\bar{\ell} = \frac{\bar{\ell}}{5}, \bar{\ell} \in \mathcal{E}$  has a unique fixed point with  $\bar{\Psi}(y) = y, \hat{\eta}(y) = \frac{y}{3}$ , for any  $y \in [0, +\infty]$ .

*Proof.* As  $\min(\bar{\ell}_1, \bar{\ell}_2)(y) = \min\{\bar{\ell}_1(y), \bar{\ell}_2(y)\}$  is continuous and all other assumptions of Theorem 33 are fulfilled for  $s = 2$ . Therefore,  $0 \in \mathcal{E}$  is a unique fixed point  $\mathcal{F}$ .

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## Competing interests

The authors declare that they have no competing interests.

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