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Interval Estimation of a $P(X_1 < X_2)$ Model with General Form Distributions for Unknown Parameters

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Abstract: In this paper, we present interval estimators of $P(X_1<X_2)$, when both X_1 and X_2 follow some distributions with general exponential or general inverse exponential forms, with different unknown parameters. Different interval estimators are derived. Since many distributions in the literature belong to the general exponential and the general inverse exponential forms discussed, the results obtained may directly be applied to a numerous number of distributions. To compare the different interval estimators obtained, a simulation study is performed with applications on Weibull, and inverse Weibull distributions. The comparison is based on length, probability coverage, and tail errors.

Keywords: Generalized variable method, Markov chain Monte Carlo method, bootstrap method, interval estimator, average length; probability coverage, tail error.

1 Introduction

The estimation of $R=P(X_1<X_2)$ has been widely used in the fields of aeronautical, civil, mechanical and electronic engineering. For example, X_1 may be the voltage output of a transformer (power supply), while X_2 may represent the breakdown voltage of a capacitor, Hall [1]. Reiser and Guttman [2] presented a rocket motor experiment data where X_1 represents the operating pressure and X_2 represents the chamber burst strength. Due to the practical importance of $R=P(X_1<X_2)$ model, a numerous number of researches are presented in the literature concerning inferences on R. Kotz et al. [3] compiled the work done on R until year 2003, after year 2003; see for example, Rezaei et al [4], Amiri et al. [5], and Al-Mutairi et al. [6].

Mokhils et al. [7] introduced point and interval estimation of $R=P(X_1<X_2)$ when X_1 and X_2 have a general exponential form or a general inverse exponential form with the survival functions given respectively by either

$$\bar{F}_{X_i}(x;\theta_i)=exp\left[-\theta_i\, g_1(x;c)\right],$$

or

$$\bar{F}_{X_i}(x;\eta_i)=1-exp\left[-\eta_i\, g_2(x;c)\right];\ i\ =\ 1,\ 2,$$

where, the function $g_1(x;c)$ is continuous, monotone increasing, differentiable function such that, $g_1(x;c)\to 0$ as $x\to 0$ and $g_1(x;c)\to\infty$ as $x\to\infty$, the function $g_2(x;c)$ is a continuous, monotone decreasing, differentiable function, such that, $g_2(x;c)\to\infty$ as $x\to 0$ and $g_2(x;c)\to 0$ as $x\to\infty$, θ_i and η_i are unknown parameters, while c is common known parameter.

In the present article, we obtain interval estimators of $R=P(X_1<X_2)$, where X_1 and X_2 are non-negative independent and continuous random variables, having the same general forms discussed by Mokhils et al. [7], with the survival functions given by either

$$\bar{F}_{X_i}(x;b_i,c)=exp\left[-\theta_i(b_i,c)\, g_1(x;c)\right], \tag{1}$$

or,

$$\bar{F}_{X_i}(x;b_i,c)=1-exp\left[-\eta_i(b_i,c)\, g_2(x;c)\right];\qquad i\ =\ 1,2, \tag{2}$$

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where, $\theta_i(b_i,c)$ and $\eta_i(b_i,c)$ are differentiable functions in two unknown parameters b_i and c; $i = 1,2$. Of course, they could be functions of just b_i; $i=1,\ 2$. Consequently, if X_1 and X_2 follow the forms in (1) or (2), then R will take the following forms

$$R=P\left(X_1<X_2\right)=\frac{\theta_1(b_1,c)}{\theta_1(b_1,c)+\theta_2(b_2,c)}, \tag{3}$$

or,

$$R=P\left(X_1<X_2\right)=\frac{\eta_2(b_2,c)}{\eta_1(b_1,c)+\eta_2(b_2,c)}. \tag{4}$$

For simplicity, we shall refer to $\theta_i\left(b_i,c\right)$ and $\eta_i(b_i,c)$ by θ_i and η_i; $i=1,\ 2$, respectively.

We construct approximate confidence intervals for R, using the maximum likelihood estimator (MLE) of R. Generalized confidence intervals are obtained, using the generalized variable (GV) approach. Two bootstrap confidence intervals (percentile and t) are also presented. Bayesian credible intervals of R are obtained, using Markov chain Monte Carlo method (MCMC) in two cases. The different interval estimators are compared via a simulation study.

2 Confidence limits of $R=P(X_1<X_2)$

In this section, we present different confidence intervals of R namely: the approximate, generalized, bootstrap (percentile and t) and Bayesian with different priors.

2.1 Approximate confidence interval of R (ACI)

Suppose that $\underline{X}_i=\left(X_{i1},X_{i2},\ldots,X_{in_i}\right)$; $i=1,\ 2$, be two independent random samples from populations with survivor function given by (1). The likelihood function is

$$L_1\left(\underline{x}_1,\underline{x}_2|\,b_1,b_2,c\right)=\exp\left[\sum_{i=1}^{2}n_i\ln\theta_i+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}\theta_i\sum_{j=1}^{n_i}g_1(x_{ij};c)\right], \tag{5}$$

where, $g_1'\left(x_{ij};c\right)$ is the first derivative of $g_1\left(x_{ij};c\right)$ w.r.t x_{ij}. The log-likelihood function is

$$l_1\left(\underline{x}_1,\underline{x}_2|\,b_1,b_2,c\right)=\sum_{i=1}^{2}n_i\ln\theta_i+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}\theta_i\sum_{j=1}^{n_i}g_1(x_{ij};c). \tag{6}$$

Differentiating l_1 with respect to the parameters c, b_1, b_2 and equating with zero, we get

$$\frac{\partial l_1}{\partial c}=\sum_{i=1}^{2}\frac{n_i}{\theta_i}\frac{\partial\theta_i}{\partial c}+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\frac{\partial}{\partial c}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}\frac{\partial\theta_i}{\partial c}\sum_{j=1}^{n_i}g_1(x_{ij};c)-\sum_{i=1}^{2}\theta_i\sum_{j=1}^{n_i}\frac{\partial}{\partial c}g_1(x_{ij};c)=0 \tag{7}$$

$$\frac{\partial l_1}{\partial b_i}=\left(\frac{n_i}{\theta_i}-\sum_{j=1}^{n_i}g_1(x_{ij};c)\right)\frac{\partial\theta_i}{\partial b_i}=0;\ i=1,2. \tag{8}$$

The MLE $\hat{c}$ of c can be obtained by solving (7) numerically. Solving (8), the MLEs $\hat{\theta}_i$ of θ_i; $i=1,2$, are given by

$$\hat{\theta}_i=\frac{n_i}{\sum_{j=1}^{n_i}g_1(x_{ij};\hat{c})};\ i=1,\ 2, \tag{9}$$

see [7]. The corresponding MLE $\hat{R}$ of R is

$$\hat{R}=\frac{\hat{\theta}_1(\hat{b}_1,\hat{c})}{\hat{\theta}_1(\hat{b}_1,\hat{c})+\hat{\theta}_2(\hat{b}_2,\hat{c})} \tag{10}$$

It is known that, the MLE $\hat{R}$ is asymptotically normal with mean R and variance $\sigma^2_{\hat{R}}=N^tV^{-1}N$, where, V^{-1} the inverse of the Fisher information matrix V of (c,b_1,b_2), N^t is the transpose of matrix N, (see, Rao [8]), where,

$$V=-E\begin{bmatrix}\frac{\partial^2 l_1}{\partial c^2} & \frac{\partial^2 l_1}{\partial c\partial b_1} & \frac{\partial^2 l_1}{\partial c\partial b_2}\\ \frac{\partial^2 l_1}{\partial b_1\partial c} & \frac{\partial^2 l_1}{\partial b_1^2} & \frac{\partial^2 l_1}{\partial b_1\partial b_2}\\ \frac{\partial^2 l_1}{\partial b_2\partial c} & \frac{\partial^2 l_1}{\partial b_2\partial b_1} & \frac{\partial^2 l_1}{\partial b_2^2}\end{bmatrix},\qquad N=\begin{bmatrix}\frac{\partial R}{\partial c}\\ \frac{\partial R}{\partial b_1}\\ \frac{\partial R}{\partial b_2}\end{bmatrix},$$

$$\frac{\partial^2 l_1}{\partial c^2} = \sum_{i=1}^{2} \frac{n_i}{\theta_i}\frac{\partial^2 \theta_i}{\partial c^2} - \sum_{i=1}^{2} \frac{n_i}{\theta_i^2}\left(\frac{\partial \theta_i}{\partial c}\right)^2 + \sum_{i=1}^{2}\sum_{j=1}^{n_i} \frac{\partial^2}{\partial c^2}\ln g_1'(x_{ij};c) - \sum_{i=1}^{2}\theta_i \sum_{j=1}^{n_i}\frac{\partial^2}{\partial c^2} g_1(x_{ij};c)$$
$$-\sum_{i=1}^{2}\frac{\partial^2 \theta_i}{\partial c^2}\sum_{J=1}^{n_i} g_1(x_{ij};c) - 2\sum_{i=1}^{2}\frac{\partial \theta_i}{\partial c}\sum_{j=1}^{n_i}\frac{\partial}{\partial c} g_1(x_{ij};c),$$
$$\frac{\partial^2 l_1}{\partial c \partial b_i} = \frac{\partial^2 l_1}{\partial b_i \partial c} = \frac{n_i}{\theta_i}\frac{\partial^2 \theta_i}{\partial c \partial b_i} - \frac{n_i}{\theta_i^2}\frac{\partial \theta_i}{\partial c}\frac{\partial \theta_i}{\partial b_i} - \left(\frac{\partial \theta_i}{\partial b_i}\right)\sum_{j=1}^{n_i}\frac{\partial}{\partial c} g_1(x_{ij};c) - \left(\frac{\partial^2 \theta_i}{\partial c \partial b_i}\right)\sum_{j=1}^{n_i} g_1(x_{ij};c),$$
$$\frac{\partial^2 l_1}{\partial b_i^2} = \frac{n_i}{\theta_i}\frac{\partial^2 \theta_i}{\partial b_i^2} - \frac{n_i}{\theta_i^2}\left(\frac{\partial \theta_i}{\partial b_i}\right)^2 - \left(\frac{\partial^2 \theta_i}{\partial b_i^2}\right)\sum_{j=1}^{n_i} g_1(x_{ij};c); \qquad i=1,2, \quad \frac{\partial^2 l_1}{\partial b_1 \partial b_2} = \frac{\partial^2 l_1}{\partial b_2 \partial b_1} = 0,$$
$$\frac{\partial R}{\partial b_1} = \frac{\theta_2 \frac{\partial \theta_1}{\partial b_1}}{(\theta_1+\theta_2)^2}, \quad \frac{\partial R}{\partial b_2} = \frac{-\theta_1 \frac{\partial \theta_2}{\partial b_2}}{(\theta_1+\theta_2)^2}, \quad \text{and} \quad \frac{\partial R}{\partial c} = \frac{\theta_2 \frac{\partial \theta_1}{\partial c} - \theta_1 \frac{\partial \theta_2}{\partial c}}{(\theta_1+\theta_2)^2}.$$

The approximate $(1-\alpha)100\%$ confidence interval for R is $\left(\hat{R} \pm z_{(1-\alpha/2)}\sqrt{\hat{\sigma}_{\hat{R}}^2}\right)$, where, $z_{(1-\alpha/2)}$ is the $(1-\alpha/2)$th quantile of the standard normal distribution and $\hat{\sigma}_{\hat{R}}^2$ is the estimator of $\sigma_{\hat{R}}^2$, and it is obtained by replacing c, θ_i and R with $\hat{c}$, $\hat{\theta}_i$ and $\hat{R}$, respectively. It is important to mention that, the explicit expression of $\sigma_{\hat{R}}^2$ depends on θ_i, $g_1'(x_{ij};c)$ and $g_1(x_{ij};c)$; $j=1, \ldots, n_i$, $i=1, 2$.

Similarly, if $\underline{X}_i$; $i=1, 2$, are two independent random samples from populations with survivor function given by (2), the MLE $\hat{c}$ of c can be obtained numerically by solving the following equation

$$\sum_{i=1}^{2}\frac{\partial \eta_i}{\partial c}\frac{n_i}{\eta_i} + \sum_{i=1}^{2}\sum_{j=1}^{n_i}\frac{\partial}{\partial c}\ln\left(-g_2'(x_{ij};c)\right) - \sum_{i=1}^{2}\eta_i\sum_{j=1}^{n_i}\frac{\partial}{\partial c} g_2(x_{ij};c) - \sum_{i=1}^{2}\frac{\partial \eta_i}{\partial c}\sum_{j=1}^{n_i} g_2(x_{ij};c) = 0,$$

where, $g_2'(x_{ij};c)$ is the first derivative of $g_2(x_{ij};c)$ w.r.t x_{ij}. The MLEs $\hat{\eta}_i$ of η_i will be $\hat{\eta}_i = \frac{n_i}{\sum_{j=1}^{n_i} g_2(x_{ij};\hat{c})}$; $i=1, 2$. The corresponding MLE $\hat{R}$ of R will be $\hat{R} = \frac{\hat{\eta}_2(\hat{b}_2,\hat{c})}{\hat{\eta}_1(\hat{b}_1,\hat{c})+\hat{\eta}_2(\hat{b}_2,\hat{c})}$, and hence, the approximate $(1-\alpha)100\%$ confidence interval for R will be easily obtained in a similar manner as that of the case of the general exponential form (1).

2.2 Generalized confidence interval of R (GCI)

The generalized pivotal quantity (GPQ) is a function of observed statistics and random variables whose distribution is free of unknown parameters. The useful feature of the GV approach is that the GPQ for a function of unknown parameters can be obtained by simply plugging their GPQs in the function. Let $\underline{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$; $i=1, 2$, be two independent random samples from populations with survivor function (1) or (2) having unknown parameters θ_i or η_i; $i=1, 2$, respectively, and a common unknown parameter c. The GPQ for R given respectively by

$$G_R = R\left(G_{\theta_1}, G_{\theta_2}\right) = \frac{G_{\theta_1}}{G_{\theta_1} + G_{\theta_2}}, \tag{11}$$

or,

$$G_R = R\left(G_{\eta_1}, G_{\eta_2}\right) = \frac{G_{\eta_2}}{G_{\eta_1} + G_{\eta_2}}. \tag{12}$$

where, $G_{\theta_i} = \theta_i(G_{b_i}, G_c)$ and $G_{\eta_i} = \eta_i(G_{b_i}, G_c)$; G_{θ_i}, G_{η_i}, G_{b_i}, and G_c denote the GPQs for θ_i, η_i, b_i, and c; $i=1, 2$, respectively. It is necessary to mention that, G_{θ_i} and G_{η_i} may be depend on G_{b_i} only. The $(1-\alpha)100\%$ generalized confidence interval of R can be obtained as $\left(G_{R(\alpha/2)}, G_{R(1-\alpha/2)}\right)$, where, $G_{R(\alpha/2)}$ and $G_{R(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R.

2.3 Bootstrap confidence interval of R (boot)

Suppose that $\underline{X}_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$; i = 1, 2 are two independent random samples from populations with survivor function (1) having unknown parameters θ_i; $i=1, 2$, respectively, and a common unknown parameter c. For generating bootstrap samples, we apply the following algorithm, (see, Efron [9]).

Algorithm 1.

1.From the original data $\underline{X}_i$; $i=1,\ 2$, compute the MLEs $\left(\hat{c},\hat{\theta}_1,\hat{\theta}_2,\hat{R}\right)$ of (c,θ_1,θ_2,R) using (7), (9) and (10).
2.Resample two independent random samples $\underline{X}_i^{**}$; $i=1,2$, with replacement from the original samples $\underline{X}_i$; $i=1,2$, respectively; compute the MLEs $\left(\hat{c}^{**},\hat{\theta}_1^{**},\hat{\theta}_2^{**},\hat{R}^{**}\right)$ of (c,θ_1,θ_2,R) from (7), (9) and (10).
3.Repeat the step 2, N times to obtain a set of bootstrap samples of R, say $\left\{\hat{R}_j^{**};\ j=1,\ldots,N\right\}$, and order $\hat{R}_j^{**};j=1,\ldots,\ N$, ascending as $\hat{R}_j^{**1}\leq\cdots\leq\hat{R}_j^{**(N)}$.
4.Construct two different bootstrap intervals of R.
a.The $(1-\alpha)100\%$ percentile bootstrap confidence interval of R (P-boot) given by $\left(\hat{R}^{**}_{(\alpha/2)},\ \hat{R}^{**}_{(1-\alpha/2)}\right)$, where, $\hat{R}^{**}_{(\alpha/2)}$ and $\hat{R}^{**}_{(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R, respectively.
b.The $(1-\alpha)100\%$ t-bootstrap confidence interval of R (T-boot) given by $\left(\hat{R}-\hat{t}_{(1-\alpha/2)}S^{**},\ \hat{R}-\hat{t}_{(\alpha/2)}S^{**}\right)$, where, S^{**} is the sample standard deviation of $\left\{\hat{R}_j^{**};j=1,\ldots,N\right\}$ and $\hat{t}_{(\alpha)}$ be the (α)th quantile of $\left\{\frac{\hat{R}_j^{**}-\hat{R}}{S^{**}};j=1,\ldots,N\right\}$.

The two different bootstrap intervals of R for the form (2) can be obtained, using a similar algorithm as Algorithm 1, if $\underline{X}_i$; i = 1, 2 being two independent random samples from populations with survivor function (2).

2.4 Bayesian Credible Interval of R (BCI)

To explore the sensitivity of prior distributions of the unknown parameters, we apply MCMC method for estimating the Bayesian credible interval of R in two cases. In the first case we assume gamma priors for θ_1,θ_2, and c, while in the second case we consider independent gamma priors for θ_1,θ_2 and uniform prior for c as the available prior information is weak for c. In Bayesian statistics, there are generally two MCMC algorithms that use the Gibbs sampling and the Metropolis-Hastings algorithm. If the full conditional distribution for each parameter is known, the Gibbs sampling can be used. If the full conditional doesn't look like any known distribution, in this case the Metropolis-Hastings algorithm can be useful.

2.4.1 Gamma priors (G-BCI)

Suppose that $\underline{X}_i$; $i=1,\ 2$ are two independent random samples from populations with survivor function (1), and also suppose that, θ_i; $i=1,\ 2$ having independent gamma prior distributions with probability density function $f(\theta_i)=\frac{h_i^{d_i}}{\Gamma d_i}\theta_i^{d_i-1}e^{-h_i\theta_i}$; $\theta_i,\ d_i,\ h_i>0$, and the prior distribution of c follows the gamma distribution with probability density function $f(c)=\frac{h_3^{d_3}}{\Gamma d_3}c^{d_3-1}e^{-h_3c}$; $c,\ d_3,\ h_3>0$. From the likelihood function in (5), and the prior density functions of θ_1,θ_2, and c. The joint posterior density function of θ_1,θ_2, and c is given by

$$\pi_1\left(\theta_1,\theta_2,c|\underline{x}_1,\underline{x}_2\right)\propto\exp\left[\sum_{i=1}^{2}(n_i+d_i-1)\ln\theta_i+(d_3-1)\ln c-ch_3+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}\theta_i\left(h_i+\sum_{j=1}^{n_i}g_1(x_{ij};c)\right)\right].$$

We find the marginal posterior distribution of θ_i is gamma with parameters $\left((n_i+d_i),\ \left(h_i+\sum_{j=1}^{n_i}g_1\left(x_{ij};c\right)\right)\right)$; $i=1,2$, respectively, and the marginal posterior distribution of c is

$$\pi_1\left(c|\underline{x}_1,\underline{x}_2\right)=K_1^{-1}\exp\left[(d_3-1)\ln c-ch_3+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}(n_i+d_i)\ln\left(h_i+\sum_{j=1}^{n_i}g_1(x_{ij};c)\right)\right],$$

where,

$$K_1=\int_{-\infty}^{\infty}\exp\left[(d_3-1)\ln c-ch_3+\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c)-\sum_{i=1}^{2}(n_i+d_i)\ln\left(h_i+\sum_{j=1}^{n_i}g_1(x_{ij};c)\right)\right]dc.$$

However, the marginal posterior distribution of c doesn't look like any known distribution, in order to solve our problem we shall use the Gibbs sampling and Metropolis-Hastings (see, Asgharzadeh et al. [10]). The Metropolis-Hastings with Gibbs sampling algorithm follows the following steps.

Algorithm 2.

1.Choose a starting value $c^{(0)}$.
2.For j=1 to N times.
3.Generate $\theta_i^{(j)}$ from Gamma $\left((n_i{+}d_i),\left(h_i{+}\sum_{j=1}^{n_i} g_1\left(x_{ij};c^{(j-1)}\right)\right)\right)$; i=1, 2, respectively.
4.Generate $c^{(j)}$ from $\pi_1\left(c \mid \underline{x}_1, \underline{x}_2\right)$ using the Metropolis-Hastings algorithm with the normal proposal distribution $\pi \sim N(c^{(j-1)},1)$.
a.Generate ξ from the proposal distribution π.
b.Define $Q = min\left\{1, \frac{\pi_1(\xi|\,\underline{x}_1,\ \underline{x}_2)\ \pi\left(c^{(j-1)}\right)}{\pi_1\left(c^{(j-1)}\right|\underline{x}_1,\ \underline{x}_2)\ \pi(\xi)}\right\}$.
c.Generate u from Uniform (0, 1). Take $c^{(j)} = \begin{cases} \xi & ; u \le Q, \\ c^{(j-1)}; & otherwise \end{cases}$.
5.Compute the $R^{(j)}$ at $\left(\theta_1^{(j)}, \theta_2^{(j)}\right)$ from (3).
6.End j loop.
7.Repeat the steps 2-6, N times, and order $R^j; j = 1, \ldots, N$, as $R^{j1} < \cdots < R^{j(N)}$.
8.Construct the $(1-\alpha)100\%$ Bayesian credible interval of R as $\left(\widetilde{R}_{g(\alpha/2)},\ \widetilde{R}_{g(1-\alpha/2)}\right)$, where, $\widetilde{R}_{g(\alpha/2)}$ and $\widetilde{R}_{g(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R, respectively.

2.4.2 Mixed priors (M-BCI)

Let $\underline{X}_i; i = 1,\ 2$ be two independent random samples from populations with survivor function (1). Let θ_i have independent gamma prior distributions with parameters (d_i, h_i), $i = 1, 2$, respectively, and c has a non-informative uniform prior distribution with probability density function $f(c) = 1; c > 0$. From the likelihood function in (5) and the prior density functions of θ_1, θ_2, and c, so the joint posterior density function of θ_1, θ_2, and c can be obtained as

$$\pi_2\left(\theta_1, \theta_2, c|\underline{x}_1, \underline{x}_2\right) \propto \exp\left[\sum_{i=1}^{2}(n_i + d_i - 1)\ln\theta_i + \sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c) - \sum_{i=1}^{2}\theta_i\left(h_i + \sum_{j=1}^{n_i} g_1(x_{ij};c)\right)\right].$$

The marginal posterior distribution of θ_i will be gamma with parameters $\left((n_i{+}d_i),\ \left(h_i{+}\sum_{j=1}^{n_i} g_1(x_{ij};c)\right)\right)$; i = 1, 2, respectively, while the marginal posterior distribution of c will be

$$\pi_2\left(c|\underline{x}_1, \underline{x}_2\right) = K_2^{-1}\exp\left[\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c) - \sum_{i=1}^{2}(n_i + d_i)\ln\left(h_i + \sum_{j=1}^{n_i} g_1(x_{ij};c)\right)\right],$$

where,

$$K_2 = \int_{-\infty}^{\infty}\exp\left[\sum_{i=1}^{2}\sum_{j=1}^{n_i}\ln g_1'(x_{ij};c) - \sum_{i=1}^{2}(n_i + d_i)\ln\left(h_i + \sum_{j=1}^{n_i} g_1(x_{ij};c)\right)\right]dc.$$

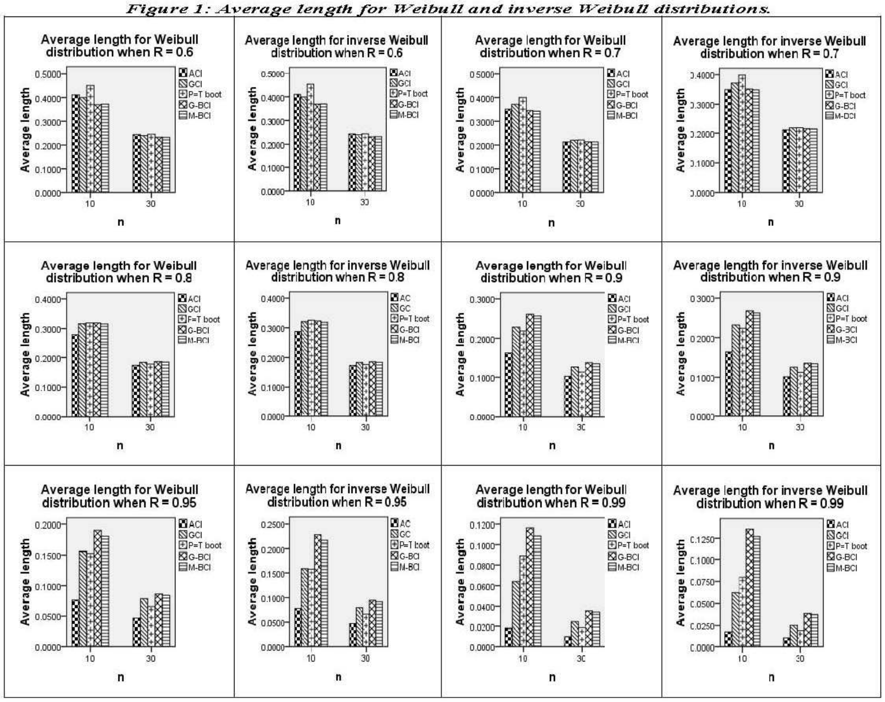
It is observed that, the marginal posterior distribution of c is not known. Using Algorithm 2 of the Metropolis-Hastings with Gibbs sampling, the $(1-\alpha)100\%$ Bayesian credible interval of R can be obtained as $\left(\widetilde{R}_{m(\alpha/2)},\ \widetilde{R}_{m(1-\alpha/2)}\right)$, where, $\widetilde{R}_{m(\alpha/2)}$ and $\widetilde{R}_{m(1-\alpha/2)}$ are the $(\alpha/2)$th and $(1-\alpha/2)$th quantiles of R.

Similarly, for the case of the inverse exponential form in (2), the $(1-\alpha)100\%$ Bayesian credible intervals for R can be obtained assuming gamma priors and mixed priors.

3 Simulation

In this section we present a simulation study, to observe the behavior of the estimators obtained by different methods for different sample sizes and different parameter values. We compare different interval estimators of $R = P(X_1 < X_2)$, namely approximate, generalized, bootstrap (percentile and t) and Bayesian with gamma priors and mixed priors when $\underline{X}_i;\ i = 1,\ 2$, have the general exponential or the general inverse exponential forms in (1) or (2), respectively. We generate

Figure 1: Average length for Weibull and inverse Weibull distributions.

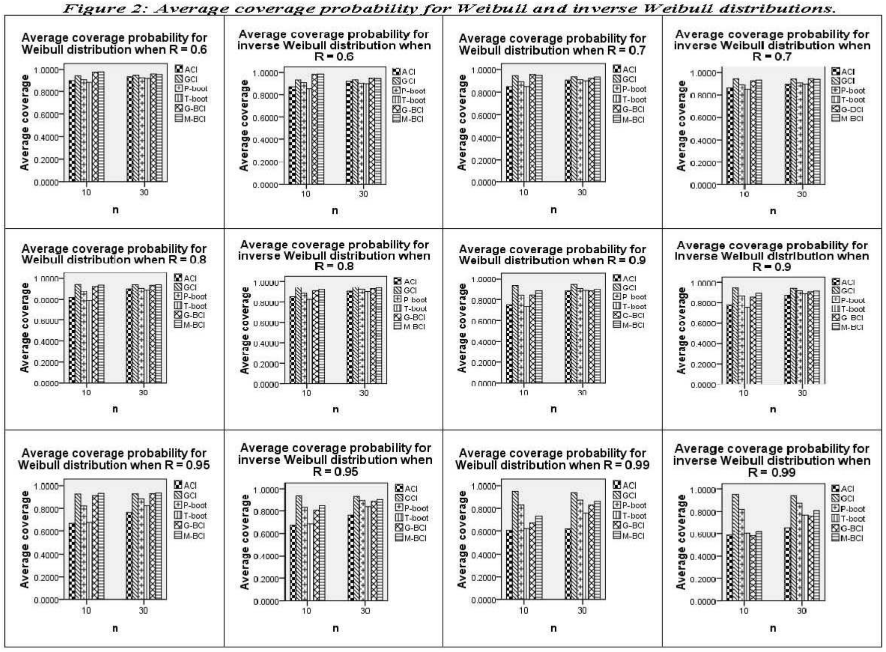


1000 samples of sample sizes (n_1, n_2) = (10, 10) (small) and (30, 30) (large) from the underlying distributions of X_1 and X_2, with unknown parameters. The Weibull distribution is considered as an example of the general exponential form, and the inverse Weibull distribution as an example of the general inverse exponential form. Taking α = 0.05, average length, average coverage probability, left tail and right tail errors of the $(1-\alpha)100\%$ confidence intervals are calculated. We select the parameter values that produce the values of R = 0.6, 0.7, 0.8, 0.9, 0.95, and 0.99.

Let $\underline{X}_i$; $i = 1,\ 2$, be two independent random samples from Weibull distributions with the survival function given as $\bar{F}_{X_i}(x;b_i,c) = exp\left[-\theta_i(b_i,c)\ g_1(x;c)\right]$; $i = 1,2$, where, $\theta_i = \frac{1}{b_i^c}$; $i = 1,2$, and $g_1(x;c) = x^c$. For the approximate $(1-\alpha)100\%$ confidence interval for R, using the MLEs $\left(\hat{c},\ \hat{\theta}_1, \hat{\theta}_2, \hat{R}\right)$, where, the MLE $\hat{c}$ of c is obtained from (7) by the Newton–Raphson iterative method, and the MLEs $\hat{\theta}_i$, and $\hat{R}$ can be expressed from (9) and (10) as $\hat{\theta}_i = \frac{1}{\hat{b}_i^{\hat{c}}} = \frac{n_i}{\sum_{j=1}^{n_i} x_{ij}^{\hat{c}}}$;$i = 1,\ 2$, and $\hat{R} = \frac{1}{1+\frac{\hat{\theta}_2}{\hat{\theta}_1}} = \frac{1}{1+\left(\frac{\hat{b}_1}{\hat{b}_2}\right)^{\hat{c}}}$. For the generalized confidence interval, the G_R can be obtained from (11), where, $G_{\theta_i} = \left(\frac{1}{G_{b_i}}\right)^{G_c}$, $G_c = \left(\frac{c}{\hat{c}}\right)\hat{c}_0 = \frac{\hat{c}_0}{\hat{c}^*}$, and $G_{b_i} = \left(\frac{b_i}{\hat{b}_i}\right)^{\frac{1}{G_c}}\hat{b}_{0i} = \left(\frac{1}{\hat{b}_i^*}\right)^{\frac{1}{G_c}}\hat{b}_{0i}$;$i = 1,\ 2$, and $\left(\hat{c}_0,\ \hat{b}_{01}, \hat{b}_{02}\right)$ denotes the observed value of the MLEs $\left(\hat{c},\ \hat{b}_1, \hat{b}_2\right)$. Thoman et al. [11] showed that the distributions of these quantities $\hat{c}^* = \left(\frac{\hat{c}_1}{c_1}\right)$ and $\hat{b}_i^* = \left(\frac{\hat{b}_i}{b_i}\right)$;$i = 1,2$, do not depend on any unknown parameters, and so they are pivotal quantities. The MLEs $\hat{c}^*$, $\hat{b}_i^*$ of c, b_i can be obtained respectively by generating independent samples from Exp(1) distribution (see, Krishnamoorthy et al. [12]). We introduce the following algorithm to estimate the generalized confidence interval of R, using any programming language as R-language (see, Krishnamoorthy and Lin [13]).

Algorithm 3.

Figure 2: Average coverage probability for Weibull and inverse Weibull distributions.



1.Generate two independent random samples $\underline{X}_i$ from $Weibull(b_i,\ c)$; $i=1,\ 2$, respectively, compute the MLEs $\left(\hat{c}_0,\, \hat{b}_{01}, \hat{b}_{02}\right)$ of $(c,\ b_1, b_2)$.

2.Generate two independent random samples $\underline{X}_i^*$ from $Exp(1)$; $i=1,\ 2$, compute the MLEs $\left(\hat{c}^*,\, \hat{b}_1^*, \hat{b}_2^*\right)$.

3.Compute the GPQs, $G_c, G_{b_i}, G_{\theta_i}$, *and* G_R;$i=1,\ 2$.

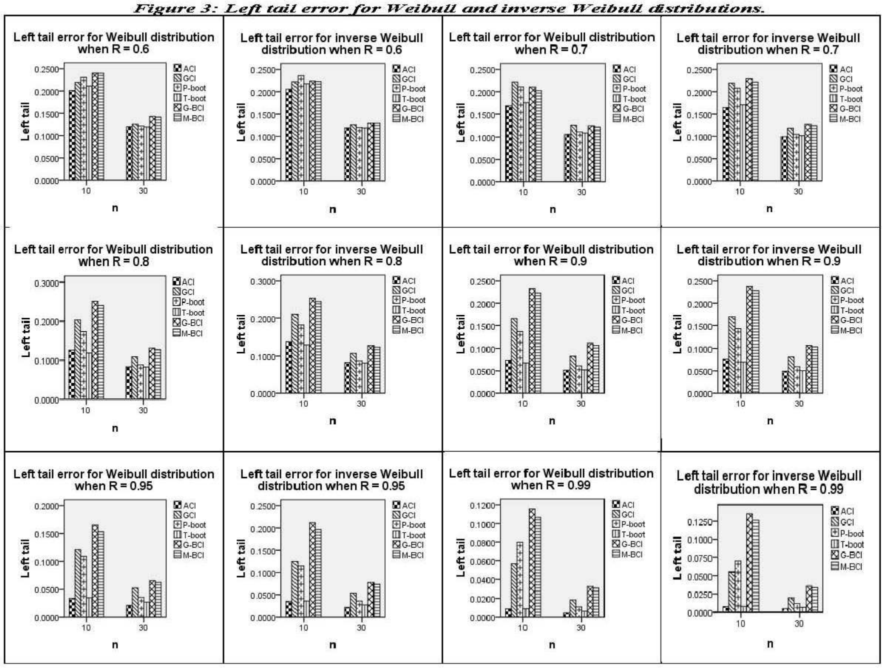
4.Repeat the steps 2-3, N times to obtain a set of samples of G_R, say$\left\{G_{R_j}; j=1,\ldots,N\right\}$, and the ordered $G_{R_j}; j=1,\ldots,\ N$, will be denoted as $G_{R_j}^{(1)} < \cdots < G_{R_j}^{(N)}$.

5.Construct the $(1-\alpha)100\%$ generalized confidence interval of R as $\left(G_{R(\alpha/2)},\, G_{R(1-\alpha/2)}\right)$.

We can also obtain the $(1-\alpha)100\%$ bootstrap and Bayesian confidence intervals of R, using Algorithm 1 and 2, respectively.

If $\underline{X}_1$ and $\underline{X}_2$ are two independent random samples from inverse Weibull distributions $F_{X_i}(x; b_i, c) = exp\left[-\eta_i(b_i, c)\, g_2(x; c)\right]$;$i=1,\ 2$, respectively, where, $\eta_i = \frac{1}{b_i^c}$;$i=1,2$, and $g_2(x;c) = \frac{1}{x^c}$. We used the MLEs $\left(\hat{c},\, \hat{\eta}_1, \hat{\eta}_2, \hat{R}\right)$ to obtain the approximate $(1-\alpha)100\%$ confidence interval for R, where, the MLE $\hat{c}$ of c is obtained numerically, using the Newton–Raphson iterative method, and the MLEs $\hat{\eta}_i$, and $\hat{R}$ can be obtained as $\hat{\eta}_i = \frac{1}{\hat{b}_i^{\hat{c}}} = \frac{n_i}{\sum_{j=1}^{n_i} x_{ij}^{-\hat{c}}}$;$i=1,\ 2$, and $\hat{R} = \frac{1}{1+\frac{\hat{\eta}_1}{\hat{\eta}_2}} = \frac{1}{1+\left(\frac{\hat{b}_2}{\hat{b}_1}\right)^{\hat{c}}}$. The G_R given from (12), where, $G_{\eta_i} = \left(\frac{1}{G_{b_i}}\right)^{G_c}$, $G_c = \left(\frac{c}{\hat{c}}\right)\hat{c}_0 = \frac{\hat{c}_0}{\hat{c}^*}$, and $G_{b_i} = \left(\frac{b_i}{\hat{b}_i}\right)^{\frac{1}{G_c}} \hat{b}_{0i} = \left(\frac{1}{\hat{b}_i^*}\right)^{\frac{1}{G_c}} \hat{b}_{0i}$;$i=1,\ 2$, and $\left(\hat{c}_0,\, \hat{b}_{01}, \hat{b}_{02}\right)$ is the observed value of the MLEs $\left(\hat{c},\, \hat{b}_1, \hat{b}_2\right)$, and $\hat{c}^* = \left(\frac{\hat{c}}{c}\right)$ and $\hat{b}_i^* = \left(\frac{\hat{b}_i}{b_i}\right)$;$i=1,2$, are pivotal quantities. The MLEs $\hat{c}^*, \hat{b}_i^*$ of c, b_i can be obtained respectively by generating independent samples from inverse exponential distribution $F_{X_i}(x) = exp\left[-\frac{1}{x}\right]$;$i=1,\ 2$. Using the same techniques in Algorithms (1-3), we can obtain the $(1-\alpha)100\%$ bootstrap, Bayesian, and generalized confidence intervals of R, respectively.

Figure 3: Left tail error for Weibull and inverse Weibull distributions.



In the Bayesian estimation, we choose the values of the hyper-parameters in both cases (gamma priors and mixed priors) for both general forms on basis of same means, but different variances.

For gamma priors: let $(d_1, h_1) = (3, 3/2)$, $(d_2, h_2) = (2, 1)$, and $(d_3, h_3) = (1, 1/2)$.

For mixed priors: let $(d_1, h_1) = (3, 3/2)$, $(d_2, h_2) = (2, 1)$.

The comparison on the basis of average length, average coverage, left and right tail errors are introduced for the Weibull and the inverse Weibull distributions. Figure 1 and 2 present the average lengths and the average coverage probabilities of the different intervals (ACI, GCI, P-boot, T-boot, G-BCI, and M-BCI) for both Weibull & inverse Weibull distributions. Figures 3 and 4 present the left and right tail errors of the same intervals for the same distributions. From Figure 1, we see that the boot is the largest average length when $R = 0.6\&0.7$, at $R = 0.8 - 0.99$, G-BCI and ACI have the largest and the smallest average length, respectively. In Figure 2, the average coverage probability of GCI is roundly the anticipated$(1-\alpha)\,100\%$, the P-boot gives better results than T-boot, we see also from Figure 2 that, ACI and T-boot affected by n and R. We observe in Figure 3 that, the G-BCI has the largest left tail error when $R = 0.8 - 0.99$. From Figure 4 we see that, the right tail error of T-boot is the largest and G-BCI is the smallest. We note that in Figures 1-4, the G-BCI and M-BCI are very close to each other. In Figures 1, 3, and 4, R and n affect average length, and tail errors of all confidence intervals except BCI.

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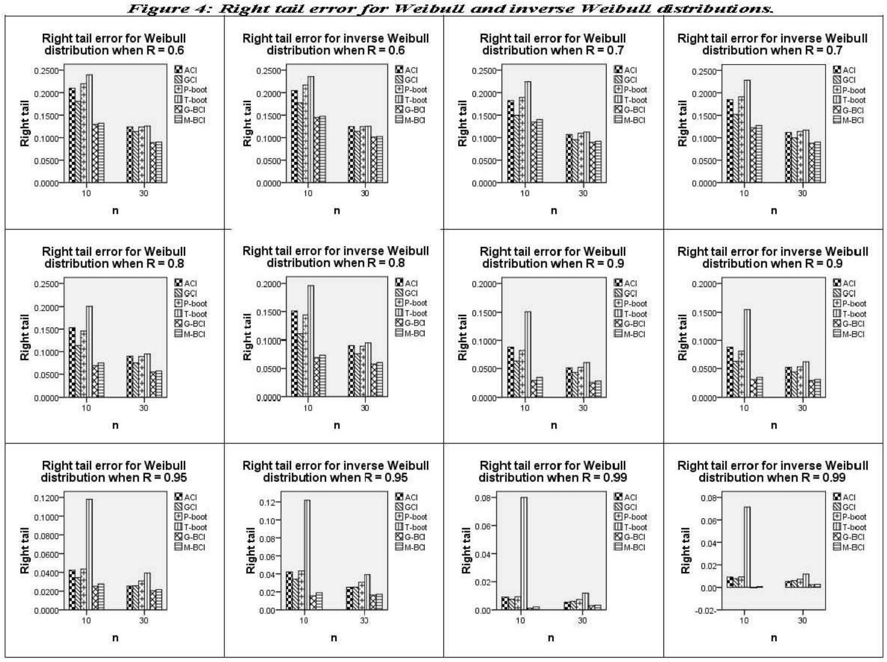
Figure 4: Right tail error for Weibull and inverse Weibull distributions.



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