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A Generalization of the BurrXII-Poisson Distribution and its Applications

Mustapha Muhammad[*]

Department of Mathematical Science, Bayero University Kano, PMB 3011, Nigeria.

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Abstract: In this paper we generalize the BurrXII-poisson distribution and we refer to this generalization as the generalize BurrXII-poisson distribution (GBXIIP). Several properties and inferences of the generalize BurrXII-poisson distribution are obtained and studied including the shapes properties of its probability density and hazard rate functions. Moreover the existence of its MLEs under some certain conditions are analyzed. Two real data applications are used to demonstrate the performance and effectiveness of the proposed distribution.

Keywords: BurrXII; BurrXII-Poisson distribution; maximum likelihood estimates.

1 Introduction

A new family of distribution known as exponential geometric (EG) was proposed by [1], by compounding the exponential and geometric distribution. [2] introduced the complementary exponential-geometric (CEG) as a complementry to the exponential-gometric(EG) distribution. In the same way, [3] introduced a new class of distribution known as the Exponential Poisson distribution (EP) and recently [4] proposed the Generalize exponencial poisson (GEP) distribution as the generalization of (EP) distribution, by exponentiating the cdf of the Exponential Poisson (EP) distribution. [5] proposed a new family of distribution called the BurrXII-power series (BXIIPS), this distribution is obtained using the procedure follows by the [1], the BurrXII-power series distribution includes the BurrXII-poisson (BXIIP) distribution as its sub model. The cumulative distribution function of the BXIIP distribution with parameters $\alpha > 0$, $\beta > 0$ and $\lambda > 0$ is defined as

$$H(x) = \frac{1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1))}{(1-\exp(-\lambda))}, \qquad x > 0. \tag{1}$$

Following the same approach, in this paper we introduces a new four parameter lifetime model named the generalized BurrXII-poisson (GBXIIP) distribution by exponentiating the cdf (in Eq. (1)) of BXIIP distribution. In section 2 we give the density, hazard rate function and the quantile function of the GBXIIP distribution. Moreover we derive the moments, orderstatistics, moment of orderstatistics and the Renyi entropy. In section 3 Estimation by maximum likelihood and inferences are analyzed. Section 4 provides applications to two real data sets and section 5 concludes the paper.

2 The GBXIIP distribution

A random variable X has the generalize BurrXII-poisson distribution with parameter a, α, β and λ if its cdf is

$$F(x) = \left(\frac{1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1))}{(1-\exp(-\lambda))}\right)^{a}, \qquad x > 0. \tag{2}$$

[*] Corresponding author e-mail: mmmahmoud12@sci.just.edu.jo, mmustaphamuhd@yahoo.com

Notice that when $a=1$ the GBXIIP be come BXIIP distribution. The pdf, hazard rate function (hrf) and the survival function of the GBXIIP distribution are given respectively, by

$$f(x;a,\alpha,\beta,\lambda)=\frac{a\,\alpha\beta\lambda\,x^{\alpha-1}(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{a-1}\exp(\lambda((1+x^{\alpha})^{-\beta}-1)}{(1-\exp(-\lambda))^{a}\,(1+x^{\alpha})^{(\beta+1)}}, \tag{3}$$

$$s(x)=\frac{(1-\exp(-\lambda))^{a}-(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{a}}{(1-\exp(-\lambda))^{a}}, \tag{4}$$

$$h(x)=\frac{a\alpha\beta\lambda x^{\alpha-1}(1+x^{\alpha})^{-(\beta+1)}(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{a-1}\exp(\lambda((1+x^{\alpha})^{-\beta}-1))}{(1-\exp(-\lambda))^{a}-(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{a}}. \tag{5}$$

The quantile function $\psi(u)$ of the generalize BurrXII-poisson (GBXIIP) distribution given by Eq. (6), is straightforward to be computed by inverting Eq. (2), and it can be used to generate random data.

$$\psi(u)=\left(\left(\frac{\log(1-u^{\frac{1}{a}}(1-\exp(-\lambda)))}{\lambda}+1\right)^{-\frac{1}{\beta}}-1\right)^{\frac{1}{\alpha}}, \qquad u\in(0,1). \tag{6}$$

Theorem 2.1 *The limiting distribution of the GBXIIP(a,α,β,λ) given by Eq. (2) when $\lambda\to 0^{+}$ and $a\in\mathbb{N}$ or $a=\frac{1}{k}$, $k(even)\in\mathbb{N}$, is, $\lim_{\lambda\to 0^{+}}F(x)=\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{a}$, i.e. It converges to exponentiated BurrXII distribution with parameter a,α and β.*

Theorem 2.2 *The pdf of the generalize BurrXII-poisson (GBXIIP) given by Eq. (3) is monotone decreasing function for $0<\alpha\leq 1$ and $0<a\leq 1$, and unimodal for $\alpha>1$ and $a>1$.*

Proof:

$$\log(f(x))=\log\left(\frac{a\,\alpha\beta\lambda}{(1-\exp(-\lambda))^{a}}\right)+(\alpha-1)\log x-(\beta+1)\log(1+x^{\alpha})$$
$$+(a-1)\log(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))+(\lambda((1+x^{\alpha})^{-\beta}-1))$$

and

$$(\log f(x))'=\frac{\alpha-1}{x}-\frac{\alpha(\beta+1)x^{\alpha-1}}{(1+x^{\alpha})}+\frac{(a-1)\alpha\beta\lambda\,x^{\alpha-1}\exp(\lambda((1+x^{\alpha})^{-\beta}-1))}{(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))\,(1+x^{\alpha})^{\beta+1}}-\frac{\alpha\beta\lambda\,x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}.$$

When $0<\alpha\leq 1$ and $0<a\leq 1$, $(\log f(x))'<0$, this implies $f(x)$ is monotone decreasing function. Suppose that $\alpha>1$ and $a>1$, then $f(x)$ has exactly one root say x_o. Then, for $x<x_o$, $f(x)>0$ and for $x>x_o$, $f(x)<0$ so, $f(x)$ gives a unimodal shape with mode value $x=x_o$.

Theorem 2.3 *If $\alpha\leq 1$ and $a\leq 1$, the hrf of the generalize BurrXII-poisson (GBXIIP) given by Eq. (5) is monotone decreasing function.*

Proof:

Following the theorem of [6], we let

$$\eta(x)=\frac{-f'(x)}{f(x)}=-\frac{(\alpha-1)}{x}+\frac{\alpha(\beta+1)x^{\alpha-1}}{(1+x^{\alpha})}-\frac{(a-1)\alpha\beta\lambda\,x^{\alpha-1}\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))}{(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))(1+x^{\alpha})^{\beta+1}}+\frac{\alpha\beta\lambda\,x^{\alpha-1}}{(1+x^{\alpha})^{\beta+1}}.$$

Where $f'(x)$ is the first derivative of $f(x)$ in Eq. (3). Then, $\eta'(x)$ is given by

$$\eta'(x) = \frac{(\alpha-1)}{x^2} + \frac{\alpha(\alpha-1)(\beta+1)x^{\alpha-2}}{(1+x^{\alpha})} - \frac{\alpha^2(\beta+1)x^{2(\alpha-1)}}{(1+x^{\alpha})^2}$$
$$- \frac{(a-1)(\alpha-1)\alpha\beta\lambda x^{\alpha-2} z(x)}{(1-z(x))(1+x^{\alpha})^{\beta+1}} + \frac{(a-1)\alpha^2\beta^2\lambda^2 x^{2(\alpha-1)} z(x)}{(1-z(x))(1+x^{\alpha})^{2(\beta+1)}}$$
$$+ \frac{(a-1)\alpha^2\beta^2\lambda^2 x^{2(\alpha-1)} (z(x))^2}{(1-z(x))^2(1+x^{\alpha})^{2(\beta+1)}} + \frac{(a-1)\alpha^2\beta\lambda x^{2(\alpha-1)} z(x)}{(1-z(x))(1+x^{\alpha})^{2(\beta+2)}}$$
$$+ \frac{(\alpha-1)\alpha\beta\lambda x^{\alpha-2}}{(1+x^{\alpha})^{\beta+1}} - \frac{\alpha^2\beta(\beta+1)x^{2(\alpha-1)}}{(1+x^{\alpha})^{\beta+2}}.$$

Where $z(x) = \exp(\lambda((1+x^{\alpha})^{-\beta}-1))$, thus, for $\alpha \le 1$ and $a \le 1$, $\eta'(x) < 0$, hence the proof. We can also see from Figure 1 that Eq. (5) can take other shapes

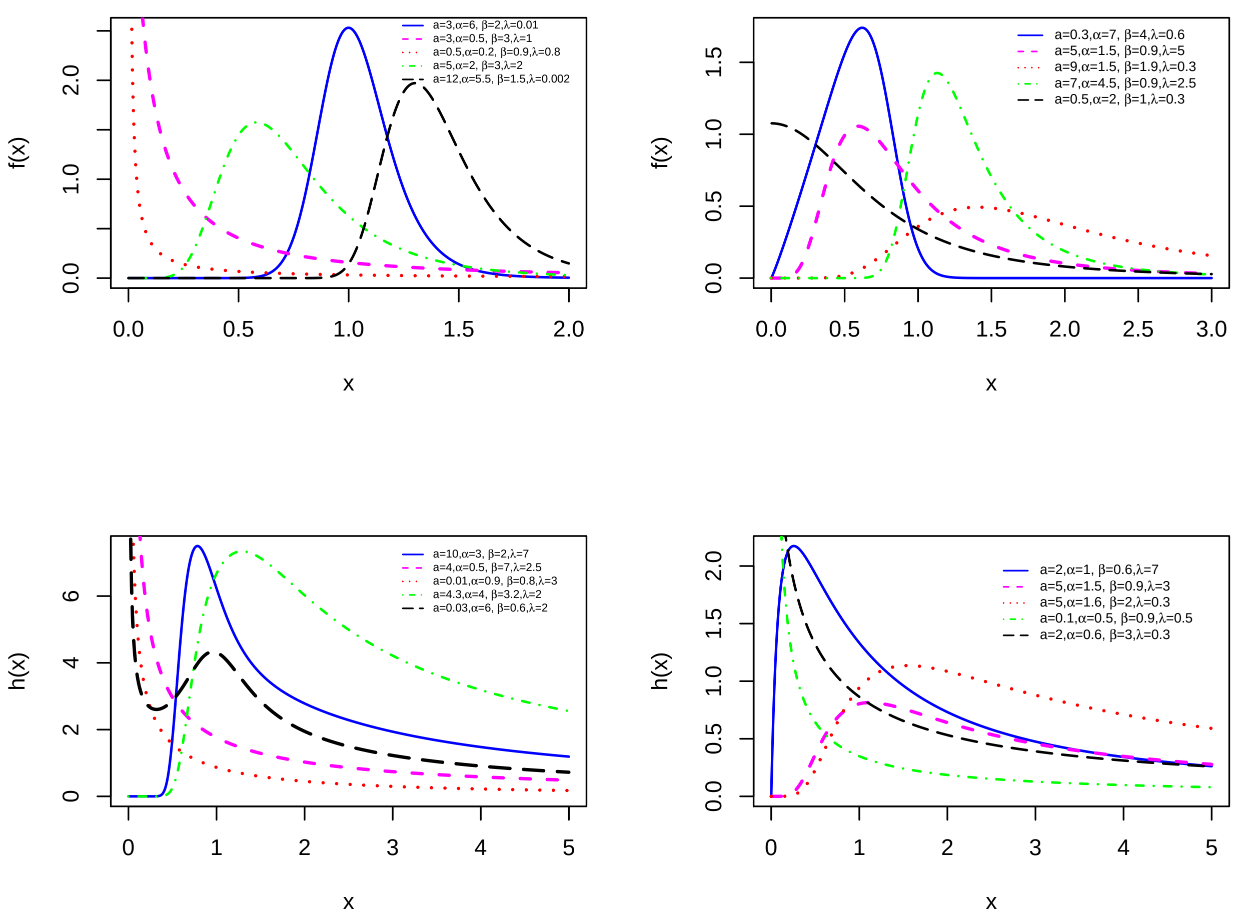

Fig. 1: plots of Probability density (f(x)) and hazard rate (h(x)) functions of the generalize BurrXII-poisson distribution for different values of parameter.

2.1 Useful expansions

We demonstrate that the Pdf of the generalize BurrXII-poisson distribution (GBXIIP) can be written as a infinite mixture of the BXIIP or BXII densities. Using the following series representation

$$(1-u)^{a-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j)\, j!} u^j \tag{7}$$

where $|u| < 1$, $a > 0$ real and non-intiger, then, we have an infinite mixture as

$$f(x) = \sum_{j=0}^{\infty} \varphi_j\, g(x; \alpha, \beta, \lambda(j+1)), \tag{8}$$

where

$$\varphi_j = \frac{(-1)^j\, a!\, (1-\exp(-\lambda(j+1)))}{(1-\exp(-\lambda))^a\, (j+1)\, \Gamma(a-j)\, j!}$$

and $g(x; \alpha, \beta, \lambda(j+1))$ is the BurrXII-poisson probability density function with parameter α, β and $\lambda(j+1)$. Also by applying the exponential expansion in Eq. (8) above and some algebraic manipulation the GBXIIP density function can be written as an infinite double mixture of BXII$(\alpha, \beta(i+1))$ as

$$f(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \varphi^*_{i,j}\, g^*(x; \alpha, \beta(i+1)), \tag{9}$$

where

$$\varphi^*_{i,j} = \frac{(-1)^j\, a!\, \lambda^{i+1} (j+1)^i (\exp(-\lambda(j+1)))}{(1-\exp(-\lambda))^a\, (i+1)\, \Gamma(a-j)\, j!\, i!}$$

and $g^*(x; \alpha, \beta(i+1))$ is the pdf of the BurrXII distribution with parameters α and $\beta(i+1)$.

2.2 Moments

Moments of a distribution are extremely essential in various statistical analysis, particularly in practical applications. Most of the features and characteristics of a probability model can be analyzed through its skewness, kurtusis, tendency and dispertion. The following lemma provide the r^{th} central moment of the generalize BurrXII-poisson (GBXIIP) distribution which is extremely useful in computing some other properties of the proposed distribution.

Lemma 2.4 *if X has GBXIIP$(a, \alpha, \beta, \lambda)$ and for $a > 0$ real and non-intigers, the r^{th} central moment of X, say μ^r is given as*

$$E(X^r) = \frac{a!\, \beta}{(1-\exp(-\lambda))^a} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^j\, \lambda^{i+1} (j+1)^i\, \exp(-\lambda(j+1))}{\Gamma(a-j)\, j!\, i!} B(\beta(i+1) - \frac{r}{\alpha}, \frac{r}{\alpha} + 1).$$

Proof:

$$E(X^r) = \int_0^{\infty} x^r f(x) dx$$

Let $u = (1+x^{\alpha})^{-\beta}$ then, for $a > 0$ real and non-integers we can apply the series representation given by Eq. (7), therefore,

$$E(X^r) = \frac{a\Gamma(a)\, \lambda}{(1-\exp(-\lambda))^a} \sum_{j=0}^{\infty} \frac{(-1)^j\, \exp(-\lambda(j+1))}{\Gamma(a-j)\, j!} \int_0^1 (1-u^{\frac{1}{\beta}})^{\frac{r}{\alpha}} u^{-\frac{r}{\alpha\beta}} \exp(\lambda(j+1)u) du,$$

by apllying the exponential expansion for $\exp(\lambda(j+1)u)$ and letting $u=v^{\beta}$, we have

$$E(X^r)=\frac{a\Gamma(a)\,\beta\,\lambda}{(1-\exp(-\lambda))^a}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j\,\lambda^i(j+1)^i\,\exp(-\lambda(j+1))}{\Gamma(a-j)\,j!\,i!}\int_0^1(1-v)^{\frac{r}{\alpha}}v^{\beta(i+1)-\frac{r}{\alpha}-1}dv.$$

Thus,

$$E(X^r)=\frac{a!\,\beta}{(1-\exp(-\lambda))^a}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j\,\lambda^{i+1}(j+1)^i\,\exp(-\lambda(j+1))}{\Gamma(a-j)\,j!\,i!}B(\beta(i+1)-\frac{r}{\alpha},\frac{r}{\alpha}+1). \tag{10}$$

The moment generating function (mgf) of GBXIIP distribution is computed by $M_x(t)=E(e^{tX})$ which can be express as

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(X^r). \tag{11}$$

Putting Eq. (10) in Eq. (11) we have

$$M_x(t)=\frac{a!\beta}{(1-\exp(-\lambda))^a}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j t^r\lambda^{i+1}(j+1)^i\exp(-\lambda(j+1))}{\Gamma(a-j)\,i!\,j!\,r!}B(\beta(i+1)-\frac{r}{\alpha},\frac{r}{\alpha}+1). \tag{12}$$

While the skewness (γ_3) and kurtosis (γ_4) of the GBXIIP distribution can be obtained by substituting Eq. (10) in the followings below.

$$\gamma_3=\frac{1}{\sigma^3}\sum_{r=0}^{3}\binom{3}{r}(-1)^{r+1}\mu^{3-r}E(X^r), \tag{13}$$

and

$$\gamma_4=\frac{1}{\sigma^4}\sum_{r=0}^{4}\binom{4}{r}(-1)^{r}\mu^{4-r}E(X^r). \tag{14}$$

Where μ and σ are the mean and standard deviation of the GBXIIP distribution. Futhermore, the skewness and kurtosis of the generalize BurrXII-poisson (GBXIIP) distribution can be observed using its quantile function. The Bowley skewness (B) and Moores kurtosis (M) provides the measure of asymmetry and the degree of peakedness of a distribution with respect to one of its parameter respectively. The Bowley skewness and Moores kurtosis are defined, respectively by

$$B=\frac{\psi(3/4)+\psi(1/4)-2\,\psi(2/4)}{\psi(3/4)-\psi(1/4)}, \tag{15}$$

and

$$M=\frac{\psi(3/8)-\psi(1/8)+\psi(7/8)-\psi(5/8)}{\psi(6/8)-\psi(2/8)}. \tag{16}$$

Where $\psi(u)$ is a quantile fuction given by Eq. (6). Figure 2 shows the plots of the Bowley skewness and Moores kurtosis of generalize BurrXII-poisson (GBXIIP) distribution.

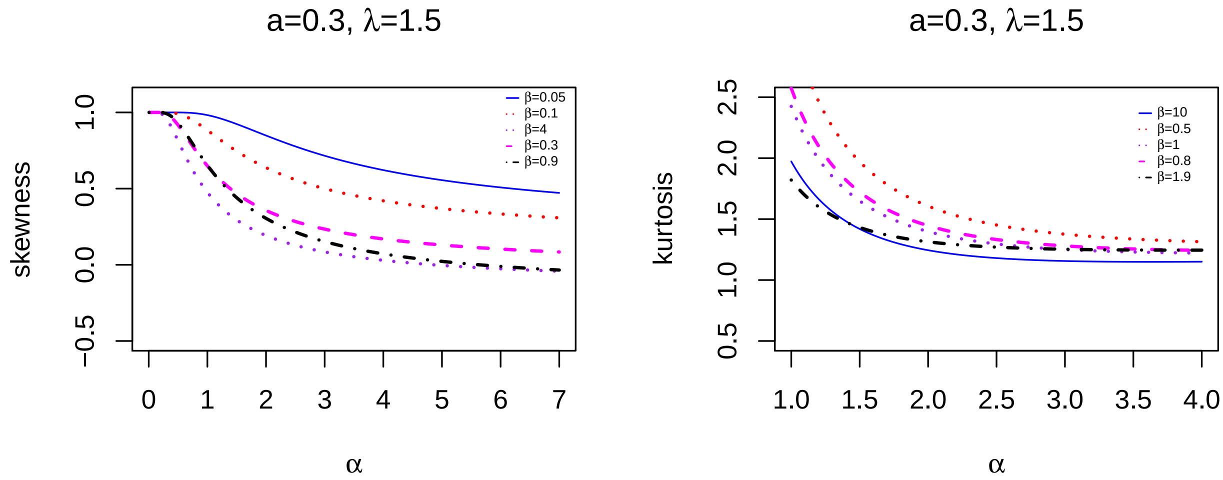

Fig. 2: plots of the Bowley skewness (B) and Moores kurtosis (M) of generalize BurrXII-poisson distribution for different values of parameter β

2.3 Order statistics

Let $X_1, X_2, \cdots, X_n$ be a random sample obtained from GBXIIP(a,α,β,λ) distribution with cdf and pdf given by Eqs. (2) and (3), respectively. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics observed from this sample, then, the pdf of $X_{j:n}$ that is $f_{j:n}(x), \;\; j = 1,2,\cdots,n$ can be computed by

$$f_{j:n}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\, F(x)^{j-1}(1-F(x))^{n-j}, \tag{17}$$

where F(x) and f(x) are given by Eqs. (2) and (3) respectively. By the binomial expansion for

$$\begin{aligned}((1-\exp(-\lambda))^{a} - (1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{a})^{n-j} = &\sum_{k=0}^{n-j}\frac{(-1)^{k}\,(n-j)!}{k!\,(n-j-k)!}\\ &\times(1-\exp(-\lambda))^{a(n-j-k)}\\ &\times(1-\exp(\lambda((1+x^{\alpha})^{-\beta}-1)))^{ak},\end{aligned}$$

then, subtitute in Eq. (17) above and after some algebraic manipulation, finally we obtain the following

$$f_{j:n}(x) = \sum_{k=0}^{n-j}\frac{(-1)^{k}\,n!}{(n-j-k)!\,(j+k)\,(j-1)!\,k!}\; f(x;a(j+k),\alpha,\beta,\lambda) \tag{18}$$

where $f(x;a(j+k),\alpha,\beta,\lambda)$ is the pdf of the generalize BurrXII-poisson (GBXIIP) model with parameter $a(j+k),\alpha,\beta$ and λ. The r^{th} ordinary moment of the j^{th} order statistics of the GBXIIP(a,α,β,λ) distribution can be computed from

$$E(X_{j:n}^{r}) = \int_{0}^{\infty} x^{r} f_{j:n}(x)dx. \tag{19}$$

where $f_{j:n}(x)$ is the pdf of order statistics of the generalize BurrXII-poisson (GBXIIP) distribution given by Eq. (18). Following the same approach as in (10), thus,

$$E(X_{j:n}^{r}) = \sum_{k=0}^{n-j}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{k+w}\,a\,\beta\,n!\,\lambda^{i+1}\,(w+1)^{i}\,\Gamma(a(j+k))\,\exp(-\lambda(w+1))}{(1-\exp(-\lambda))^{a(j+k)}(j-1)!\,(n-j-k)!\,\Gamma(a(j+k)-w)\,i!\,k!\,w!}B(\beta(i+1)-\frac{r}{\alpha},\frac{r}{\alpha}+1).$$

2.4 Renyi entropy

An entropy of a random variable X can be defined as a measure of variation of uncertainty. If a random variable X is distributed according to generalize BurrXII-poisson (GBXIIP) distribution, then, the renyi entropy can be obtain from $I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\int_0^\infty f(x)^\rho dx\right]$, where $\rho > 0$ and $\rho \neq 1$, therefor, we start by computing

$$\int_0^\infty f(x)^\rho dx = \left(\frac{a\alpha\beta\lambda}{(1-\exp(-\lambda))^a}\right)^\rho \int_0^\infty \frac{x^{\rho(\alpha-1)}(1-\exp(\lambda((1+x^\alpha)^{-\beta}-1)))^{\rho(a-1)}\exp(\lambda\rho((1+x^\alpha)^{-\beta}-1))}{(1+x^\alpha)^{\rho(\beta+1)}}dx.$$

Setting $u=(1+x^\alpha)^{-\beta}$ and for $\rho(a-1)+1>0$ real and non-integers we can apply the series representation in Eq. (7), then,

$$\int_0^\infty f(x)^\rho dx = \frac{(a\lambda)^\rho(\alpha\beta)^{\rho-1}}{(1-\exp(-\lambda))^{a\rho}}\sum_{j=0}^\infty \frac{(-1)^j\,\Gamma(\rho(a-1))\,\exp(-\lambda(\rho+j))}{\Gamma(\rho(a-1)-j)\,j!} \times\int_0^1 (1-u^{\frac{1}{\beta}})^{(1-\frac{1}{\alpha})(\rho-1)}\,u^{(\frac{1}{\alpha\beta}+1)(\rho-1)}\,\exp(\lambda(\rho+j)u)\,du.$$

Apllying the exponential expansion for $\exp(\lambda(\rho+j)u)$ and letting $u=v^\beta$, we have

$$\int_0^\infty f(x)^\rho dx = \frac{(a\beta\lambda)^\rho\,\alpha^{\rho-1}}{(1-\exp(-\lambda))^{a\rho}}\sum_{j=0}^\infty\sum_{i=0}^\infty \frac{(-1)^j\,\lambda^i\,(\rho+j)^i\,\Gamma(\rho(a-1))\,\exp(-\lambda(\rho+j))}{\Gamma(\rho(a-1)-j)\,i!\,j!} \times\int_0^1 (1-v)^{(1-\frac{1}{\alpha})(\rho-1)}\,v^{\beta(i+1)+(\frac{1}{\alpha}+\beta)(\rho-1)-1}\,dv.$$

$$\int_0^\infty f(x)^\rho dx = \frac{(a\beta\lambda)^\rho\,\alpha^{\rho-1}}{(1-\exp(-\lambda))^{a\rho}}\sum_{j=0}^\infty\sum_{i=0}^\infty \frac{(-1)^j\,\lambda^i\,(\rho+j)^i\,\Gamma(\rho(a-1))\,\exp(-\lambda(\rho+j))}{\Gamma(\rho(a-1)-j)\,i!\,j!} \times B(\beta(i+1)+(\frac{1}{\alpha}+\beta)(\rho-1),(1-\frac{1}{\alpha})(\rho-1)+1).$$

Thus, the renyi entropy is

$$I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\sum_{j=0}^\infty\sum_{i=0}^\infty \phi_{i,j}(\rho)\,B(\beta(i+1)+(\frac{1}{\alpha}+\beta)(\rho-1),(1-\frac{1}{\alpha})(\rho-1)+1)\right].$$

Where

$$\phi_{i,j}(\rho) = \frac{(a\beta\lambda)^\rho\,\alpha^{\rho-1}\,(-1)^j\,\lambda^i\,(\rho+j)^i\,\Gamma(\rho(a-1))}{(1-\exp(-\lambda))^{a\rho}\,\Gamma(\rho(a-1)-j)\,i!\,j!}.$$

3 Estimation and inference

Let X_i, $i=1,2,\cdots n.$, be a random sample of size n obtained from the generalize BurrXII-poisson (GBXIIP) distribution, let $\theta=(a,\alpha,\beta,\lambda)^T$ be the vetor of parameters of the generalize BurrXII-poisson (GBXIIP) distribution. Then we can express the log-likelihood function for the vetor of parameters as

$$\begin{aligned}\log\ell(\theta) &= n\log a+n\log\alpha+n\log\beta+n\log\lambda-an\log(1-\exp(-\lambda))\\ &+(\alpha-1)\sum_{i=1}^n\log x_i-(\beta+1)\sum_{i=1}^n\log(1+x_i^\alpha)\\ &+(a-1)\sum_{i=1}^n\log(1-\exp(\lambda((1+x_i^\alpha)^{-\beta}-1)))+\lambda\sum_{i=1}^n((1+x_i^\alpha)^{-\beta}-1).\end{aligned}$$

And the first partial derivative of $\log \ell(\theta)$, that is $\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}$ are computed as

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - n\log(1-\exp(-\lambda)) + \sum_{i=1}^{n} \log\left(1-\exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))\right), \tag{20}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i - (\beta+1)\sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(1+x_i^{\alpha})} - \beta\lambda \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(1+x_i^{\alpha})^{\beta+1}} \\ &+ (a-1)\beta\lambda \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i\ \exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))}{(1+x_i^{\alpha})^{\beta+1}\left(1-\exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))\right)},\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n} \log(1+x_i^{\alpha}) - \lambda \sum_{i=1}^{n} \frac{\log(1+x_i^{\alpha})}{(1+x_i^{\alpha})^{\beta}} \\ &+ (a-1)\lambda \sum_{i=1}^{n} \frac{\log(1+x_i^{\alpha})\ \exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))}{(1+x_i^{\alpha})^{\beta}\left(1-\exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))\right)},\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} - \frac{a\,n\exp(-\lambda)}{(1-\exp(-\lambda))} + \sum_{i=1}^{n} ((1+x_i^{\alpha})^{-\beta}-1) \\ &- (a-1)\sum_{i=1}^{n} \frac{((1+x_i^{\alpha})^{-\beta}-1)\ \exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))}{\left(1-\exp(\lambda((1+x_i^{\alpha})^{-\beta}-1))\right)}.\end{aligned} \tag{23}$$

The maximum likelihood estimate (MLEs) $\hat{\theta} = (\hat{a}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})^T$ of $\theta = (a, \alpha, \beta, \lambda)^T$ is obtained simultaneously by solving Eqs. (20), (21), (22) and (23) equated to zero. These nonlinear equations can be solved numerically using existing mathematical or statistical packages. For the asymptotic interval estimation and hypothesis tests of the four parameters a, α, β and λ, we need 4×4 Fisher information matrix denoted by $(J(\theta))$, under the usual condition that are fulfilled for the parameters a, α, β, and λ in the interior of the parameter space but not on the boundary. The asymptotic distribution of $\sqrt{n}(\hat{\theta}-\theta)$ is $N_4(0, I^{-1}(\theta))$, which is a Normal $4-$variate with zero mean and variance covariance $I(\theta)$. This condition is also applicable if $I(\theta)$ is substitute by the information matrix evaluated at $\hat{\theta}$, that is $J(\hat{\theta})$. The Normal $4-$variate distribution $N_4(0, J^{-1}(\theta))$ can be used to establish an approximate confidence interval and region for the model parameter a, α, β, and λ. The 4×4 informtion matrix is given as $J(\theta) = -[\partial^2 \ell/\partial\theta\partial\theta^T]$, and the element of $J(\theta)$ are given in Appendix$(D4)$.

Theorem 3.1 *Let $g_1(a;\alpha,\beta,\lambda,x_i)$ denote the function on the right of Eq. (20), where α, β and λ are the exact values of the parameters, then, the equation $g_1(a;\alpha,\beta,\lambda,x_i) = 0$ has at least one root for $\log(1-e^{-\lambda}) > \sum_{i=1}^{n} \frac{\log(1-e^{-\lambda((1+x_i^{\alpha})^{-\beta}-1)})}{n}$.*

Theorem 3.2 *Let $g_2(\alpha;a,\beta,\lambda,x_i)$ denote the function on the right of Eq. (21) where a,β and λ are the exact value of the parameters, then, the equation $g_2(\alpha;a,\beta,\lambda,x_i) = 0$ can take one of the following forms.*
(i) For $\min\{X_i\} > 1$ and $a = 1$.
(ii) For $\max\{X_i\} < 1$ and $a = 1$.

Theorem 3.3 *Let $g_3(\beta;a,\alpha,\lambda,x_i)$ be the function on the right of Eq. (22), where a, α and λ are the exact values of the parameters , then, the equation $g_2(\beta;a,\alpha,\lambda,x_i) = 0$, has at least one root for $a \neq 1$ and for $a = 1$, the root of $g_3(\beta;a,\alpha,\lambda,x_i) = 0$ lies in the interval $(n((1+\lambda)\sum_{i=1}^{n}\log(1+x_i^{\alpha}))^{-1}, n(\sum_{i=1}^{n}\log(1+x_i^{\alpha}))^{-1})$.*

Theorem 3.4 *Let $g_4(\lambda;a,\alpha,\beta,x_i)$ be the function on the right of Eq. (23), where a, α and β are the exact values of the parameters, then, the equation $g_4(\lambda;a,\alpha,\beta,x_i) = 0$, has at least one root.*

Proofs. for theorem 3.1, 3.2 and 3.3 see Appendix $D1$, $D2$ and $D3$ respectively. Theorem 3.4 is similar to theorem 3.3 for $a \neq 1$.

4 Applications

In this, section, we fit the GBXIIP distribution to two distinct real data set and we compare the performance with those of the BXIIP, BXIIG by [5] and BurrXII by [7] distributions. The Akaike information criteria (AIC) and Bayesian information criteria (BIC) are used to assess the effectiveness of the models, the model with the smallest value of these measures gives a better representation of the data set than the others. The histogram, empirical cdf and the fitted distributions are plotted for each data set.

First data set.

This data set represent the marks for the pace slow program in mathematics 2003, provided by [8], and recently studied by [9]. the data set are: 29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21,15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31. As you can see from Table 1 the GBXIIP has the AIC=409.9288 and BIC=408.6538, this show that GBXIIP fit this data better then the other distributions. Also Figure 3 provide the plots of the histogram and empirical cdf of the first data with the estimated densities obtained using MLE procedure.

Table 1: MLEs, $\ell(\theta)$, AIC and BIC for the first data

Models	Estimated parameters	$\ell(\theta)$	AIC	BIC
GBXIIP(a, α,β,λ)	$\hat{a}=39.0759$, $\hat{\alpha}=22.8287$ $\hat{\beta}=0.0607$, $\hat{\lambda}=0.03404$	-200.9644	**409.9288**	**408.6538**
BXIIP(α,β,λ)	$\hat{\alpha}=6.9462$, $\hat{\beta}=0.0478$ $\hat{\lambda}=1.4\times10^{-4}$	-245.5362	497.0724	496.1161
BXIIG(α,β,p)	$\hat{\alpha}=10.2248$, $\hat{\beta}=0.0325$ $\hat{p}=1.0\times10^{-4}$	-245.5352	497.07	496.114
BXII(α,β)	$\hat{\alpha}=9.994$, $\hat{\beta}=0.0332$	-245.5353	495.071	494.433

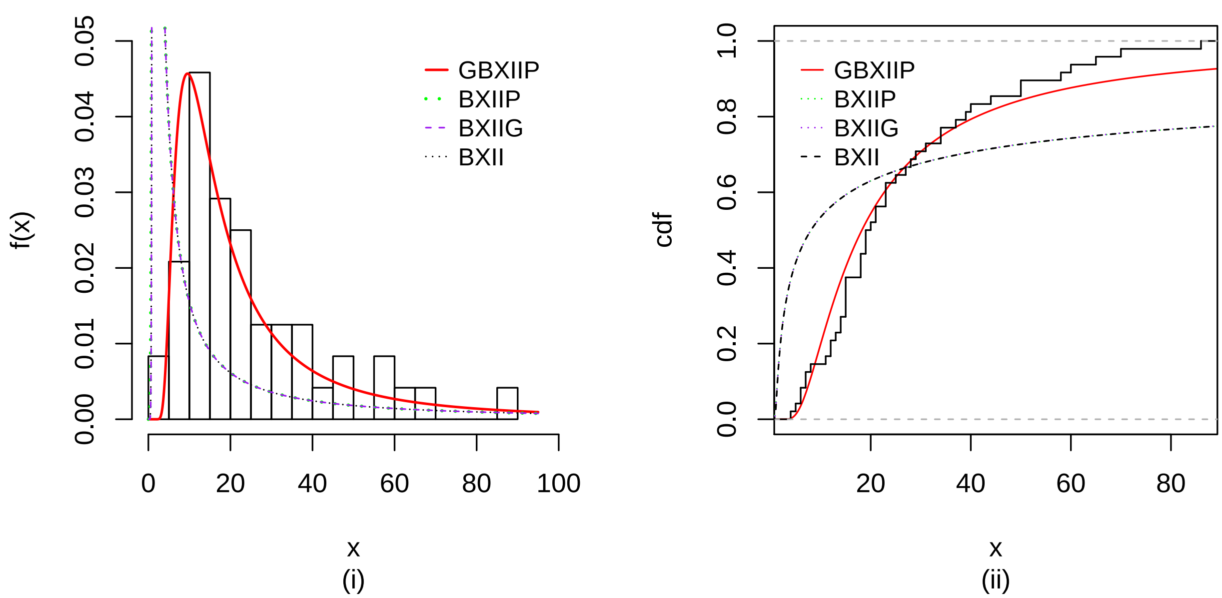

Fig. 3: (i) Histogram (ii) Empirical cdf of the first data and the fitted GBXIIP, BXIIP, BXIIG & BXII densities.

Second data set.

This data set is provided by [10] and recently analyzed by [11]. It is the measured in GPa for the strength of single carbon fibers and impregnated one thousand carbon fibers tows. Each single fiber of carbon was examined under the tension at gauge length of ten millimeters. 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397,2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. The values presented in Table 2 shows that the smallest AIC and BIC belongs to the GBXIIP distribution. Therefore, GBXIIP fit the data better than the other distributions. Also Figure 4 provide the plots of the histogram and empirical cdf of the first data with the estimated densities obtained by MLE method.

Table 2: MLEs, $\ell(\theta)$, AIC and BIC for the second data.

Models	Estimated parameters	$\ell(\theta)$	AIC	BIC
GBXIIP(a, α,β,λ)	$\hat{a}=243.327$, $\hat{\alpha}=159.5948$ $\hat{\beta}=0.03397$, $\hat{\lambda}=0.1397$	-58.9453	**125.8906**	**125.088**
BXIIP(α,β,λ)	$\hat{\alpha}=17.194$, $\hat{\beta}=0.05294$ $\hat{\lambda}=4.03\times10^{-4}$	-138.1271	282.2542	281.6522
BXIIG(α,β,p)	$\hat{\alpha}=22.6217$, $\hat{\beta}=0.0402$ $\hat{p}=1.0\times10^{-4}$	-138.1267	282.2534	281.6514
BXII(α,β)	$\hat{\alpha}=22.293$, $\hat{\beta}=0.0408$	-138.1267	280.2534	279.8521

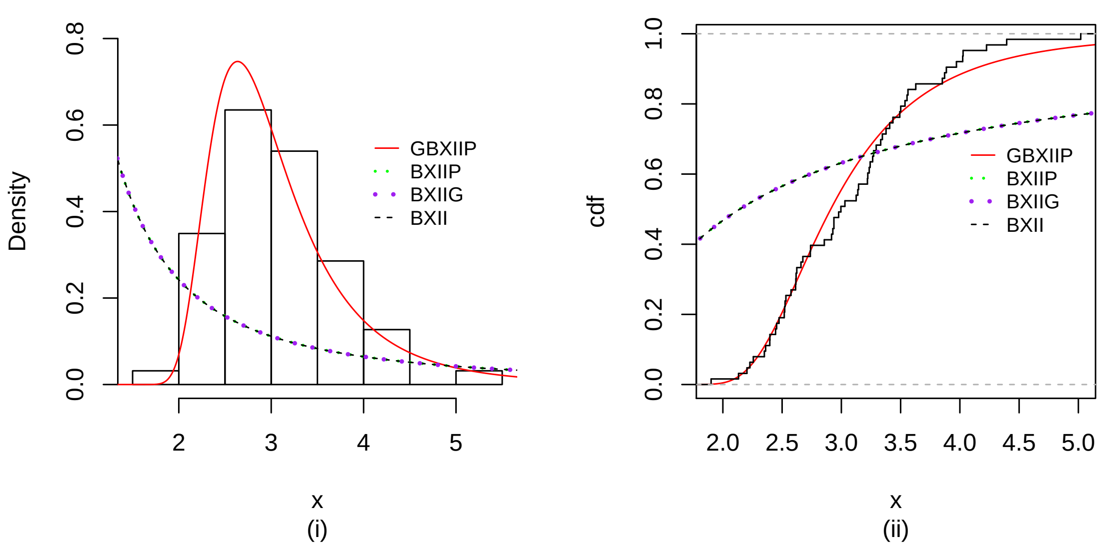

Fig. 4: (i) Histogram (ii) Empirical cdf of the first data and the fitted GBXIIP, BXIIP, BXIIG & BXII densities.

5 Conclusion

We have introduce a new probability model called generalize BurrXII-poisson (GBXIIP) distribution. The GBXIIP distributions consist of the BXIIP distribution as its special case. We provide explicit mathematical formulas for the r^{th} moment, order statistics, moment of order statistics and the Renyi entropy. The existence of its MLEs are investigated under some certain conditions. Finally we fitted the generalize BurrXII-poisson (GBXIIP) distribution to two real life data set, in which the GBXIIP fit better than BXIIP, BXIIG and BXII distributions as measured by AIC and BIC.

Appendix

D1:

$\lim_{a\to 0} g_1 = \infty$, we show that $\lim_{a\to\infty} g_1 < 0$.

$\lim_{a\to\infty} g_1 = -n\log(1-e^{-\lambda}) + \sum_{i=1}^{n}\log(1-e^{-\lambda((1+x_i^{\alpha})^{-\beta}-1)})$, thus, $\lim_{a\to\infty} g_1 < 0$ only

if $\log(1-e^{-\lambda}) > \sum_{i=1}^{n}\frac{\log(1-e^{-\lambda((1+x_i^{\alpha})^{-\beta}-1)})}{n}$. therefore, $g_1(\beta;a,\alpha,\lambda,x_i) = 0$ has at least one root, since it is a contineous function and monotone which decreases from positive values to negetive values.

D2:

It is clear that, $\lim_{\alpha\to 0} g_2(\alpha;a,\beta,\lambda,x_i) = \infty$, we show that, $\lim_{\alpha\to\infty} g_2(\alpha;a,\beta,\lambda,x_i) < 0$.

thus, $\lim_{\alpha\to\infty} g_2(\alpha;a,\beta,\lambda,x_i) = \sum_{x_i<1}^{n}\log x_i - \beta\sum_{x_i>1}^{n}\log x_i$.

To show that $g_2(\alpha;\beta,\lambda,x_i) < 0$ as $\alpha\to\infty$, we consider the following cases.

(i) If $\min\{X_i\} > 1$, $a = 1$, then, $g_2(\alpha;a,\beta,\lambda,x_i) = -\beta\sum_{x_i>1}^{n}\log x_i < 0$.

(ii) If $\max\{X_i\} < 1$, $a = 1$, then, $g_2(\alpha;a,\beta,\lambda,x_i) = \sum_{x_i<1}^{n}\log x_i < 0$.

(iii) If $\max\{X_i\} > 1$ and $\min\{X_i\} < 1$ then,

$g_2(\alpha;a,\beta,\lambda,x_i) = \sum_{x_i<1}^{n}\log x_i - \beta\sum_{x_i>1}^{n}\log x_i < 0$. Thus, $g_2(\alpha;a,\beta,\lambda,x_i) < 0$ for all the cases, if and only if $x_i \neq 1$ for some $i = 1,2,\cdots,n$. Since, $g_2(\alpha;a,\beta,\lambda,x_i)$ is a continuous function which decreases monotonically from positive values to negative values, hence, $g_2(\alpha;a,\beta,\lambda,x_i) = 0$ has at least one root.

D3:

For $a = 1$, let $w_3 = -\lambda\sum_{i=1}^{n}\frac{\log(1+x_i^{\alpha})}{(1+x_i^{\alpha})^{\beta}}$, clearly w_3 is strictly decreasing in β,

$\lim_{\beta\to 0} w_3 = -\lambda\sum_{i=1}^{n}\log(1+x_i^{\alpha})$, thus,

$g_3(\beta;a,\alpha,\lambda,x_i) > \frac{n}{\beta} - \sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \lim_{\beta\to 0} w_3 = \frac{n}{\beta} - (\lambda+1)\sum_{i=1}^{n}\log(1+x_i^{\alpha})$,

then, $g_3(\beta;a,\alpha,\lambda,x_i) > 0$ when $\beta < \frac{n}{(\lambda+1)\sum_{i=1}^{n}\log(1+x_i^{\alpha})}$.

And

$\lim_{\beta\to\infty} w_3 = 0$,

$g_3(\beta;a,\alpha,\lambda,x_i) < \frac{n}{\beta} - \sum_{i=1}^{n}\log(1+x_i^{\alpha}) + \lim_{\beta\to\infty} w_3 = \frac{n}{\beta} - \sum_{i=1}^{n}\log(1+x_i^{\alpha})$,

thus, $g_3(\beta;a,\alpha,\lambda,x_i) < 0$ when $\beta > \frac{n}{\sum_{i=1}^{n}\log(1+x_i^{\alpha})}$.

Hence $g_3(\beta;a,\alpha,\lambda,x_i) = 0$ has at least one root in the interval

$\left(\frac{n}{(1+\lambda)\sum_{i=1}^{n}\log(1+x_i^{\alpha})}, \frac{n}{\sum_{i=1}^{n}\log(1+x_i^{\alpha})}\right)$.

For $a \neq 1$

$\lim_{\beta\to 0} g_3 = \infty$. We show that, $\lim_{\beta\to\infty} g_3 < 0$.

$\lim_{\beta\to\infty} g_3 = -\sum_{i=1}^{n}\log(1+x_i^{\alpha}) < 0$, thus $g_3(\beta;a,\alpha,\lambda,x_i) = 0$, has at least one root, since $g_3(\beta;a,\alpha,\lambda,x_i)$ is contineous function and monotone which decreases from positive to negetive values.

D4:

The elements of $J(\theta)$ are given by

$$\begin{aligned}
\frac{\partial^2\ell}{\partial\alpha^2} &= -\frac{n}{\alpha^2} - (\beta+1)\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1+x_i^{\alpha})} + (\beta+1)\sum_{i=1}^{n}\frac{x_i^{2\alpha}\log x_i}{(1+x_i^{\alpha})^2} \\
&+ (a-1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2 z_i}{(1+x_i^{\alpha})^{\beta+1}(1-z_i)} - (a-1)(\beta+1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2 z_i}{(1+x_i^{\alpha})^{\beta+2}(1-z_i)} \\
&- (a-1)\beta^2\lambda^2\sum_{i=1}^{n}\frac{x_i^{2\alpha}(\log x_i)^2 z_i}{(1+x_i^{\alpha})^{2(\beta+1)}(1-z_i)} - (a-1)\beta^2\lambda^2\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)z_i^2}{(1+x_i^{\alpha})^{2(\beta+1)}(1-z_i)^2} \\
&- \beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1+x_i^{\alpha})^{\beta+1}} + (\beta+1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^{\alpha}(\log x_i)^2}{(1+x_i^{\alpha})^{\beta+2}}
\end{aligned}$$

$$\frac{\partial^2\ell}{\partial\beta^2}=-\frac{n}{\beta^2}-(a-1)\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))^2 z_i}{(1+x_i^\alpha)^\beta(1-z_i)}-(a-1)\lambda^2\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))^2 z_i}{(1+x_i^\alpha)^{2\beta}(1-z_i)}$$
$$-(a-1)\lambda^2\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))^2 z_i^2}{(1+x_i^\alpha)^{2\beta}(1-z_i)}-(a-1)\lambda^2\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))^2 z_i^2}{(1+x_i^\alpha)^{2\beta}(1-z_i)^2}$$
$$+\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))}{(1+x_i^\alpha)^\beta}$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{n}{\lambda^2}+\frac{a\,n\exp(-\lambda)}{(1-\exp(-\lambda))^2}-(a-1)\sum_{i=1}^{n}\frac{((1+x_i^\alpha)^{-\beta}-1)^2 z_i}{(1-z_i)}$$
$$-(a-1)\sum_{i=1}^{n}\frac{((1+x_i^\alpha)^{-\beta}-1)^2 z_i^2}{(1-z_i)^2}$$

$$\frac{\partial^2\ell}{\partial a^2}=-\frac{n}{a^2}$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial a}=\beta\lambda\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)z_i}{(1+x_i^\alpha)^{\beta+1}(1-z_i)}$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\beta}=-\sum_{i=1}^{n}\frac{x_i^\alpha\log x_i)}{(1+x_i^\alpha)}+(a-1)\lambda\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)z_i}{(1+x_i^\alpha)^{\beta+1}(1-z_i)}$$
$$-(a-1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)\,(\log(1+x_i^\alpha))\,z_i}{(1+x_i^\alpha)^{\beta+1}(1-z_i)}$$
$$-(a-1)\lambda^2\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)\,(\log(1+x_i^\alpha))\,z_i^2}{(1+x_i^\alpha)^{2\beta+1}(1-z_i)^2}$$

$$\frac{\partial^2\ell}{\partial\alpha\,\partial\lambda}=(a-1)\beta\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)z_i}{(1+x_i^\alpha)^{\beta+1}(1-z_i)}$$
$$+(a-1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)\,((1+x_i^\alpha)^{-\beta}-1)\,z_i}{(1+x_i^\alpha)^{\beta+1}(1-z_i)}$$
$$+(a-1)\beta\lambda\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)\,((1+x_i^\alpha)^{-\beta}-1)\,z_i^2}{(1+x_i^\alpha)^{\beta+1}(1-z_i)^2}-\beta\sum_{i=1}^{n}\frac{x_i^\alpha(\log x_i)}{(1+x_i^\alpha)^{\beta+1}}$$

$$\frac{\partial^2\ell}{\partial\beta\,\partial\lambda}=(a-1)\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))z_i}{(1+x_i^\alpha)^\beta(1-z_i)}-\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))}{(1+x_i^\alpha)^\beta}$$
$$+(a-1)\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))\,((1+x_i^\alpha)^{-\beta}-1)\,z_i}{(1+x_i^\alpha)^\beta(1-z_i)}$$
$$+(a-1)\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))\,((1+x_i^\alpha)^{-\beta}-1)\,z_i^2}{(1+x_i^\alpha)^\beta(1-z_i)^2}$$

$$\frac{\partial^2\ell}{\partial\beta\,\partial a}=\lambda\sum_{i=1}^{n}\frac{(\log(1+x_i^\alpha))z_i}{(1+x_i^\alpha)^\beta(1-z_i)}$$

$$\frac{\partial^2\ell}{\partial\lambda\,\partial a}=-\frac{n\exp(-\lambda)}{(1-\exp(-\lambda))}-\sum_{i=1}^{n}\frac{((1+x_i^\alpha)^{-\beta}-1)z_i}{(1-z_i)}.$$

Where $z_i=\exp(\lambda((1+x_i^\alpha)^{-\beta}-1))$.

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Mustapha Muhammad is currently a teacher and researcher in the Department of Mathematical Science at Bayero University Kano, Nigeria. His main research interests are: Probability Modeling (Mixture of Distributions and Linear Models), Survival Analysis, Bayesian Statistics (Inference and Prediction), Model Checking and Data Analysis (Censoring and Missing). He received his first and second degree in Mathematics from Bayero University Kano (Nigeria) and Jordan University of Science and Technology (Jordan) respectively.