Applied Mathematics & Information Sciences
An International Journal

Taylor series coefficients of the HP-polynomial as an invariant for links in the solid torus

K. Bataineh

Jordan University of Science and Technology, Jordan and University of Bahrain, Bahrain

Received: 22 Jul. 2012; Revised 17 Oct. 2012; Accepted 27 Oct. 2012
Published online: 1 Jan. 2013

Abstract: We show that the coefficients of a reformulation of a Taylor series expansion of the Hoste and Pryzyticki polynomial are Vassiliev invariants. We also show that many other reformulations of the Taylor series expansion have coefficients that are Vassiliev invariants. We charecterize the first two coefficients $b_0^1(t)$ and $b_1^L(t)$ for one of these expansions. Moreover, the second coefficient $b_1^L(t)$, which is a type-one Vassiliev invariant, is given two explicit computational formulas, which are easy to calculate. $b_1^L(t)$ is used to give a lower bound for the crossing number of a knot of zero winding number in the solid torus.

Keywords: Taylor series; Solid torus

1. Introduction

Discovering computable formulas of Vassiliev invariants started by the appearance of the combinatorial definition by Birman and Lin in [5]. Since the richest source of invariants continues to be polynomial invariants, much of work has been done to characterize Vassiliev invariants of a given type by exploring these polynomial invariants. The difference relation in the combinatorial definition of the Vassiliev invariants is what stands behind the interest of the derivatives and hence the Taylor series expansion coefficients of the Laurent polynomial invariants. See [5] and [12].

The two-variable polynomial for dichromatic links by Hoste and Przytycki in [7] was considered in [4] as an invariant of links in the solid torus. We refer to this polynomial invariant by the *HP-polynomial.* Moreover, the definition of Vassiliev invariants was extended to knots and links in the solid torus. In this paper we show that the coefficients of a reformulation of the Taylor series expansion of the HP-polynomial are Vassiliev invariants and we show that many other choices of Taylor series expansion lead to the same result. We characterize the first two coefficients of this expansion $b_0^L(t)$ and $b_1^L(t)$. The second coefficient $b_1^L(t)$, which is a type-one Vassiliev invariant, is given explicitly by two computational formulas. Both of these two formulas are finite sum formulas that are easy to calculate. Finally, as an application, we analyze $b_1^L(t)$ to give a lower bound for one of the most important geometric knot invariants, the crossing number, which has special importance in natural sciences, especially in studying knotted DNA.

In section 2 we give preliminaries needed in the later sections. In section 3 we show that the coefficients of a Taylor series expansion of the HP-polynomial are Vassiliev invariants and we show that many other Taylor series expansions of the HP-polynomial have this property too. In section 4 we show that $b_1^L(t)$ can be computed separately and easily through two different formulas, and we list some remarks on these formulas. In section 5 we extract a lower bound for the crossing number of a knot of zero winding number, and we show how this bound could be used to compute the crossing number of a knot.

2. Preliminaries

Links (knots) and singular links (knots) in the solid torus are usually represented by what is called *punc-*

* Corresponding author: khaledb@just.edu.jo

tured diagrams, which is a usual diagram of the singular knot or link but in $\mathbb{R}^2 - \{(0,0)\}$.

Like in the case of links in $\mathbb{R}^3$, we shall call a point of intersection of a singular link L a *singular point* of L. A singular link in ST might also be given an orientation for each of its components.

Definition 1. *Two singular knots (links) are said to be* isotopy equivalent *if we can get from a diagram of one of them to a diagram of the other by a finite sequence of the known five generalized Reidemeister moves. See [11].*

Let $\mathcal{L}_i$ be the set of singular link types with i singular points. Let $\mathcal{L} = \bigcup\limits_{i=0}^{\infty} \mathcal{L}_i$.

Definition 2. *A function $v : \mathcal{L} \to G$, where G is an abelian group, is said to be a Vassiliev invariant or a finite-type invariant if it satisfies the following two axioms:*

(i) $v(L_\times) = v(L_+) - v(L_-)$ and

(ii) There exists $n \in \mathbb{N}$ such that $v(L) = 0$ for any $L \in \mathcal{L}_i$ with $i \geq n+1$.

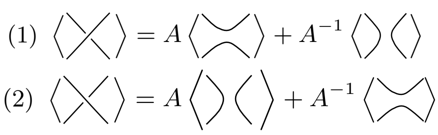
Moreover, the least such non negative integer n is called the type *or* order *of v. L_+ and L_- are usually called the* positive *and* negative resolutions of $L_\times$*, respectively, and equation (i) is usually denoted by* resolving *this double point in $L_\times$.*

The two axioms of the definition above were first given by Birman and Lin [5] for knots, and then the definition of finite-type invariants was extended to links in a similar way by Stanford [12]. The definition of finite-type invariants was extended to knots and links in general 3-manifolds by Kalfagianni [8], and in the solid torus by Goryunov [6], Aicardi [1], and Bataineh and Abu Zaytoon [2] in a similar way.

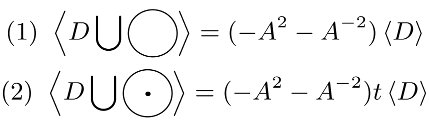
Hoste and Przytycki in [7] introduced the two-variable Laurent polynomial invariant $\widetilde{d}_L$ of 1-trivial dichromatic links with oriented 2-sublink in $\mathbb{R}^3$. In [4] we viewed this invariant as an invariant of oriented links in the solid torus instead of 1-trivial dichromatic links in $\mathbb{R}^3$. The notation we use for $\widetilde{d}_L$ as an invariant in the solid torus is Y_L.

Theorem 1. *Let D be a diagram of a link L in the solid torus, and let $\langle D\rangle$ be determined by the following formulas:*

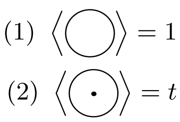
(I) The smoothing formulas



(II) The reduction formulas



(III) The finishing formulas



Then $Y_L(A,t) = (-A^3)^{-w(D)}\langle D\rangle$ is a Laurent polynomial invariant in $\mathbb{Z}[A, A^{-1}, t]$, where $w(D)$ is the sum of the signs of all crossings of D.

Hoste and Przytycki in [7] give the following skein relation for this invariant.

Lemma 1. *The Y_L invariant satisfies the skein relation given by*

$$A^4 Y_{L_+}(A,t) - A^{-4} Y_{L_-}(A,t) = (A^{-2} - A^2) Y_{L_0}(A,t)$$

The proof of this Lemma goes along the lines of the proof of Lemma 2.6 in [10].

Let L be a link in the solid torus, and $n_i(L)$ be the number of copies of the component in L whose winding number is i. Let $n(L)$ be the number of components of L. Note that $\sum\limits_i n_i(L) = n(L)$. For the proof of the following theorem see [3].

Theorem 2. *For a link L in the solid torus*

$$Y_L(1,t) = (-2)^{n(L)-1} \prod_i [T_i(t)]^{n_i(L)},$$

where the set $\{T_k(t) : k \in \mathbb{Z}\}$ is the set of the extended Chebyshev Polynomials of the First Kind given recursively by:

$T_0(t) = 1, \quad T_1(t) = t,$

$T_k(t) = 2t\, T_{k-1}(t) - T_{k-2}(t)$ *for* $k \geq 0$.

Moreover $T_k(t) = T_{-k}(t)$ *for* $k \leq 0$.

The reader can easily deduce the following corollary.

Corollary 1. *If K is just a knot in the solid torus, then $Y_K(1,t) = T_i(t)$, where i is the winding number of the knot.*

3. Power Series Coefficients of the HP-Polynomial

Theorem 3. *Let L be a link in the solid torus and let $Y_L(A,t)$ be the HP-polynomial. Let $\overline{Y}_L(e^{-\frac{1}{4}x}, t)$ be obtained from $Y_L(A,t)$ by replacing the variable A by $e^{-\frac{1}{4}x}$. Let $\overline{\overline{Y}}_L(x,t)$ be obtained by replacing $e^{-\frac{1}{4}x}$ by its Taylor series expansion about $x = 0$, then*

$$\overline{\overline{Y}}_L(x,t) = \sum_{i=0}^{\infty} b_i^L(t) x^i,$$

where $b_0^L(t) = (-2)^{n(L)-1} \prod\limits_i [T_i(t)]^{n_i(L)}$ and each $b_i^L(t)$, $i \geq 0$ is a Vassiliev invariant of order i.

Proof. Let $L^j_\times$ be a singular link with j singular points one of which is referred to by $\times$. We define

$$Y_{L^j_\times}(A,t) = Y_{L^{j-1}_+}(A,t) - Y_{L^{j-1}_-}(A,t). \qquad (1)$$

If we resolve the j crossings using (1) we end up with a sum of 2^j $Y-$polynomials of non-singular links. The skein relation

$$A^4 Y_{L^{j-1}_+}(A,t) - A^{-4} Y_{L^{j-1}_-}(A,t) = (A^{-2} - A^2) Y_{L^{j-1}_0}(A,t)$$

can be written as

$$Y_{L^{j-1}_+}(A,t) = A^{-8} Y_{L^{j-1}_-}(A,t) + (A^{-6} - A^{-2}) Y_{L^{j-1}_0}(A,t) \quad (2)$$

Substituting from equation (2) into equation (1) we obtain

$$Y_{L^j_\times}(A,t) = A^{-8} Y_{L^{j-1}_-}(A,t) + (A^{-6} - A^{-2}) Y_{L^{j-1}_0}(A,t) - Y_{L^{j-1}_-}(A,t),$$

or equivalently

$$Y_{L^j_\times}(A,t) = (A^{-8} - 1) Y_{L^{j-1}_-}(A,t) + (A^{-6} - A^{-2}) Y_{L^{j-1}_0}(A,t) \quad (3)$$

Let $L^j_{1,2,\ldots,j}$ be a singular link with j singular points labeled $1, 2, \ldots, j$. Resolving the j crossings of $L^j_{1,2,\ldots,j}$ using (3) we get a sum involving the $Y-$polynomials of the following 2^j non-singular knots

$$\{K_\sigma : \sigma = (\sigma_1, \ldots, \sigma_j), \text{ where } \sigma_i \text{ is either a minus sign or a zero}\}.$$

This sum is

$$Y_{L^j_{1,2,\ldots,j}}(A,t) = \sum_\sigma (A^{-8} - 1)^{p_\sigma} (A^{-6} - A^{-2})^{q_\sigma} Y_{L_\sigma}(A,t), \quad (4)$$

where p_σ is the number of minus signs in σ, and q_σ is the number of zeros in σ. Substituting $A = e^{-\frac{1}{4}x}$ we obtain

$$\overline{Y}_{L^j_{1,2,\ldots,j}}(e^{-\frac{1}{4}x}, t) = \sum_\sigma (e^{2x} - 1)^{p_\sigma} (e^{\frac{3}{2}x} - e^{\frac{1}{2}x})^{q_\sigma} Y_{L_\sigma}(e^{-\frac{1}{4}x}, t).$$

Note that the Taylor series expansion about $x = 0$ gives

$$e^{2x} - 1 = 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$
$$e^{\frac{3}{2}x} - e^{\frac{1}{2}x} = x + x^2 + \frac{13}{24}x^3 + \cdots,$$

and hence

$$\overline{\overline{Y}}_{L^j_{1,2,\ldots,j}}(x,t) =$$
$$\sum_\sigma (2x + 2x^2 + \frac{4}{3}x^3 + \cdots)^{p_\sigma} (x + x^2 + \frac{13}{24}x^3 + \cdots)^{q_\sigma}$$
$$Y_{L_\sigma}(e^{-\frac{1}{4}x}, t)$$
$$= \sum_\sigma (2x + 2x^2 + \frac{4}{3}x^3 + \cdots)^{p_\sigma} (x + x^2 + \frac{13}{24}x^3 + \cdots)^{q_\sigma} (c_0 + c_1 x + \cdots), \quad (5)$$

where $c_0 = (-2)^{n(L)-1} \prod_i [T_i(t)]^{n_i(L)}$, because $\overline{\overline{Y}}_L(0,t) = Y_L(1,t) = (-2)^{n(L)-1} \prod_i [T_i(t)]^{n_i(L)}$. Note that $c_0 \neq 0$. The first non-zero term of the sum in (5) is

$$(2x)^{p_\sigma} (x)^{q_\sigma} (c_0) = 2^{p_\sigma} c_0 x^{p_\sigma + q_\sigma} = (2^{p_\sigma} c_0) x^j \text{ since } p_\sigma + q_\sigma = j.$$

So the coefficient of x^i in the sum above is zero for all $i < j$. Therefore

$$\overline{\overline{Y}}_{L^j_{1,2,\ldots,j}}(x,t) = \sum_{i=0}^\infty b_i^{L^j_{1,2,\ldots,j}}(t) x^i,$$

where $b_i^{L^j_{1,2,\ldots,j}}(t) = 0$ for all $i < j$.

Let i be fixed and let $j > i$, then $b_i^{L^j_{1,2,\ldots,j}}(t) = 0$, hence $b_i^L(t)$ satisfies the first condition in the definition of a Vassiliev invariant of order i. Moreover, $b_i^L(t)$ is an invariant that satisfies $b_i^{L\times}(t) = b_i^{L+}(t) - b_i^{L-}(t)$ since Y does. Hence $b_i^L(t)$ is a Vassiliev invariant of order i.

Next we show that we do not have to use the substitution $A = e^{-\frac{1}{4}x}$ in order to prove the previous theorem. In fact we have a wide range of choices.

Proposition 1. *If $f(0) = 1$, $f'(0) \neq 0$ and $f(x)$ has a convergent Taylor series about $x = 0$, then each coefficient $d_i^L(t)$ of the Taylor series expansion of $Y_L(A,t)$ determined by the substitution $A = [f(x)]^{-\frac{1}{2}}$ is a Vassiliev invariant of order i.*

Proof. Recall equation (4) in the proof above

$$Y_{L^j_{1,2,\ldots,j}}(A,t) = \sum_\sigma (A^{-8} - 1)^{p_\sigma} (A^{-6} - A^{-2})^{q_\sigma} Y_{L_\sigma}(A,t).$$

Substituting $A = [f(x)]^{-\frac{1}{2}}$ we obtain

$$\overline{Y}_{L^j_{1,2,\ldots,j}}([f(x)]^{-\frac{1}{2}}, t)$$
$$= \sum_\sigma (f^4(x) - 1)^{p_\sigma} (f^3(x) - f(x))^{q_\sigma} Y_{L_\sigma}([f(x)]^{-\frac{1}{2}}, t).$$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Note that $f(0) = 1$, $f'(0) \neq 0 \Rightarrow f(x) = 1 + a_1 x + a_2 x^2 + \cdots$, such that $a_1 \neq 0$. Hence, the sum in (6) becomes

$$\sum_\sigma ((1 + a_1 x + \cdots)^4 - 1)^{p_\sigma} ((1 + a_1 x + \cdots)^3 - (1 + a_1 x + \cdots))^{q_\sigma}$$
$$Y_{L_\sigma}([f(x)]^{-\frac{1}{2}}, t)$$

$$= \sum_\sigma [(1 + 4a_1 x + \cdots) - 1]^{p_\sigma} [(1 + 3a_1 x + \cdots)$$
$$-(1 + a_1 x + \cdots)]^{q_\sigma} Y_{L_\sigma}([f(x)]^{-\frac{1}{2}}, t)$$

$$= \sum_\sigma [4a_1 x + \cdots]^{p_\sigma} [2a_1 x + \cdots]^{q_\sigma}$$
$$(c_0 + c_1 x + \cdots),$$

where $c_0 = (-2)^{n(L)-1} \prod_i [T_i(t)]^{n_i(L)}$. Note that $c_0 \neq 0$. The first non-zero term of this sum is

$$(4a_1 x)^{p_\sigma} (2a_1 x)^{q_\sigma} (c_0)$$
$$= (4^{p_\sigma} 2^{q_\sigma} a_1^{p_\sigma} a_1^{q_\sigma} c_0) x^{p_\sigma} x^{q_\sigma} =$$
$$(2^{j+p_\sigma} a_1^j c_0) x^j \text{ since } p_\sigma + q_\sigma = j.$$

Note that $(2^{j+p_\sigma}a_1^j c_0) \neq 0$. So the coefficient of x^i in the sum above is zero for all $i < j$. Therefore

$$\overline{\overline{Y}}_{L^j_{1,2,\dots,j}}(x,t) = \sum_{i=0}^{\infty} d_i^{L^j_{1,2,\dots,j}}(t)x^i,$$

where $d_i^{L^j_{1,2,\dots,j}}(t) = 0$ for all $i < j$.

The rest of the proof is similar to that in the proof of the previous theorem, because if i is fixed and $j > i$, then $d_i^{L^j_{1,2,\dots,j}}(t) = 0$, hence $d_i^L(t)$ satisfies the first condition in the definition of a Vassiliev invariant of order i.

4. Two formulas for the invariant $b_1^L(t)$

In the previous section we gave the explicit formula of the type-zero invariant $b_0^L(t)$. In this section we give two computational explicit formulas for the type-one Laurent polynomial invariant $b_1^L(t)$. The first one is a sum given in terms of the extended Chebyshev Polynomials, and the second one is a sum given in terms of the states of the link on Kauffman's way.

Let L be a link in the solid torus. If p is a crossing in L between two strands in the same component, then smoothing this crossing produces two components. We denote the winding numbers of these two components by $\{m(p), l(p)\}$. We also define $j(p) = m(p) + l(p)$. On the other hand, If q is a crossing in L between two strands from two different components, we denote the winding numbers of the two components by $\{m(q), l(q)\}$. Smoothing this crossing connects these two components into one component. We define $j(q) = m(q) + l(q)$ and this is the total winding number of the resulting component. We refer to the sign of the crossing p by $e(p)$, and the sign of the crossing q by $e(q)$.

For the proof of the following theorem see [3].

Theorem 4. $\left[\frac{\partial}{\partial A}Y_L(A,t)\right]_{A=1}$ *is given by*

$$Y_L(1,t)\left(\sum_p e(p)\left[4\frac{T_{m(p)}(t)T_{l(p)}(t)}{T_{j(p)}(t)} - 4\right] + \sum_q e(q)\left[\frac{T_{j(q)}(t)}{T_{m(q)}(t)T_{l(q)}(t)} - 4\right]\right).$$

The explicit formula of $b_1(t)$ is now given.

Theorem 5. *The type-one invariant $b_1^L(t)$ is given by*

$$b_1^L(t) = b_0^L(t)\left(\sum_p e(p)\left[1 - \frac{T_{m(p)}(t)T_{l(p)}(t)}{T_{j(p)}(t)}\right] + \sum_q e(q)\left[1 - \frac{T_{j(q)}(t)}{T_{m(q)}(t)T_{l(q)}(t)}\right]\right).$$

Proof. Recall that we have $Y_L(A,t)$ and $A = e^{-\frac{1}{4}x}$ and the power series

$$\overline{\overline{Y}}_L(x,t) = \sum_{i=0}^{\infty} b_i^L(t)x^i.$$

One can easily see that $b_0^L(t) = \overline{\overline{Y}}_L(0,t) = Y_L(1,t)$. Moreover

$$\begin{aligned}
b_1^L(t) &= \left[\frac{\partial}{\partial A}\overline{\overline{Y}}_L(x,t)\right]_{x=0} \\
&= \left[\frac{\partial}{\partial A}Y_L(A,t)\frac{dA}{dx}\right]_{x=0} \\
&= \left[\frac{\partial}{\partial A}Y_L(A,t)\right]_{A=1}\left[\frac{dA}{dx}\right]_{x=0} \\
&= \left[\frac{\partial}{\partial A}Y_L(A,t)\right]_{A=1}\left[-\frac{1}{4}\right] \\
&= -\frac{1}{4}Y_L(1,t)\left(\sum_p e(p)\left[4\frac{T_{m(p)}(t)T_{l(p)}(t)}{T_{j(p)}(t)} - 4\right]\right. \\
&\quad \left.+ \sum_q e(q)\left[\frac{T_{j(q)}(t)}{T_{m(q)}(t)T_{l(q)}(t)} - 4\right]\right) \\
&= b_1^L(t)\left(\sum_p e(p)\left[1 - \frac{T_{m(p)}(t)T_{l(p)}(t)}{T_{j(p)}(t)}\right]\right. \\
&\quad \left.+ \sum_q e(q)\left[1 - \frac{T_{j(q)}(t)}{T_{m(q)}(t)T_{l(q)}(t)}\right]\right)
\end{aligned}$$

This completes the proof.

The following corollary follows from the last theorem and the fact that when L is a knot, we have $b_0^L(t) = T_{j(p)}(t)$.

Corollary 2. *For a knot K in the solid torus, we have*

$$b_1^K(t) = \sum_p e(p)[T_{m(p)+l(p)}(t) - T_{m(p)}(t)T_{l(p)}(t)].$$

The other explicit computational formula in this section is a formula that we will derive from the following state sum formula given in [4].

Theorem 6. *Let D be a diagram of a link L in the solid torus. Then*

$$Y_L(A,t) = (-A^3)^{-w(D)}\sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}t^{|T|}$$

where the sum runs over all possible states S of the link diagram, $|S|$ is the total number of circles and dotted circles in the state S, $|T|$ is the number of the dotted circles, $a(S)$ is the number of A-splits in the state S, and $b(S)$ is the number of B-splits in the state S.

Now we give the state sum formula for the type-one invariant $b_1^L(t)$.

Theorem 7.
$b_1^L(t) = \frac{(-1)^{w(D)}}{4}\sum_S [3w(D) + b(S) - a(S)]\,[-2]^{|S|-1}t^{|T|}.$

Proof. Note that

$$\begin{aligned}
&\frac{\partial}{\partial A}Y_L(A,t) = (-A^3)^{-w(D)} \\
&\sum_S [A^{a(S)-b(S)}(|S|-1)(-A^2 - A^{-2})^{|S|-2}(-2A + 2A^{-3}) \\
&+ (a(S) - b(S))A^{a(S)-b(S)-1}(-A^2 - A^{-2})^{|S|-1}]t^{|T|} \\
&+ [-w(D)(-A^3)^{-w(D)-1}(-3A^2)] \\
&\left[\sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}t^{|T|}\right]
\end{aligned}$$

Hence

$$\left[\frac{\partial}{\partial A}Y_L(A,t)\right]_{A=1} = (-1)^{-w(D)}$$
$$\left[\sum_S \left[(a(S)-b(S))(-2)^{|S|-1}\right] t^{|T|}\right]$$
$$+\left[-w(D)\,(-1)^{-w(D)-1}\,(-3)\right]$$
$$\left[\sum_S (-2)^{|S|-1} t^{|T|}\right]$$
$$= (-1)^{-w(D)}$$
$$\sum_S \left[(a(S)-b(S))(-2)^{|S|-1}\right]$$
$$t^{|T|} - 3w(D)(-2)^{|S|-1}t^{|T|}$$
$$= (-1)^{w(D)}$$
$$\sum_S \left[(a(S)-b(S)-3w(D))(-2)^{|S|-1}\right] t^{|T|}$$

Now recall from the last proof that

$$\begin{aligned} b_1^L(t) &= \left[-\frac{1}{4}\right]\left[\frac{\partial}{\partial A}Y_L(A,t)\right]_{A=1} \\ &= \left[-\frac{1}{4}\right](-1)^{w(D)}\sum_S \left[(a(S)-b(S)-3w(D))(-2)^{|S|-1}\right] t^{|T|} \\ &= \frac{(-1)^{w(D)}}{4}\sum_S \left[3w(D)+b(S)-a(S)\right][-2]^{|S|-1}t^{|T|}. \end{aligned}$$

This completes the proof.

We give the following useful remarks.

Remark. If $X_L(A)$ is the well-known form of the Jones polynomial discovered by Kauffman using his bracket polynomial, then $Y_L(A,1) = X_L(A)$. This can be easily seen by definition of $Y_L(A,t)$.

Remark. If K is a knot in the three space, then

$$\frac{d}{dA}\left[X_K(A)\right]_{A=1} = 0,$$

because a type-one invariant of knots in the three space is of type zero; that is a constant.

Remark. In a similar power series expansion of $X_K(A)$, the second coefficient would be $b_1^K(1)$, and the corresponding state sum formula would be

$$\frac{(-1)^{w(D)}}{4}\sum_S \left[3w(D)+b(S)-a(S)\right][-2]^{|S|-1}.$$

However, this formula is of no use, because it vanishes by the previous remark. Therefore, The formula in our last theorem has no corresponding one known for knots in the three space. But it is useful in computation for a type-one invariant for knots and links in the solid torus.

Example 1. The calculation of $b_1^K(t)$ for the knot in (Figure 1) using both of the two formulas given in this section yields the same result $b_1^K(t) = 2 - 2t^2$.

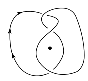


Figure 1 Knot in the Solid Torus

5. A lower bound on the crossing number

For a given knot K the *crossing number of* K, denoted by *crossing(K)*, is known to be the minimal number of crossings of all diagrams of the knot K. Like all geometric invariants of knots and links, the crossing number is hard to compute, while it is so important especially in knotted DNA, where the crossing number is instrumental in understanding the information coded in the DNA. Let KST_0 be the set of knots with zero winding number in the solid torus. In this section we explore a lower bound for the crossing number of $K \in KST_0$. This lower bound is coded in $b_1^K(t)$ in an implicit way.

For the following extended relation of Chebyshev polynomials see [3].

$$T_n(t) = \frac{1}{2}\left(t-\sqrt{t^2-1}\right)^n + \frac{1}{2}\left(t+\sqrt{t^2-1}\right)^n \text{ for all } n \in \mathbb{Z}$$

Lemma 2. *If $K \in KST_0$, and $z = t - \sqrt{t^2-1}$, then*

$$b_1^K(z) = \frac{1}{2}\sum_p e(p) - \frac{1}{4}\sum_p e(p)[z^{2m(p)} + z^{-2m(p)}].$$

Proof. Note that if $K \in KST_0$, then $m(p) + l(p) = 0$, therefore

$$T_{m(p)+l(p)}(t) = T_0(t) = 1,$$

$$\begin{aligned} T_{m(p)}(t)T_{l(p)}(t) &= T_{m(p)}(t)T_{-m(p)}(t) \\ &= T_{m(p)}(t)T_{m(p)}(t) \\ &= T^2_{m(p)}(t). \end{aligned}$$

Hence

$$b_1^K(t) = \sum_p e(p)[1 - T^2_{m(p)}(t)].$$

When $z = t - \sqrt{t^2-1}$, we have

$$T_{m(p)}(z) = \frac{1}{2}z^{m(p)} + \frac{1}{2}z^{-m(p)} \Rightarrow$$

$$T^2_{m(p)}(z) = \frac{1}{2} + \frac{1}{4}z^{2m(p)} + \frac{1}{4}z^{-2m(p)}.$$

Hence

$$\begin{aligned} b_1^K(z) &= \sum_p e(p)[1 - T^2_{m(p)}(z)] \\ &= \sum_p e(p)[1 - \frac{1}{2} - \frac{1}{4}z^{2m(p)} - \frac{1}{4}z^{-2m(p)}] \\ &= \frac{1}{2}\sum_p e(p) - \frac{1}{4}\sum_p e(p)[z^{2m(p)} + z^{-2m(p)}]. \end{aligned}$$

Note that the previous lemma expresses $b_1^K(z)$ as a symmetric Laurent polynomial in z with a constant term $M \equiv \frac{1}{2}\sum_p e(p)$. Therefore, we have the following corollary.

Corollary 3. *$b_1^K(z)$ can be written as*

$$b_1^K(z) = \left[B_{m_1}z^{2m_1} + \cdots + B_{m_s}z^{2m_s}\right] + \left[B_{m_1}z^{-2m_1} + \cdots + B_{m_s}z^{-2m_s}\right] + M,$$

where $B_{m_i} = -\frac{1}{4}\sum_{p_{m_i}} e(p_{m_i}) \neq 0$, is the sum of signs of all crossings with winding numbers $\{m_i, -m_i\}$.

Theorem 8.
If $K \in KST_0$, then

$$crossing(K) \geq 4\left|B_{m_1}\right| + \cdots + 4\left|B_{m_s}\right|.$$

Proof. Let D be any diagram for a knot $K \in KST_0$. Since $B_{m_i} = -\frac{1}{4}\sum_{p_{m_i}} e(p_{m_i})$, we have $\sum_{p_{m_i}} e(p_{m_i}) = -4B_{m_i}$. So we have at least $\left|-4B_{m_i}\right| = 4\left|B_{m_i}\right|$ crossings of winding numbers $\{m_i, -m_i\}$. Since the sets of crossings of different pairs of winding numbers in D are pairwise disjoint, we have at least $4\left|B_{m_1}\right| + \cdots + 4\left|B_{m_s}\right|$ crossings in D. Hence $crossing(K) \geq 4\left|B_{m_1}\right| + \cdots + 4\left|B_{m_s}\right|$.

Example 2. For the knot K in Figure 1, we have

$$b_1^K(z) = \frac{1}{2}[2] - \frac{1}{4}[(1)[z^2 + z^{-2}] + (1)[z^2 + z^{-2}]] = 1 - \frac{1}{2}z^2 - \frac{1}{2}z^{-2}.$$

Hence $crossing(K) \geq 4\left|-\frac{1}{2}\right| = 2$. We leave it as an easy exercise for the reader to find a diagram D_1 of K in FIGURE 1 that has exactly two crossings. Finding such a diagram together with the fact that $crossing(K) \geq 2$ implies that $crossing(K) = 2$.

References

[1] F. Aicardi, Topological invariants of knots and framed knots in the solid torus, C. R. Acad. Sci. Paris Sr. I Math., **321** (1995), 81-86.

[2] K. Bataineh and M. Abu Zaytoon, Vassiliev invariants of type one for links in the solid torus, Topology and its applications, **157** (2010) pp. 2495-2504.

[3] K. Bataineh, H. Belkhirat, The derivatives of the Hoste and Przytycki polynomial for oriented links in the solid torus, Houston Journal of Mathematics. In press.

[4] K. Bataineh and M. Hajij, Jones polynomial for links in the handlebody, Rocky Mountain Journal of Mathematics. In press.

[5] J. Birman and X-S. Lin, Knot polynomials and Vassiliev's invariants, Invent. Math., **111** (1993), 225-270.

[6] V. Goryunov, Vassiliev invariants of knots in $\mathbb{R}^3$ and in a solid torus, Amer. Math. Soc. Transl., **190** (1999), 37-59.

[7] J. Hoste and J. Przytycki, An invariant of dichromatic links, Proc. Amer. Math. Soc., **105** (1989), 1003-1007.

[8] E. Kalfagianni, *Finite type invariants for knots in 3-manifolds,* Topology, **37** (1998), 673-707.

[9] T. Kanenobu, Kauffman polynomials as Vassiliev link invariants, Proceedings of Knots 96, (1997), pp. 411-431.

[10] L. Kauffman, New invariants in the theory of knots, American Mathematical Monthly, **95** (1988), 195-242.

[11] A. Kawauchi, A survey of knot theory (Birkhauser Verlag, 1996).

[12] T. Stanford, Finite-type invariants of knots, links, and graphs, Topology, **35** (1996), 1027-1050.

K. Bataineh is presently employed as an assistant professor; currently at the University of Bahrain, and permanently at Jordan University of Science and Technology. He obtained his PhD from New Mexico State University (USA) in the fields of Topology and Knot Theory. He is active in research and teaching and he supervised a number of master's students, who are following their study as PhD students at the USA and Canada. He has published a number of research articles in reputed international journals of mathematics. He has presented many talks in international conferences.