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Some Couette flows of a Maxwell fluid with wall slip condition

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Abstract: Couette flows of a Maxwell fluid produced by the motion of a flat plate are analyzed under the slip condition at boundaries. The bottom plate is assumed to be translated in its plane with a given velocity. The flow of the fluid is studied in the assumption that the relative velocity between the fluid at the wall and the wall is proportional to the shear rate at the wall. Exact expressions for velocity and shear stress are determined by means of a Laplace transform. The velocity fields corresponding to both slip and non slip conditions for Maxwell and Newtonian fluids are obtained. Two particular cases, namely translation with constant velocity and sinusoidal oscillations of the bottom plate, are studied. Results for Maxwell fluids are compared with those of Newtonian fluids in both cases with slip and non slip conditions. Some properties of the flow are also presented.

Keywords: Maxwell fluid, Couette flows, wall slip condition.

1. Introduction

Since 1867 J.C.Maxwell (1831-1879) observed that some fluids, such as air, exhibit both viscous and elastic behaviours. The constitutive relation, in modern notations, proposed by Maxwell for these fluids is given by [1-3]

$$\mathbf{S} + \lambda(\dot{\mathbf{S}} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T) = \mu\mathbf{A}, \tag{1}$$

where S is the extra stress tensor, L is the velocity gradient, $\mathbf{A} = \mathbf{L}+\mathbf{L}^T$ is the first Rivlin-Erickson tensor, $\lambda(\geq 0)$ and $\mu(> 0)$ are the relaxation time and dynamic viscosity, respectively, and the superposed dot indicates the material time derivative. Maxwell fluids also can be considered as a special case of a Jeffreys-Oldroyd B fluid, which contain both relaxation and retardation time coefficients [1]. Maxwell's constitutive relation can be recovered from that corresponding to Jeffreys-Oldroyd B fluids by setting the retardation time to be zero. The fluids described by (1) are referred to as viscoelastic fluids of Maxwell type, or simply Maxwell fluids. Several fluids, such as glycerin, crude oils or some polymeric solutions, behave as Maxwell fluids. The reference [4] contains more examples of this type of fluids. The Maxwell model has been the subject of study for many researchers. The first exact solution of Stokes' first problem, also known as Rayleigh's problem, for Maxwell fluids was given by Tanner [5]. Other solutions of Stokes' first problem for Maxwell fluids, together some interesting properties, have been obtained by Jordan et al [6], Jordan and Puri [7] and, for Oldroyd B fluid, by Christov [8]. The unsteady Couette flow of a Maxwell fluid between two infinite parallel plates was studied by Denn and Porteous [9] while, for second grade dipolar fluids, by Jordan [10] and Jordan and Puri [11]. Interesting subjects and solutions regarding the Couette or Stokes flows of non-Newtonian fluids can be found in references [12-15]. In aforementioned papers the assumption that a liquid adheres to the solid boundary, so called nonslip boundary condition, was considered fulfilled. The nonslip boundary condition is one of the basic principles in which the mechanics of the linearly viscous fluids was built. Many experiments are in favour of the nonslip boundary condition for a large class of flows. An interesting discussion regarding the acceptance of the nonslip condition can be found in [16]. Even if the nonslip condition has proved to be successful for a great variety of flows, it has been found to be inadequate in several situations, such as problems involving multiple interfaces, flows in micro channels or in wavy tubes, flows of polymeric liquids or flows

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of rarefid fluids. Many years ago, Navier [17] proposed a slip boundary condition wherein the relative velocity (the slip velocity) depended linearly on the shear stress. A large number of models have been proposed for describing the slip that occurs at solid boundaries. Many of them can be found in the reference [18]. One of the early studies of the slip at the boundary was under taken by Monney [19]. Recently, several papers regarding flows of Newtonian or non-Newtonian fluids with slip at the boundary have been published. Khalid and Vafai [20] were studied the effect of the slip condition on Stokes' and Coutte flows due to an oscillating wall; Vieru and Rauf [21] analyzed Stokes' flows of a Maxwell fluid with wall slip condition; the Coutte flow of a third grade fluid with rotating frame and slip condition was studied by Abelman et al [22]. Many interesting and useful results regarding solutions for flows of non-Newtonian fluids with slip effects are in references [23-25].

In this study, Couette flows of a Maxwell fluid produced by the motion of a flat plate are analyzed under the assumption of the slip boundary conditions between the plates and the fluid. The motion of the bottom plate is a rectilinear translation in its plane with velocity $u_w(t) = U_o f(t)$, while the upper plate is at rest. Exact expressions for velocity and shear stress are determined by means of a Laplace transform for Maxwell and Newtonian fluids. Expressions of the relative velocity are determined, and the solutions corresponding to flows with nonslip at the boundary are also presented. Two particular cases, namely the translation of the bottom plate with a constant velocity and sinusoidal oscillations are studied. In each case, the expression of the velocity is written as a sum between "the permanent solution" and the transient solution. For large values of time t the transient solution tends to zero and the fluid flows according to the "permanent solution". Some relevant properties of the velocity and comparisons between solutions with slip and nonslip at the boundaries are presented.

2. Problem formulation and solution

Consider an infinite solid plane wall situated in the (x,z)-plane of Cartesian coordinate system with the positive $y-axis$ in the upward direction. The second infinite solid plane wall occupies the plane $y = h > 0$. Let an incompressible, homogeneous Maxwell fluid fill the slab $y \in (0, h)$. Initially, the fluid and plates are at rest. At the moment $t = 0^+$, the fluid is set in motion by the bottom plate, which begins to translate along the $x-axis$ with the velocity $u_w(t) = U_o f(t)$, where $U_o > 0$ is a constant and $f(t)$ is a piecewise continuous function defined on $[0, \infty)$ and $f(0) = 0$. Also, we suppose that the Laplace transform of the function $f(t)$ exists. In the case of parallel flow along the $x - axis$, the velocity vector is $\mathbf{v} = (u(y, t), 0, 0)$ while the constitutive relation and the governing equation are given by [7, 8, 21]

$$\tau + \lambda\frac{\partial\tau}{\partial t} = \mu\frac{\partial u}{\partial y},\ (y, t) \in (0, h) \times (0, \infty), \tag{2}$$

$$\rho\frac{\partial u}{\partial t} + \rho\lambda\frac{\partial^2 u}{\partial t^2} = \mu\frac{\partial^2 u}{\partial y^2},\ (y, t) \in (0, h) \times (0, \infty), \tag{3}$$

where $\tau(y, t) = S_{xy}(y, t)$ is one of the nonzero component of the extra stress tensor $\mathbf{S}$ and ρ is the constant density of the fluid. In this paper, we consider the existence of slip at the walls and assume that the relative velocity between the velocity of the fluid at the wall and wall is proportional to the shear rate at the wall [20,21]. The boundary conditions due to wall slip as well as the initial conditions are

$$u(0, t) - \beta\frac{\partial u(0, t)}{\partial y} = U_o f(t),\ \ t > 0, \tag{4}$$

$$u(h, t) + \beta\frac{\partial u(h, t)}{\partial y} = 0 \quad ,\ t > 0, \tag{5}$$

$$\begin{aligned} &u(y, 0) = 0,\ \tfrac{\partial u(y,0)}{\partial t} = 0,\\ &\tau(y, 0) = 0,\ y \in [0, h], \end{aligned} \tag{6}$$

where β is the slip coefficient. We introduce the following non-dimensionalization:

$$\begin{aligned} &t^* = \tfrac{t}{T},\ y^* = \tfrac{y}{h},\ u^* = \tfrac{u}{U_o},\ \tau^* = \tfrac{\tau}{(\rho h U_o/T)},\\ &\lambda^* = \tfrac{\lambda}{T},\ \beta^* = \tfrac{\beta}{h}, \end{aligned} \tag{7}$$

$T > 0$ being a characteristic time. Equations (2)-(6), in non-dimensional form are (dropping the "*" notation)

$$\tau + \lambda\frac{\partial\tau}{\partial t} = \frac{1}{R}\frac{\partial u}{\partial y},\ \ y \in (0, 1) \times (0, \infty) \tag{8}$$

$$\lambda\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{1}{R}\frac{\partial^2 u}{\partial y^2},\ \ y \in (0, 1) \times (0, \infty) \tag{9}$$

$$u(0, t) - \beta\frac{\partial u(0, t)}{\partial y} = g(t),\ \ t > 0 \tag{10}$$

$$u(1, t) + \beta\frac{\partial u(1, t)}{\partial y} = 0 \quad ,\ t > 0 \tag{11}$$

$$\begin{aligned} &u(y, 0) = 0,\ \tfrac{\partial u(y,0)}{\partial t} = 0,\\ &\tau(y, 0) = 0, y \in [0, 1], \end{aligned} \tag{12}$$

where $R = \frac{h^2}{\nu T}$ is the Reynolds number,$\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, $g(t)$ is given by $f(Tt^*)$.

2.1. Velocity field

By applying the temporal Laplace transform, $L\{.\}$ [26], to Eqs. (9)-(11) and using the initial conditions $(12)_{1,2}$ we obtain the following set of equations:

$$\frac{\partial^2\bar{u}(y,q)}{\partial y^2} - R(\lambda q^2+q)\bar{u}(y,q) = 0 \tag{13}$$

$$\bar{u}(0,q) - \beta\frac{\partial\bar{u}(0,q)}{\partial y} = G(q) = L\{g(t)\} \tag{14}$$

$$\bar{u}(1,q) + \beta\frac{\partial\bar{u}(1,q)}{\partial y} = 0 \tag{15}$$

where q is the Laplace transform parameter and $\bar{u}(y,q) = L\{u(y,t)\}$. The solution of differential equation (13) with the boundary conditions (14) and (15) is given by

$$\bar{u}(y,q) = G(q)G_1(y,q) \tag{16}$$

where

$$G_1(y,q) = \frac{sh[(1-y)\sqrt{Rw(q)}] + \beta\sqrt{Rw(q)}ch[(1-y)\sqrt{Rw(q)}]}{[1+\beta^2Rw(q)]sh[\sqrt{Rw(q)}] + 2\beta\sqrt{Rw(q)}ch[\sqrt{Rw(q)}]} \tag{17}$$

and

$$w(q) = (\lambda q^2+q) = \lambda[(q+\frac{1}{2\lambda})^2 - (\frac{1}{2\lambda})^2]. \tag{18}$$

In order to determine the inverse Laplace transform of function $G_1(y,q)$, we consider the auxiliary function

$$F_1(y,q) = \frac{sh[(1-y)\sqrt{Rq}] + \beta\sqrt{Rq}ch[(1-y)\sqrt{Rq}]}{(1+\beta^2Rq)sh(\sqrt{Rq}) + 2\beta\sqrt{Rq}ch(\sqrt{Rq})} \tag{19}$$

Observing that the singular points of $F_1(y,q)$ are simple poles located at

$$q_n = -\frac{{p_n}^2}{R},\;\; n = 1,2,... \tag{20}$$

where $p_n \neq 0$ are the real roots of the equation

$$\tan(p_n) = \frac{2\beta p_n}{\beta^2{p_n}^2 - 1}, \tag{21}$$

we invert function $F_1(y,q)$ by using the residue theorem to evaluate the Laplace inversion integral [26]. Such that, after simplifying, we obtain

$$\begin{aligned} f_1(y,t) &= L^{-1}\{F_1(y,q)\} \\ &= \sum_{n=1}^{\infty}\mathrm{Res}[F_1(y,q)e^{qt},q_n] \\ &= \sum_{n=1}^{\infty}A_n(y)\exp(-\frac{{p_n}^2}{R}t) \end{aligned} \tag{22}$$

where

$$\begin{aligned} A_n(y) &= \\ &\frac{\sin(1-y)p_n + \beta p_n\cos(1-y)p_n}{\beta(\beta+1)R\sin p_n - \frac{R}{2p_n}(1+2\beta-\beta^2{p_n}^2)\cos p_n} \\ &= \frac{2p_n}{R}\frac{\sin(yp_n)+\beta p_n\cos(yp_n)}{(1+2\beta)+\beta^2{p_n}^2}. \end{aligned} \tag{23}$$

By comparing (17) and (19) we observe that $G_1(y,q) = F_1[y,w(q)]$ and, using the inverse Laplace transform for composed functions (see (A1) and (A2) from the Appendix A), we obtain

$$g_1(y,t) = L^{-1}\{G_1(y,q)\} = \int_0^{\infty} f_1(y,s)h(s,t)ds \tag{24}$$

where

$$\begin{aligned} h(s,t) &= L^{-1}\{e^{-sw(q)}\} = \\ &= \frac{t}{2}e^{\frac{s-2t}{4\lambda}}\sum_{k=0}^{\infty}\frac{(-\lambda s)^k}{(k+1)!(2k+1)!}\int_0^{\infty} z^{2k+1}J_2(2\sqrt{zt})dz \end{aligned} \tag{25}$$

and $J_\nu(\cdot)$ is the Bessel function of first kind and order ν . Replacing (22) and (25) into (24) we find that

$$\begin{aligned} g_1(y,t) &= \int_0^{\infty}[\sum_{n=1}^{\infty}A_n(y)e^{-\frac{{p_n}^2 s}{R}}][\frac{t}{2}e^{\frac{s-2t}{4\lambda}} \\ &\times\sum_{k=0}^{\infty}\frac{(-\lambda s)^k}{(k+1)!(2k+1)!}\int_0^{\infty} z^{2k+1}J_2(2\sqrt{zt})dz]ds \\ &= \frac{t}{2}e^{-\frac{t}{2\lambda}}\sum_{n=1}^{\infty}A_n(y) \\ &\times\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(k+1)!(2k+1)!}\int_0^{\infty} z^{2k+1}J_2(2\sqrt{zt})dz \\ &\times\int_0^{\infty} s^k e^{-(\frac{{p_n}^2}{R}-\frac{1}{4\lambda})s}ds \\ &= \frac{t}{2}e^{-\frac{t}{2\lambda}}\sum_{n=1}^{\infty}A_n(y) \\ &\times\int_0^{\infty} J_2(2\sqrt{zt})\sum_{k=0}^{\infty}\frac{(-\lambda)^k\Gamma(k+1)}{(k+1)!(2k+1)!}\frac{z^{2k+1}}{{b_n}^{k+1}}dz, \end{aligned} \tag{26}$$

where $b_n = \frac{{p_n}^2}{R} - \frac{1}{4\lambda} > 0$ and Γ is the Gamma function.

By using (A3) from the Appendix A we obtain a new expression of the function $g_1(y,t)$, namely

$$\begin{aligned} &g_1(y,t) \\ &= \frac{2t}{\lambda}e^{-\frac{t}{2\lambda}}\sum_{n=1}^{\infty}A_n(y)\int_0^{\infty}\frac{1}{z}\sin^2(\frac{\sqrt{\lambda z}}{2\sqrt{b_n}})J_2(2\sqrt{zt})dz \end{aligned} \tag{27}$$

Now, using the properties of the Bessel functions [27] we can show that

$$g_1(y,t)=L^{-1}[G_1(y,q)] \tag{28}$$

$$=e^{-\frac{t}{2\lambda}}\sum_{n=1}^{\infty}\frac{A_n(y)}{\sqrt{\lambda b_n}}\sin(t\sqrt{\frac{b_n}{\lambda}}) \tag{29}$$

Finally, we obtain:

a. The velocity field corresponding to the flow of a Maxwell fluid with slip at the boundary is given by

$$u_{Ms}(y,t)=(g*g_1)(t)=\int_0^t g(t-s)g_1(y,s)ds$$

$$=\sum_{n=1}^{\infty}\frac{A_n(y)}{\sqrt{\lambda b_n}}\int_0^t g(t-s)e^{-\frac{s}{2\lambda}}\sin(s\sqrt{\frac{b_n}{\lambda}})ds. \tag{30}$$

b. For the flow of a Maxwell fluid with a nonslip boundary condition, that is $\beta=0$, the function $A_n(y)$ given by Eq. (23) becomes

$$A_{1n}(y)=\frac{2n\pi}{R}\sin(n\pi y),\ \ n=1,2,..... \tag{31}$$

and the velocity field has the expression

$$u_M(y,t)=\frac{2\pi}{R}\sum_{n=1}^{\infty}\frac{n\sin(n\pi y)}{\sqrt{\lambda c_n}}\times$$
$$\times\int_0^t g(t-s)e^{-\frac{s}{2\lambda}}\sin(s\sqrt{\frac{c_n}{\lambda}})ds, \tag{32}$$

where $c_n=\frac{n^2\pi^2}{R}-\frac{1}{4\lambda}>0$.

c. The solution in the transform domain corresponding to the flow of a viscous fluid with slip at the wall, that is for $\lambda=0$ and $\beta\neq 0$, is given by

$$\bar{u}(y,t)=G(q)F_1(y,q) \tag{33}$$

and the (y,t)-domain solution is

$$u_{Ns}(y,t)=\int_0^t g(t-s)f_1(y,s)ds$$
$$=\sum_{n=1}^{\infty}A_n(y)\int_0^t g(t-s)\exp(-\frac{{p_n}^2}{R}s)ds, \tag{34}$$

where $A_n(y)$is given by Eq. (23). For $f(t)=\sin(\omega t)$ or $f(t)=\cos(\omega t)$, the velocity field given by Eq.(33) was determined in equivalent forms by Khalid and Vafai [20].

d. For flows of viscous Newtonian fluids with a nonslip boundary condition, that is $\lambda=0$ and $\beta=0$, the transform domain solution is

$$\bar{u}_N(y,q)=G(q)G_2(y,q), \tag{35}$$

where

$$G_2(y,q)=\frac{sh[(1-y)\sqrt{Rq}]}{sh[\sqrt{Rq}]}, \tag{36}$$

and the (y,t)-domain solution is given by

$$u_N(y,t)=\frac{2\pi}{R}\sum_{n=1}^{\infty}n\sin(n\pi y)\int_0^t g(t-s)e^{-\frac{n^2\pi^2}{R}s}ds \tag{37}$$

The relative velocity between the fluid at the bottom wall and wall itself for Maxwell fluids is

$$u_{Mrel}(t)=u_{Ms}(0^+,t)-g(t)$$
$$=\frac{2\beta}{R}\sum_{n=1}^{\infty}\frac{{p_n}^2}{\sqrt{\lambda b_n}\{\beta^2{p_n}^2+(1+2\beta)\}}\times$$
$$\times\int_0^t g(t-s)e^{-\frac{s}{2\lambda}}\sin(s\sqrt{\frac{b_n}{\lambda}})ds-g(t) \tag{38}$$

and for a viscous Newtonian fluid is given by

$$u_{Nrel}(t)=u_{Ns}(0^+,t)-g(t)$$
$$=\frac{2\beta}{R}\sum_{n=1}^{\infty}\frac{{p_n}^2}{\beta^2{p_n}^2+(1+2\beta)}$$
$$\times\int_0^t g(t-s)e^{-\frac{{p_n}^2}{R}s}ds-g(t). \tag{39}$$

2.2. Shear Stress

In order to determine the shear stress $\tau(y,t)$ we use Eqs. (8),(16) and (28). Applying the Laplace transform to Eq. (8) with the initial condition $(12)_3$ we obtain

$$\bar{\tau}(y,q)=\frac{1}{\lambda R}G(q)G_3(y,q) \tag{40}$$

where

$$G_3(y,q)=\frac{1}{q+1/\lambda}\frac{\partial G_1(y,q)}{\partial y}. \tag{41}$$

The inverse Laplace transform of function $G_3(y,q)$ is

$$g_3(y,t)=\int_0^t e^{-\frac{t-s}{\lambda}}\frac{\partial g_1(y,s)}{\partial y}ds$$
$$=\sum_{n=1}^{\infty}\frac{B_n(y)}{\sqrt{\lambda b_n}}e^{-\frac{t}{\lambda}}\int_0^t e^{\frac{s}{2\lambda}}\sin(s\sqrt{\frac{b_n}{\lambda}})ds \tag{42}$$

where

$$B_n(y)=\frac{dA_n(y)}{dy}$$
$$=\frac{2{p_n}^2}{R}\frac{\cos(yp_n)-\beta p_n\sin(yp_n)}{(1+2\beta)-\beta^2{p_n}^2}. \tag{43}$$

Eq.(41) can be written in the simple form

$$g_3(y,t)=\sum_{n=1}^{\infty}\frac{B_n(y)}{\sqrt{\lambda b_n}}\frac{2\lambda e^{-\frac{t}{2\lambda}}}{1+4\lambda b_n}$$
$$\times[2\sqrt{\lambda b_n}e^{-\frac{t}{2\lambda}}+\sin(t\sqrt{\frac{b_n}{\lambda}})-2\sqrt{\lambda b_n}\cos(t\sqrt{\frac{b_n}{\lambda}})] \tag{44}$$

The (y,t)-domain solution for the shear stress is given by

$$\tau(y,t)=\frac{1}{\lambda R}(g*g_3)(t)=\frac{1}{\lambda R}\int_0^t g(t-s)g_3(y,s)ds \tag{45}$$

3. Some particular cases of the motion of the plate

In this section we consider two functions corresponding to the motion of the bottom plate, namely $f(t)=H(\omega t)$ and $f(t)=\sin(\omega t)$, with $\omega>0$ being a constant. We choose the characteristic time $T=\frac{1}{\omega}$, for the dimensionless variables and functions given by Eq.(7), and obtain $g(t)=H(t)$ and $g(t)=\sin t$, $H(t)$ being the Heaviside unit step function.

3.1. Solution for the translation of the bottom plate with a constant velocity

The motion of the bottom plate is given by the function $g(t) = H(t)$ and the velocity $u(y,t)$ is obtained from Eqs. (29), (31), (33) and (36) with $g(t-s) = 1$. The velocity fields corresponding to this type of the motion have the following expressions:
a. Maxwell fluids with slip at the boundary:

$$\begin{aligned} u_{Ms}(y,t) = H(t)\{4\lambda \sum_{n=1}^{\infty} \tfrac{A_n(y)}{1+4\lambda b_n} \\ -2\lambda \sum_{n=1}^{\infty} \tfrac{A_n(y)e^{-\frac{t}{2\lambda}}}{\sqrt{\lambda b_n}(1+4\lambda b_n)} \\ \times [\sin(t\sqrt{\tfrac{b_n}{\lambda}}) + 2\sqrt{\lambda b_n}\cos(t\sqrt{\tfrac{b_n}{\lambda}})]\}, \end{aligned} \quad (46)$$

which, by using (A4) from the Appendix A, can be written in the simpler form

$$\begin{aligned} u_{Ms}(y,t) = H(t)\{\tfrac{1+\beta-y}{1+2\beta} \\ -e^{-\frac{t}{2\lambda}} \sum_{n=1}^{\infty} \tfrac{\sin(yp_n)+\beta p_n\cos(yp_n)}{p_n(1+2\beta+\beta^2{p_n}^2)} \\ \times [2\cos(t\sqrt{\tfrac{b_n}{\lambda}}) + \tfrac{1}{\sqrt{\lambda b_n}}\sin(t\sqrt{\tfrac{b_n}{\lambda}})]\}. \end{aligned} \quad (47)$$

For large values of the time t the velocity field given by Eq. (46) tends to the "stationary solution"

$$u_{Ms}^{st}(y,t) = \frac{1+\beta-y}{1+2\beta} \quad (48)$$

and for $t \to 0^+$ $u_{Ms}(y,t)$ tends to zero.
b. Maxwell fluids with nonslip at the boundary:

$$\begin{aligned} u_M(y,t) = H(t)\{1-y-e^{-\frac{t}{2\lambda}} \sum_{n=1}^{\infty} \tfrac{\sin(n\pi y)}{n\pi} \\ \times[2\cos(t\sqrt{\tfrac{c_n}{\lambda}}) + \tfrac{1}{\sqrt{\lambda c_n}}\sin(t\sqrt{\tfrac{c_n}{\lambda}})]\}. \end{aligned} \quad (49)$$

c. Newtonian fluids with slip at the boundary:

$$\begin{aligned} u_{Ns}(y,t) = H(t)\{\tfrac{1+\beta-y}{1+2\beta} \\ -2\sum_{n=1}^{\infty} \tfrac{\sin(yp_n)+\beta p_n\cos(yp_n)}{p_n(1+2\beta+\beta^2{p_n}^2)} e^{-\frac{{p_n}^2}{R}t}\}. \end{aligned} \quad (50)$$

d. Newtonian fluids with nonslip at the boundary:

$$u_N(y,t) = H(t)\{1-y-2\sum_{n=1}^{\infty} \frac{\sin(n\pi y)}{n\pi} e^{-\frac{n^2\pi^2}{R}t}\}. \quad (51)$$

The relative velocity is given by

$$\begin{aligned} u_{Mrel}(t) = H(t)\{-\tfrac{\beta}{1+2\beta} - \\ -e^{-\frac{t}{2\lambda}} \sum_{n=1}^{\infty} \tfrac{\beta p_n}{p_n(1+2\beta+\beta^2{p_n}^2)} \\ \times[2\cos(t\sqrt{\tfrac{b_n}{\lambda}}) + \tfrac{1}{\sqrt{\lambda b_n}}\sin(t\sqrt{\tfrac{b_n}{\lambda}})]\}, \end{aligned} \quad (52)$$

for the Maxwell fluid, respectively,

$$\begin{aligned} u_{Nrel}(t) = H(t)[-\tfrac{\beta}{1+2\beta} \\ -2\sum_{n=1}^{\infty} \tfrac{\beta p_n}{p_n(1+2\beta+\beta^2{p_n}^2)} e^{-\frac{{p_n}^2}{R}t}], \end{aligned} \quad (53)$$

for Newtonian fluids. By using the illustrations generated with the software Mathcad, we discuss some relevant physical aspects of the flow. Also, the roots p_n , $n = 1, 2, ...$, of Eq.(21) are determined by means of the software "root $(f(x), x, a, b)$ " from Mathcad. For the dimensionless slip coefficient, $\beta \in \{0.4, 0.7\}$ the roots p_n are presented in the Table 1 from the Appendix A. In the figures, we use $\nu = 9.15255 \times 10^{-3} m^2/_s$, $\lambda = 0.55s$, $\rho = 1.050 ^{kg}/_{m^3}$ corresponding to Maxwell fluid $1\% PMMA$ in DEM (Poly(methylmetha crylate) in diethyl malonate) [4], and $h = 0.2m$, $U_o = 0.6m/s$, $\beta = 0.14m$. The Reynolds number corresponding to aforementioned values is $R = 3.4962934$, the dimensionless slip coefficient is $\beta = 0.7$ and the dimensionless relaxation time is $\lambda = 0.44$. Also, we use the following abbreviations for dimensionless velocities:u_{Ms}, the velocity for Maxwell fluid with slip at the wall; u_M, the velocity for Maxwell fluid with nonslip at the wall; u_{Ns}, the velocity for Newtonian fluid with slip at the wall; u_N, the velocity for Newtonian fluid with nonslip at the wall. In Fig. 1 we show the dimensionless velocity, $u(y,t)$, versus t for $y \in \{0.05, 0.25, 0.85\}$. For comparison, we have plotted the functions corresponding to Maxwell and Newtonian fluids with both slip and nonslip boundary conditions. For a fixed value of the spatial variable y, the velocity corresponding to a Maxwell fluid with slip at the boundary is zero for a short time, after that, is increasing and tends towards the "stationary solution" u_{Ms}^{st} given by Eq. (47). The velocity corresponding to Maxwell fluid with nonslip condition has a non uniform variation at the beginning of the motion after that approaches to the "stationary solution" $u_M^{st}(y,t) = 1-y$. For Newtonian fluids, the velocity is larger in the case of a nonslip than in the case of slip at the boundary near the bottom plate. For large values of the time t they tend to the "stationary solutions" $u_{Ns}^{st} = u_{Ms}^{st}$, respectively $u_N^{st} = u_M^{st}$. Fig. 2 shows the diagrams of velocity $u(y,t)$ corresponding to Maxwell and Newtonian fluids for both slip and non slip conditions. The velocity was plotted versus y for $t \in \{0.5, 1.5, 2.5\}$. For small values of time t the Newtonian fluid with slip at the wall is slower than the Maxwell fluid with slip condition near the lower plate and faster near upper plate. For increasing t the Newtonian fluid with slip condition is slower than the Maxwell fluid throughout domain of flow. In Fig.3 we have plotted the relative velocity corresponding to Maxwell and Newtonian fluids versus t for two values of the dimensionless slip coefficient β. In absolute value, the relative velocity decreases with increasing values of β and flatten out for large values of time t.

3.2. Solution for the sinusoidal oscillations of the bottom plate

In this section we consider for the motion of the bottom plate the function $g(t) = \sin t$. The velocity fields corresponding to this type of motion are given by Eqs. (29), (31), (33) and (36) in which $g(t-s)$ is replaced by $\sin(t-$

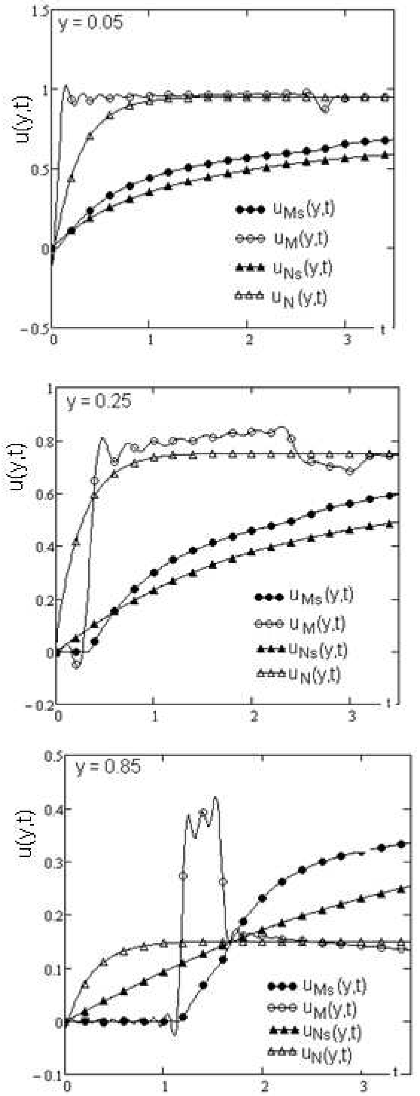

Figure 1 Plot of $u(y,t)$ versus t for both cases with slip and nonslip condition.

s).

a. Integrating by parts into Eq. (29), this yields after some simplifications, the velocity field corresponding to the flow of a Maxwell fluid with slip at the wall

$$u_{Ms}(y,t) = Q_1(y)\sin t + Q_2(y)\cos t + e^{-\frac{t}{2\lambda}}\sum_{n=1}^{\infty}[Q_{3n}(y)\cos(t\sqrt{\tfrac{b_n}{\lambda}}) + Q_{4n}(y)\sin(t\sqrt{\tfrac{b_n}{\lambda}})], \tag{54}$$

where

$$Q_1(y) = 2\sum_{n=1}^{\infty}\frac{p_n({p_n}^2-\lambda R)}{({p_n}^2-\lambda R)^2+R^2}Q_n(y), \tag{55}$$

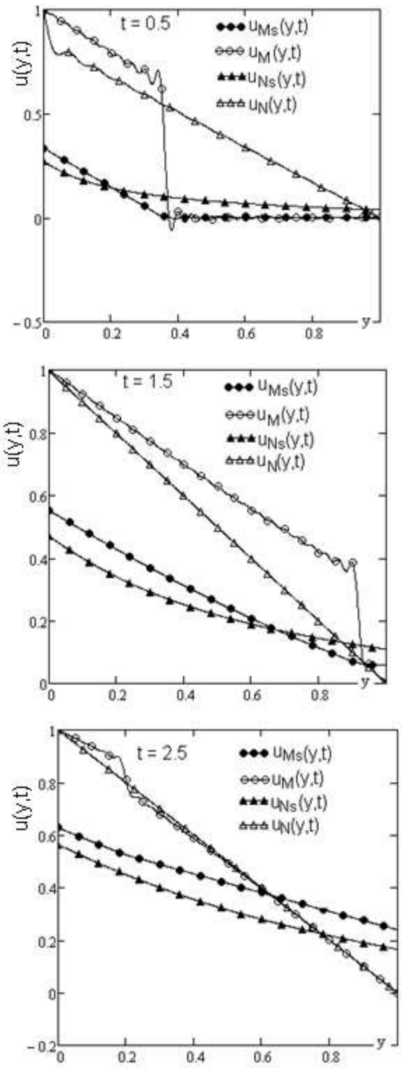

Figure 2 Plot of $u(y,t)$ versus y for both cases with slip and nonslip condition.

$$Q_2(y) = -2\sum_{n=1}^{\infty}\frac{Rp_n}{({p_n}^2-\lambda R)^2+R^2}Q_n(y), \tag{56}$$

$$Q_{3n}(y) = \frac{2p_nR}{({p_n}^2-\lambda R)^2+R^2}Q_n(y), \tag{57}$$

$$Q_{4n}(y) = \frac{p_n}{\sqrt{\lambda b_n}}\frac{R-2\lambda({p_n}^2-\lambda R)}{({p_n}^2-\lambda R)^2+R^2}Q_n(y), \tag{58}$$

$$Q_n(y) = \frac{\sin(yp_n)+\beta p_n\cos(yp_n)}{1+2\beta+\beta^2{p_n}^2}, \tag{59}$$

The velocity field given by Eq. (53) is the sum between the “permanent solution”

$$u_{Msp}(y,t) = Q_1(y)\sin t + Q_2(y)\cos t \tag{60}$$

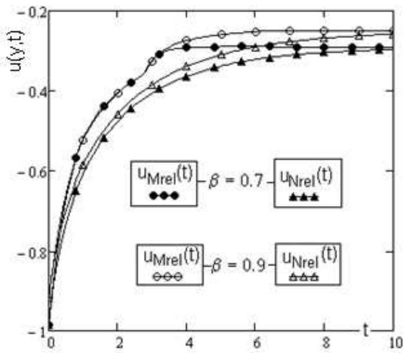

Figure 3 Plot of the relative velocity versus t for Maxwell and Newtonian fluids.

and the transient solution $u_{Mst}(y,t) = u_{Ms}(y,t) - u_{Msp}(y,t)$ which can be neglected for large values of the time t. By using the residue theorem to evaluate the inverse Laplace transform of function $\bar{u}(y,q)$ given by Eq. (16), with $G(q) = L\{\sin t\} = \frac{1}{q^2+1}$, we obtain for the "permanent solution" $u_{Msp}(y,t)$ an equivalent expression, namely

$$u_{Msp}(y,t) = \frac{1}{M}[M_1P_1(y) + M_2P_2(y)]\sin t + \frac{1}{M}[M_1P_2(y) + M_2P_1(y)]\cos t, \quad (61)$$

where

$$M_1 = [(1-\lambda\beta^2R)sh(\alpha_1\sqrt{R}) + 2\alpha_1\beta\sqrt{R}ch(\alpha_1\sqrt{R})] \times \cos(\alpha_2\sqrt{R}) - [\beta^2Rch(\alpha_1\sqrt{R}) + 2\alpha_2\beta\sqrt{R}sh(\alpha_1\sqrt{R})]\sin(\alpha_2\sqrt{R}), \quad (62)$$

$$M_2 = [(1-\lambda\beta^2R)ch(\alpha_1\sqrt{R}) + 2\alpha_1\beta\sqrt{R}sh(\alpha_1\sqrt{R})]\sin(\alpha_2\sqrt{R}) + [\beta^2Rsh(\alpha_1\sqrt{R}) + 2\alpha_2\beta\sqrt{R}ch(\alpha_1\sqrt{R})] \times \cos(\alpha_2\sqrt{R}), \quad (63)$$

$$M = {M_1}^2 + {M_2}^2, \quad (64)$$

$$P_1(y) = sh[\alpha_1\sqrt{R}(1-y)]\{\cos[\alpha_2\sqrt{R}(1-y)] - \alpha_2\beta\sqrt{R}\sin[\alpha_2\sqrt{R}(1-y)]\} + \alpha_1\beta\sqrt{R}ch[\alpha_1\sqrt{R}(1-y)]\cos[\alpha_2\sqrt{R}(1-y)], \quad (65)$$

$$P_2(y) = ch[\alpha_1\sqrt{R}(1-y)]\{\sin[\alpha_2\sqrt{R}(1-y)] + \alpha_2\beta\sqrt{R}\cos[\alpha_2\sqrt{R}(1-y)]\} + \alpha_1\beta\sqrt{R}sh[\alpha_1\sqrt{R}(1-y)]\sin[\alpha_2\sqrt{R}(1-y)], \quad (66)$$

and $\alpha_{1,2} = \sqrt{(\sqrt{\lambda^2+1} \mp \lambda)/2}$.

b. The velocity field corresponding to Maxwell fluids with nonslip at the boundary is given by

$$u_M(y,t) = 2\sin t \sum_{n=1}^{\infty} \frac{(n\pi)(n^2\pi^2-\lambda R)\sin(n\pi y)}{(n^2\pi^2-\lambda R)^2+R^2} - 2\cos t \sum_{n=1}^{\infty} \frac{R(n\pi)\sin(n\pi y)}{(n^2\pi^2-\lambda R)^2+R^2} + 2e^{-\frac{t}{2\lambda}}\{\sum_{n=1}^{\infty} \frac{R(n\pi)\sin(n\pi y)}{(n^2\pi^2-\lambda R)^2+R^2}\cos(t\sqrt{\frac{c_n}{\lambda}}) + \frac{(n\pi)[R-2\lambda(n^2\pi^2-\lambda R)]\sin(n\pi y)}{\sqrt{\lambda c_n}[(n^2\pi^2-\lambda R)^2+R^2]}\sin(t\sqrt{\frac{c_n}{\lambda}})\}. \quad (67)$$

The "permanent solution" corresponding to this type of the motion, can be written in the equivalent form

$$u_{M\,p}(y,t) = P_3(y)\sin t + P_4(y)\cos t, \quad (68)$$

where

$$P_3(y) = \{sh[\alpha_1\sqrt{R}(1-y)] \times \cos[\alpha_2\sqrt{R}(1-y)]sh(\alpha_1\sqrt{R})\cos(\alpha_2\sqrt{R}) + ch[\alpha_1\sqrt{R}(1-y)]\sin[\alpha_2\sqrt{R}(1-y)] \times ch(\alpha_1\sqrt{R})\sin(\alpha_2\sqrt{R})\}\frac{1}{sh^2(\alpha_1\sqrt{R})+\sin^2(\alpha_2\sqrt{R})}, \quad (69)$$

$$P_4(y) = \{ch[\alpha_1\sqrt{R}(1-y)]\sin[\alpha_2\sqrt{R}(1-y)] \times sh(\alpha_1\sqrt{R})\cos(\alpha_2\sqrt{R}) + sh[\alpha_1\sqrt{R}(1-y)]\cos[\alpha_2\sqrt{R}(1-y)] \times ch(\alpha_1\sqrt{R})\sin(\alpha_2\sqrt{R})\}\frac{1}{sh^2(\alpha_1\sqrt{R})+\sin^2(\alpha_2\sqrt{R})}. \quad (70)$$

c. The velocity field corresponding to the flows of Newtonian fluids with slip at the wall has expression

$$u_{Ns}(y,t) = 2\sin t \sum_{n=1}^{\infty} \frac{p_n^3Q_n(y)}{p_n^4+R^2} - 2\cos t \sum_{n=1}^{\infty} \frac{Rp_nQ_n(y)}{p_n^4+R^2} + 2\sum_{n=1}^{\infty} \frac{Rp_nQ_n(y)}{p_n^4+R^2}e^{-\frac{p_n^2}{R}t}, \quad (71)$$

where $Q_n(y)$ is given by Eq. (58). The "permanent solution" corresponding to this type of flows can be written in the equivalent form

$$U_{Nsp}(y,t) = \frac{1}{D}[D_1E_1(y) + D_2E_2(y)]\sin t + \frac{1}{D}[D_1E_2(y) - D_2E_1(y)]\cos t, \quad (72)$$

where

$$D_1 = sh(\sqrt{\frac{R}{2}})\cos(\sqrt{\frac{R}{2}}) - \beta^2Rch(\sqrt{\frac{R}{2}})\sin(\sqrt{\frac{R}{2}}) + \beta\sqrt{2R}[ch(\sqrt{\frac{R}{2}})\cos(\sqrt{\frac{R}{2}}) - sh(\sqrt{\frac{R}{2}})\sin(\sqrt{\frac{R}{2}})], \quad (73)$$

$$D_2 = ch(\sqrt{\frac{R}{2}})\sin(\sqrt{\frac{R}{2}}) + \beta^2Rsh(\sqrt{\frac{R}{2}})\cos(\sqrt{\frac{R}{2}}) + \beta\sqrt{2R}[sh(\sqrt{\frac{R}{2}})\sin(\sqrt{\frac{R}{2}}) + ch(\sqrt{\frac{R}{2}})\cos(\sqrt{\frac{R}{2}})] \quad (74)$$

$$D = {D_1}^2 + {D_2}^2, \tag{75}$$

$$\begin{aligned} E_1(y) &= sh[\sqrt{\tfrac{R}{2}}(1-y)]\cos[\sqrt{\tfrac{R}{2}}(1-y)] \\ &+\beta\sqrt{\tfrac{R}{2}}\{ch[\sqrt{\tfrac{R}{2}}(1-y)]\cos[\sqrt{\tfrac{R}{2}}(1-y)] \\ &- sh[\sqrt{\tfrac{R}{2}}(1-y)]\sin[\sqrt{\tfrac{R}{2}}(1-y)]\}, \end{aligned} \tag{76}$$

$$\begin{aligned} E_2(y) &= ch[\sqrt{\tfrac{R}{2}}(1-y)]\sin[\sqrt{\tfrac{R}{2}}(1-y)]+ \\ &+\beta\sqrt{\tfrac{R}{2}}\{sh[\sqrt{\tfrac{R}{2}}(1-y)]\sin[\sqrt{\tfrac{R}{2}}(1-y)] \\ &+ch[\sqrt{\tfrac{R}{2}}(1-y)]\cos[\sqrt{\tfrac{R}{2}}(1-y)]\}. \end{aligned} \tag{77}$$

d. The Couette flow of a Newtonian fluid with nonslip boundary condition is characterized by the velocity field

$$\begin{aligned} u_N(y,t) &= 2\sin t\sum_{n=1}^{\infty}\frac{(n\pi)^3\sin(n\pi y)}{(n\pi)^4+R^2} \\ &-2\cos t\sum_{n=1}^{\infty}\frac{R(n\pi)\sin(n\pi y)}{(n\pi)^4+R^2}+ \\ &+2\sum_{n=1}^{\infty}\frac{R(n\pi)\sin(n\pi y)}{(n\pi)^4+R^2}e^{-\frac{n^2\pi^2}{R}t}. \end{aligned} \tag{78}$$

The "permanent solution" corresponding to the expression (77) can be written in the following form

$$\begin{aligned} &u_{Np}(y,t) = \\ &= \frac{\sin t}{sh^2(\sqrt{\frac{R}{2}})+\sin^2(\sqrt{\frac{R}{2}})} \\ &\times\{sh(\sqrt{\tfrac{R}{2}})\cos(\sqrt{\tfrac{R}{2}})sh[\sqrt{\tfrac{R}{2}}(1-y)]\cos[\sqrt{\tfrac{R}{2}}(1-y)] \\ &+ch(\sqrt{\tfrac{R}{2}})\sin(\sqrt{\tfrac{R}{2}})ch[\sqrt{\tfrac{R}{2}}(1-y)]\sin[\sqrt{\tfrac{R}{2}}(1-y)]\} \\ &+\frac{\cos t}{sh^2(\sqrt{\frac{R}{2}})+\sin^2(\sqrt{\frac{R}{2}})}\times \\ &\times\{sh(\sqrt{\tfrac{R}{2}})\cos(\sqrt{\tfrac{R}{2}})ch[\sqrt{\tfrac{R}{2}}(1-y)]\sin[\sqrt{\tfrac{R}{2}}(1-y)] \\ &-ch(\sqrt{\tfrac{R}{2}})\sin(\sqrt{\tfrac{R}{2}})sh[\sqrt{\tfrac{R}{2}}(1-y)]\cos[\sqrt{\tfrac{R}{2}}(1-y)]\}. \end{aligned} \tag{79}$$

Some important properties of flow due to sinusoidal oscillations of the bottom plate are presented using illustrations from Figs. 4-6.

In Fig. 4 we plotted the velocity $u(y,t)$ and the "permanent solutions" corresponding to Maxwell and Newtonian fluids with both slip and non slip boundary conditions. These functions were presented versus t for $y \in \{0.05, 0.25, 0.85\}$ and, it is evident that the difference between the velocity $u(y,t)$ and the "permanent velocity" is significant only for small values of the time . We see that in the considered case, after the moment $t = 4$ for Maxwell fluid with slip at the wall, respectively $t = 6$ for the Maxwell fluid with non slip condition the transient velocities $u^t_{Ms}(y,t) = u_{Ms}(y,t) - u_{Msp}(y,t)$, $u^t_M(y,t) = u_M(y,t) - u_{Mp}(y,t)$ can be neglected. For Newtonian fluids $t = 6$ in the case of the flow with slip at the wall and $t = 4$ in the case of nonslip at the wall. After these moments, the fluids flow according to the "permanent solution". Fig. 5 contains diagrams of velocity $u(y,t)$, versus y for six different values of time, t. The curves corresponding to the slip and nonslip boundary conditions, for Maxwell and Newtonian fluids were considered. At the

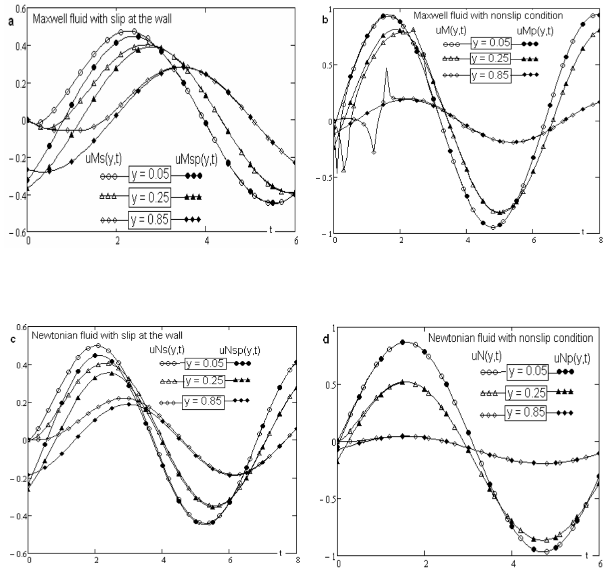

Figure 4 Plot of the $u(y,t)$ versus t for both cases with slip and nonslip condition.

small values of time the Maxwell fluid has not a monotonous flow. After the value $t = 1$, the absolute values of the velocity corresponding to both types of Maxwell fluids increase for increasing y. The absolute values of the velocity corresponding to both cases of Newtonian fluids increase for increasing y and for all values of the time t . In Fig. 6 we have plotted the relative velocity corresponding to Maxwell and Newtonian fluids versus t for two values of the dimensionless slip coefficient β . The relative velocity, in absolute terms, is an increasing function of β.

4. Conclusion

Couette flows of a Maxwell fluid were analyzed under slip conditions between the fluid and walls. The motion of the bottom plate was assumed to be a rectilinear translation in its plane while, the upper plate is at rest. Two particular cases, namely translation with constant velocity and sinusoidal oscillations of the bottom plate, were considered. The relative velocity between the fluid at the wall and the wall was assumed to be proportional to the shear rate at the wall. The exact expressions for the velocity $u(y,t)$ and shear stress, have been determined by means of Laplace transform. For a complete study and for comparisons, we presented velocity fields corresponding to both flows (with slip and nonslip conditions) for Maxwell and Newtonian fluids. The expressions of the relative velocity have also been determined. If the bottom plate translates with the constant velocity then the velocity fields corresponding to the four types of the flows were written as sums between the stationary solutions and transient solutions. For large

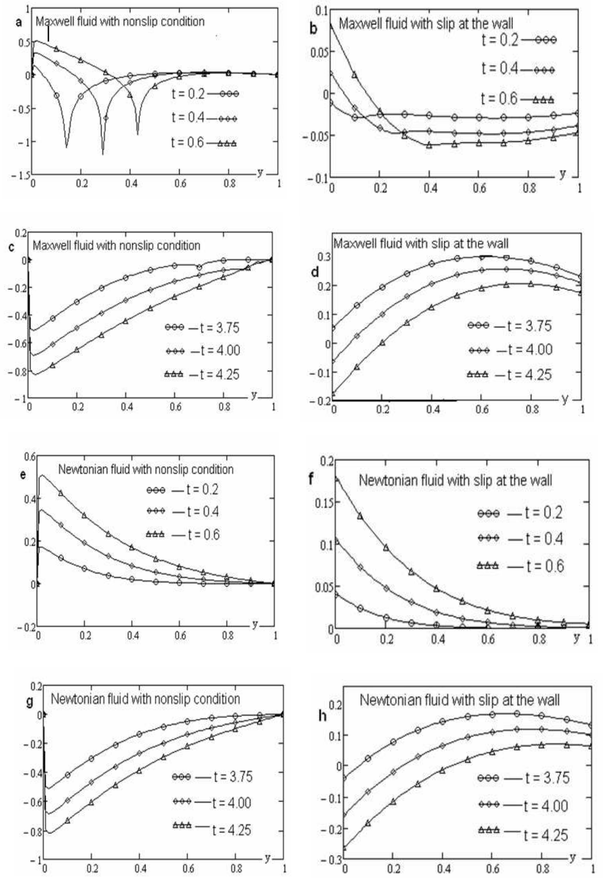

Figure 5 Plot of the $u(y,t)$ versus y for both cases with slip and nonslip condition.

values of the time t the transient solutions can be neglected and the fluid flows according to the stationary solutions. For Maxwell fluids the velocity is zero a short period after the staring of the motion. After this period the values of the velocity increase for increasing time t and tend to the values of the stationary solutions. For Newtonian fluids the velocities are increasing functions of t. The velocity corresponding to the flow with slip condition is smaller than the velocity for the flow with non slip condition for both types of fluids (see Fig. 1 and 2). The relative velocity, in absolute value, is a decreasing function of the slip coefficient β (see Fig. 3). For sinusoidal oscillations of the plate the expressions of the velocities corresponding to the four types of flows were written as the sums between the "permanent solutions" and transient solutions. In each case two equivalent forms of the permanent solution were presented. The difference between the velocity $u(y,t)$ and the permanent solution is significant only for the small values of the time t (see Fig. 4).For large values of the time t the fluids flow according to the permanent solutions. The velocity field $u(y,t)$ versus y, in absolute terms, is a decreasing function

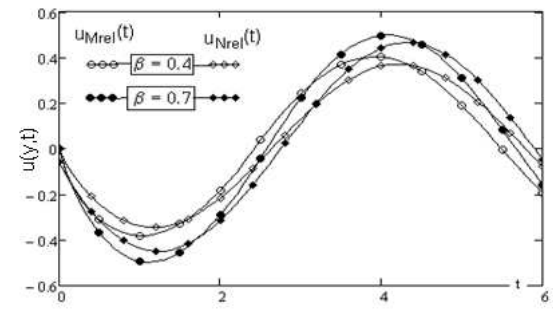

Figure 6 Plot of the relative velocity versus t for Maxwell and Newtonian fluids.

(see Fig. 5), and the relative velocity, in absolute terms is an increasing function of the slip coefficient β. The software Mathcad 14.0 was used for numerical calculations and to generate the diagrams presented herein and the roots of Eq. (21) (See Table 1 from Appendix A).

5. Appendix A

$$A_1.\, L^{-1}\{F(q)\} = f(t),\, L^{-1}\{F[w(q)]\}$$
$$= \int_0^\infty f(x)g(x,t)dx,\, g(x,t) = L^{-1}\{e^{-xw(q)}\}$$
$$A_2.\, L^{-1}\{q^b e^{ab}\}$$
$$= \frac{1}{b}\sum_{n=0}^{\infty}\frac{a^n}{(n+1)!\Gamma[b(n+1)]}\int_0^\infty x^{b(n+1)}J_o(2\sqrt{xt})dx,\, b>0$$
$$A_3.\, \sum_{k=0}^{\infty}\frac{(-\lambda)^k z^{2k+1}}{(k+1)(2k+1)!{b_n}^{k+1}}$$
$$= \frac{2}{\lambda z}[1-\cos(z\sqrt{\frac{\lambda}{b_n}})]$$
$$= \frac{4}{\lambda z}\sin^2(\frac{z\sqrt{\lambda}}{2\sqrt{b_n}})$$
$$A_4.\, 4\lambda\sum_{k=0}^{\infty}\frac{A_n(y)}{1+4\lambda b_n}$$
$$= 2\sum_{n=1}^{\infty}\frac{\sin(p_n y)+\beta p_n\cos(p_n y)}{p_n(1+2\beta+\beta^2{p_n}^2)}$$
$$= \frac{1+\beta-y}{1+2\beta}$$

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Table 1 Table 1. Roots of Eq. (21)

p_n	$\beta = 0.4$	$\beta = 0.7$
p_1	1.8615134	1.513246
p_2	4.2127514	3.8518918
p_3	6.9717948	6.7031418
p_4	9.9185957	9.7167336
p_5	12.9478517	12.7888567
p_6	16.0176262	15.887318
p_7	19.1097267	18.9996515
p_8	22.2152756	22.1201338
p_9	25.3295014	25.2457934
p_{10}	28.449632	28.3749412
p_{11}	31.5739545	31.5065484
p_{12}	34.7013567	34.6399135
p_{13}	37.8310854	37.7747121
p_{14}	40.9626254	40.9105149
p_{15}	44.0955657	44.04714
p_{16}	47.2296565	47.184424
p_{17}	50.3646768	50.3222441
p_{18}	53.5004611	53.4605064
p_{19}	56.636892	56.5991374
p_{20}	59.7738602	59.7380791

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