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On the moduli space of smooth plane quartic curves with a sextactic point

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Abstract: Let $\mathcal{M}_g$ be the moduli space of smooth algebraic curves of genus g over $\mathbb{C}$. In this paper, we prove that the set $\mathcal{S}_r \subseteq \mathcal{M}_3$ of moduli points of smooth plane quartic curves (nonhyperelliptic curves of genus 3) having at least one sextactic point of sextact multiplicity r, where $r \in \{1, 2, 3\}$, is an irreducible, closed and rational subvariety of codimensional $r-1$ of $\mathcal{M}_3 - \mathcal{H}_3$ (where $\mathcal{H}_3 \subset \mathcal{M}_3$ is the hyperelliptic locus).

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1. Introduction

On an algebraic plane curve $C \subset \mathbb{P}^2(\mathbb{C})$ of degree $d \geq 3$, we say that a flex point $P \in C$ is *i-flex* if the contact order with the tangent line T_P at P is equal to $i+2$, i.e., $i = I_P(C, T_P) - 2$. This positive integer i is called $the\ flex\ multiplicity$ of C at P. Vermeulen in [7] studied the subvariety $\mathcal{V} \subseteq \mathcal{M}_3$, where $\mathcal{M}_g$ is the moduli space of smooth algebraic curves of genus g over the complex field $\mathbb{C}$, corresponding to plane smooth quartics C having at least one hyperflex (2-flex). He proved that $\mathcal{V}$ is an irreducible, closed subvariety of dimensional 5 (recall that $\dim \mathcal{M}_g = 3g - 3$).

In analogy with the tangent lines and the flexes of plane curves, one can consider *the osculating conics* and *the sextactic points*. Let P be a non-flex smooth point on a plane curve C of degree $d \geq 3$. Then, there is a unique irreducible conic D_P with $I_P(C, D_P) \geq 5$. Such conic D_P is called $the\ osculating\ conic\ of$ C at P.

Definition 1(Cf.[1]). *A smooth, but not a flex, point P on a plane curve C is called a* $sextactic\ point$ *if the osculating conic D_P meets C at P with contact order at least six. Furthermore, a sextactic point P is called* $s\text{-}sextactic$, *if $s = I_P(C, D_P) - 5$. This positive integer s is called* $the\ sextact\ multiplicity$ *of C at P.*

Definition 2. *A sextactic point P on a plane curve C of degree $d \geq 3$ is said to be total sextactic point if the osculating conic D_P of C at P meets C only at P, i.e., if $I_P(C, D_P) = 2d$.*

Historically, the term sextactic points have been introduced by Cayley around 1859 in [2]. Cayley remarked that sextactic points has been studied before him by Plüker and Steiner without giving concrete references. He is certainly referring to papers in Crelle's Journal 32 (1847) by Plüker. One can add a paper by Hesse in volume 36 (1848) of the same journal. In all of these papers it is claimed that there are 27 sextactic points on a cubic and clearly all of them are total sextactic points. In [3], Cayley proved that a curve with ordinary flex points (1-flex points) has exactly $3d(4d-9)$ sextactic points counted with multiplicities. In [6], Thorbergsson and Umehara showed that, if C is a curve of degree d and has k flex points with multiplicities $\mu_1, ..., \mu_k$, then C has $3d(5d-11) - \sum_{i=1}^{k}(4\mu_i - 3)$ sextactic points counted up to their multiplicities.

2. Smooth plane quartics

Let C be a smooth plane quartic curve and P be a sextactic point on C. Then either P is a total sextactic point

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(3-sextactic point) or hypersextactic point (2-sextactic point) or ordinary sextactic point (1-sextactic point).

*Remark.*It is well known that sextactic points on C are nothing but 2-Weierstrass points. Geometrically, P is a 2-Weierstrass point if and only if there is a unique conic D_P with $I_P\left(C, D_P\right) \geq 6$. It turns out that either $D_P = 2T_P$ (P is a flex and T_P is the tangent line at P), or D_P is an irreducible conic (P is a sextactic point and D_P is the osculating conic at P). For more details see [1].

Let $d \in \mathbb{Z}^+$ be given and put $N = \frac{1}{2}d(d+3)$. Identify the homogenous forms of degree d in $\mathbb{C}[X,Y,Z]$ with the points $\mathbb{P}^N(\mathbb{C})$. Let, under this identification $\triangle \subset \mathbb{P}^N(\mathbb{C})$ be the closed subvariety corresponding to the forms which define singular plane curves of degree d. Hence there exists for each $d \geq 3$ a morphism

$$\phi : \mathbb{P}^N(\mathbb{C}) - \triangle \longrightarrow \mathcal{M}_g,$$

where $N = \frac{1}{2}d(d+3)$, $g = \frac{1}{2}(d-1)(d-2)$. Assigning to a smooth plane curve of degree d its moduli point. We remark that

$$\phi(\mathbb{P}^N(\mathbb{C}) - \triangle) \cap \mathcal{H}_g = \emptyset,$$

where $\mathcal{H}_g \subset \mathcal{M}_g$ is the hyperelliptic locus. If $g = 3$, there is the following well known result.

Proposition 1(Cf.[7]). *The morphism*

$$\phi : \mathbb{P}^{14}(\mathbb{C}) - \triangle \longrightarrow \mathcal{M}_3 - \mathcal{H}_3$$

is surjective. Moreover it is closed.

*Proof.*It is surjective since the canonical morphism embeds a smooth nonhyperelliptic curves of genus $g = 3$ in $\mathbb{P}^2(\mathbb{C})$ as a curve of degree $d = 4$. It is closed because ϕ establishes in fact an isomorphism

$$\left(\mathbb{P}^{14}(\mathbb{C}) - \triangle\right)/PGL(3;\mathbb{C}) \xrightarrow{\sim} \mathcal{M}_3 - \mathcal{H}_3.$$

We define

$$\mathcal{S}_r = \{m(C) \in \mathcal{M}_3 - \mathcal{H}_3 : C \text{ is a smooth plane quartic curve with at least one } r\text{-sextactic point}\},$$

where $r \in \{1,2,3\}$. The purpose of this paper is to prove the following:

Theorem 1.*The set $\mathcal{S}_r \subseteq \mathcal{M}_3$ of moduli points of smooth plane quartic curves (nonhyperelliptic curves of genus 3) having at least one sextactic point of sextact multiplicity r, where $r \in \{1,2,3\}$, is an irreducible, closed and rational subvariety of codimensional $r-1$ of $\mathcal{M}_3 - \mathcal{H}_3$ (where $\mathcal{H}_3 \subset \mathcal{M}_3$ is the hyperelliptic locus).*

In the sequel, the triple (P, T_PC, D_P) denotes to a sextactic point P on a smooth plane quartic C with its associated osculating conic $D_P : Q(X,Y,Z) = 0$ and $T_PC : \ell(X,Y,Z) = 0$ is the common tangent to C and D_P at P.

3. Total sextactic point

We now study

$$\mathcal{S}_3 = \{m(C) \in \mathcal{M}_3 - \mathcal{H}_3 : C \text{ is a smooth plane quartic curve with a total sextactic point}\}$$

Lemma 1.*A smooth plane quartic curve C has at least one total sextactic point (P, T_PC, D_P) if and only if its defining equation $F(X,Y,Z) = 0$ is given by, up to scalar multiple,*

$$F(X,Y,Z) = \alpha\ell^4 + Q(X,Y,Z)\psi(X,Y,Z), \quad (1)$$

where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\psi(X,Y,Z)$ is a complex quadratic homogenous form.

*Proof.*Suppose that the defining equation of C is given by (1). The contact order of the irreducible conic D_P, whose defining equation is $Q(X,Y,Z) = 0$, and C at the point P is given by

$$\begin{aligned} I_P(F,Q) &= I_P(\alpha\ell^4 + Q\psi, Q) \\ &= I_P(\ell^4, Q) \\ &= 4I_P(\ell, Q) = 8. \end{aligned}$$

Then (P, T_PC, D_P) is a total sextactic point.

Conversely, Let C be a smooth plane quartic curve has a total sextactic point (P, T_PC, D_P). Without loss of generality, we may assume that $P = [0:0:1]$, $T_PC : X = 0$ and $D_P : Y^2 = XZ$ (any smooth projective plane conic is isomorphic to $Y^2 = XZ$, see for example [4]). It sufficient to prove the statement in the open set where $Z \neq 0$; other open sets the argument is similar. Here C is isomorphic to the affine plane curve defined by $f(X,Y) = 0$, where $f(X,Y) = F(X,Y,1)$; moreover D_P defined by $Y^2 = X$. Since P is a total sextactic point, then $f(Y^2,Y) = \alpha Y^8$, for some constant $\alpha \neq 0$. Now consider the polynomial $g(X,Y) = f(X,Y) - \alpha X^4$, then $g(Y^2,Y) = f(Y^2,Y) - \alpha Y^8 = 0$, consequently the conic $Y^2 - X$ is a factor of $g(X,Y)$. Therefore

$$f(X,Y) = \alpha X^4 + (Y^2 - X)\psi(X,Y),$$

for some complex quadratic polynomial $\psi(X,Y)$. The homogenization of the previous equation is

$$F(X,Y,Z) = \alpha X^4 + (Y^2 - XZ)\psi(X,Y,Z). \quad (2)$$

Example 1(Cf [1]). Consider the smooth plane quartic

$$C : X^4 + Y^4 + Z^4 + 14(X^2Y^2 + Y^2Z^2 + X^2Z^2) = 0.$$

The two points $P_1 = [\omega : \omega^2 : 1]$ and $P_2 = [\omega^2 : \omega : 1]$, where $\omega = \exp(2\pi\sqrt{-1}/3)$, are total sextactic points on C and lie on a bitangent line $L : X + Y + Z = 0$. The osculating conics at these points are the following, respectively:

$$\begin{aligned} D_1 : Q_1(X,Y,Z) &= (X^2 + 5YZ) + \omega^2(Y^2 + 5XZ) \\ &\quad + \omega(Z^2 + 5XY) = 0, \\ D_2 : Q_2(X,Y,Z) &= (X^2 + 5YZ) + \omega(Y^2 + 5XZ) \\ &\quad + \omega^2(Z^2 + 5XY) = 0. \end{aligned}$$

Note that we can write the defining equation of C as

$$C : \frac{5}{9}(X+Y+Z)^4 + \frac{4}{9}Q_1(X,Y,Z)Q_2(X,Y,Z) = 0.$$

Lemma 2. *Let V_t be the subspace of $\mathbb{P}^{14}(\mathbb{C}) - \triangle$ such that its points corresponding to the forms which define smooth plane quartic curves having at least one total sextactic point (P, T_PC, D_P). Then the group of automorphisms G_t of V_t is given by*

$$G_t = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ b^2 & 2ab & a^2 \end{pmatrix} \in PGL(3;\mathbb{C}) \right\}$$

wlere $a, b \in \mathbb{C}$, $a \neq 0$.

Proof. Using Lemma 1, we can assume that

$$V_t = \{F(X,Y,Z) \in \mathbb{P}^{14}(\mathbb{C}) - \triangle : F(X,Y,Z) \text{ as in (2)}\}.$$

Then each $g \in G_t$ must fix $P = [0:0:1]$, $T_PC : X = 0$ and $D_P : Y^2 = XZ$.

Now, it is easy to prove the following proposition.

Proposition 2. *The set $\mathcal{S}_3$ is an irreducible, closed and rational subvariety of codimensional two of $\mathcal{M}_3 - \mathcal{H}_3$.*

Proof. Let $C : F(X,Y,Z) = 0$ be a smooth plane quartic curve has a total sextactic point (P, T_PC, D_P), then $F(X,Y,Z) \in V_t$. But V_t is a 6-dimensional irreducible closed subvariety of $\mathbb{P}^{14} - \triangle$. Hence $\mathcal{S}_3 = \phi(V_t)$ is irreducible and closed in $\mathcal{M}_3 - \mathcal{H}_3$. Its dimension equal 4, since each fiber of $\phi : V_t \longrightarrow \mathcal{S}_3$ has dimension $2 = \dim G_t$. Since G_t is triangular, then $\mathcal{S}_3$ is rational (see for example Theorem 1 in [5]).

4. Hypersextactic point

In this section, we study

$$\mathcal{S}_2 = \{m(C) \in \mathcal{M}_3 - \mathcal{H}_3 : C \text{ is a smooth plane quartic curve with a hypersextactic point}\}$$

Let C be a smooth plane quartic curve has a hypersextactic point (P, T_PC, D_P). In this case, the osculating conic D_P meets C transversely at another point differs from P; say R. Assume that the line $L : \ell_1(X,Y,Z) = 0$ joins P and R. We prove the following lemma:

Lemma 3. *A smooth plane quartic curve C has at least one hypersextactic point (P, T_PC, D_P) if and only if its defining equation $F(X,Y,Z) = 0$ is given by, up to scalar multiple,*

$$F(X,Y,Z) = \alpha \ell_1 \ell^3 + Q(X,Y,Z)\psi(X,Y,Z), \quad (3)$$

where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\psi(X,Y,Z)$ is a complex quadratic homogenous form.

Proof. Suppose that the defining equation of C is given by (3). The contact order of the irreducible conic D_P, whose defining equation is $Q(X,Y,Z) = 0$, and C at the point P is given by

$$\begin{aligned} I_P(F,Q) &= I_P(\alpha\ell_1\ell^3 + Q\psi, Q) \\ &= I_P(\ell_1, Q) + I_P(\ell^3, Q) \\ &= 1 + 3I_P(\ell, Q) = 7. \end{aligned}$$

Then (P, T_PC, D_P) is a hypersextactic point.
Conversely, Let C be a smooth plane quartic curve has a hypersextactic point (P, T_PC, D_P). Without loss of generality, we may assume that $P = [0:0:1]$, $T_PC : X = 0$ and $D_P : Y^2 = XZ$. Hence $L : Y = mX$ and $R = [1:m:m^2]$. It sufficient to prove the statement in the open set where $Z \neq 0$; other open sets have the same argument. Here C is isomorphic to the affine plane curve defined by $f(X,Y) = 0$, where $f(X,Y) = F(X,Y,1)$; moreover D_P defined by $Y^2 = X$. Since P is a hypersextactic point, then $f(Y^2, Y) = \alpha(1-mY)Y^7$, for some nonzero constant α. Now consider the polynomial $g(X,Y) = f(X,Y) - \alpha(Y-mX)X^3$, then $g(Y^2,Y) = f(Y^2,Y) - \alpha(1-mY)Y^7 = 0$, consequently the conic $Y^2 - X$ is a factor of $g(X,Y)$. Therefore

$$f(X,Y) = \alpha(Y-mX)X^3 + (Y^2 - X)\psi(X,Y),$$

for some complex quadratic polynomial $\psi(X,Y)$. The homogenization of the previous equation is

$$F(X,Y,Z) = \alpha(Y-mX)X^3 + (Y^2 - XZ)\psi(X,Y,Z). \quad (4)$$

Lemma 4. *Let V_h be the subspace of $\mathbb{P}^{14}(\mathbb{C}) - \triangle$ such that its points corresponding to the forms which define smooth plane quartic curves having at least one hypersextactic point (P, T_PC, D_P). Then the group of automorphisms G_h of V_h is given by*

$$G_h = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ m(1-a) & a & 0 \\ m^2(1-a)^2 & 2m(1-a)a & a^2 \end{pmatrix} \in PGL(3;\mathbb{C}) \right\}$$

where $a \in \mathbb{C}\backslash\{0\}$.

Proof. Using Lemma 3, we can assume that

$$V_h = \{F(X,Y,Z) \in \mathbb{P}^{14}(\mathbb{C}) - \triangle : F(X,Y,Z) \text{ as in (4)}\}.$$

Then each $g \in G_h$ must fix $P = [0:0:1]$, $T_PC : X = 0$, $D_P : Y^2 = XZ$ and $L : Y = mX$.

Now, it is easy to prove the following proposition.

Proposition 3. *The set $\mathcal{S}_2$ is an irreducible, closed and rational subvariety of codimensional one of $\mathcal{M}_3 - \mathcal{H}_3$.*

Proof. Follow the proof of Proposition 2, but note only that each fiber of $\phi : V_h \longrightarrow \mathcal{S}_2$ has dimension $1 = \dim G_h$.

5. Ordinary sextactic point

Finally, we study

$$\mathcal{S}_1 = \{m(C) \in \mathcal{M}_3 : C \text{ is a smooth plane quartic curve with an ordinary sextactic point}\}$$

Let C be a smooth plane quartic curve has an ordinary sextactic point (P, T_PC, D_P). In this case we have

$$D_P \cdot C = 6P + R_1 + R_2,$$

where R_1, R_2 are two points different from P but not necessarily distinct. Assume that the line $L_1 : \ell_1(X,Y,Z) = 0$ (resp. $L_2 : \ell_2(X,Y,Z) = 0$) joins P and R_1(resp. and R_2). We prove the following lemma:

Lemma 5.*A smooth plane quartic curve C has at least one ordinary sextactic point (P, T_PC, D_P) if and only if its defining equation $F(X,Y,Z) = 0$ is given by, up to scalar multiple,*

$$F(X,Y,Z) = \alpha\ell_1\ell_2\ell^2 + Q(X,Y,Z)\psi(X,Y,Z), \qquad (5)$$

where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\psi(X,Y,Z)$ is a complex quadratic homogenous form.

*Proof.*Suppose that the defining equation of C is given by (5). The contact order of the irreducible conic D_P, whose defining equation is $Q(X,Y,Z) = 0$, and C at the point P is given by

$$\begin{aligned} I_P(F,Q) &= I_P(\alpha\ell_1\ell_2\ell^2 + Q\psi, Q) \\ &= I_P(\ell_1,Q) + I_P(\ell_2,Q) + I_P(\ell^2,Q) \\ &= 1 + 1 + 2I_P(\ell,Q) = 6. \end{aligned}$$

Then (P, T_PC, D_P) is an ordinary sextactic point.
Conversely, Let C be a smooth plane quartic curve has an ordinary sextactic point (P, T_PC, D_P). Assume that $P = [0:0:1]$, $T_PC : X = 0$ and $D_P : Y^2 = XZ$. Hence $L_1 : Y = m_1X$ and $R_1 = [1 : m_1 : m_1^2]$ (resp. $L_2 : Y = m_2X$ and $R_2 = [1 : m_2 : m_2^2]$). It sufficient to prove the statement in the open set where $Z \neq 0$. Here C is isomorphic to the affine plane curve defined by $f(X,Y) = 0$, where $f(X,Y) = F(X,Y,1)$; moreover D_P defined by $Y^2 = X$. Since P is an ordinary sextactic point, then $f(Y^2,Y) = \alpha(1-m_1Y)(1-m_2Y)Y^6$, for some nonzero constant α. Now consider the polynomial $g(X,Y) = f(X,Y) - \alpha(Y-m_1X)(Y-m_2X)X^2$, then $g(Y^2,Y) = f(Y^2,Y) - \alpha(1-m_1Y)(1-m_2Y)Y^6 = 0$, consequently the conic $Y^2 - X$ is a factor of $g(X,Y)$. Therefore

$$f(X,Y) = \alpha(Y-m_1X)(Y-m_2X)X^2 + (Y^2-X)\psi(X,Y),$$

for some complex quadratic polynomial $\psi(X,Y)$. The homogenization of the previous equation is

$$\begin{aligned} F(X,Y,Z) = {}&\alpha(Y-m_1X)(Y-m_2X)X^2 \\ &+ (Y^2 - XZ)\psi(X,Y,Z). \end{aligned} \qquad (6)$$

Lemma 6.*Let V_o be the subspace of $\mathbb{P}^{14}(\mathbb{C}) - \triangle$ such that its points corresponding to the forms which define smooth plane quartic curves having at least one ordinary sextactic point (P, T_PC, D_P). Then the group of automorphisms G_o of V_o is the trivial subgroup which contains only the identity matrix.*

*Proof.*Using Lemma 5, we can assume that

$$V_o = \{F(X,Y,Z) \in \mathbb{P}^{14}(\mathbb{C}) - \triangle : F(X,Y,Z) \text{ as in } (6)\}.$$

Then each $g \in G_o$ must fix $P = [0:0:1]$, $T_PC : X = 0$, $D_P : Y^2 = XZ$, $L_1 : Y = m_1X$ and $L_2 : Y = m_2X$.

Now, it is easy to prove the following proposition.

Proposition 4.*The set $\mathcal{S}_1$ is an irreducible, closed and rational subvariety of codimensional zero of $\mathcal{M}_3 - \mathcal{H}_3$.*

*Proof.*Note only that each fiber of $\phi : V_o \longrightarrow \mathcal{S}_1$ has dimension $0 = \dim G_o$ and then follow the proof of Proposition 2.

*Remark.*Proposition 4 tells us that there is no a smooth plane quartic curve all its sextactic points of higher multiplicity.

Putting all together, we proved our main Theorem 1.

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