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Auxiliary Principle Technique for Solving Split Feasibility Problems

Muhammad Aslam Noor and Khalida Inayat Noor

Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan

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Abstract: Let K and C be nonempty, closed and convex sets in R^n and R^m respectively and A be an $m \times n$ real matrix. The split feasibility problem is to find $u \in K$ with $Au \in C$. Many problems arising in the image reconstruction can be formulated in this form. In this paper, we use the auxiliary principle technique to suggest and analyze some new iterative algorithms for solving the split feasibility problems. Our new algorithms include the previously known ones as special cases. We also study the convergence criteria of these algorithms under some weaker conditions. In this respect, our results present a refinement and improvement of the previously known results.

Keywords: Variational inequalities, split feasibility problems, convergence, iterative methods, auxiliary principle technique.

1. Introduction

In recent years much attention has been given to study the split feasibility problems, which arise in diverse fields of pure and applied sciences including image reconstruction, medical sciences (medical image) and signal processing. Many iterative projection-type algorithms have been proposed and analyzed for solving split feasibility problems, see Byrne [2,3], Yang [18,19], and the references therein. To implement these algorithms, one has to find the projection on the closed convex sets, which is not possible except in simple cases. We would like to mention that these problems can be studied by the variational inequalities approach. In fact, we have shown that the split feasibility problems are equivalent to the variational inequalities. This alternative approach is more flexible and allows to improve the convergence analysis of these iterative-type projection algorithms. In this paper, we use the auxiliary principle technique. This technique deals with finding the auxiliary variational inequality problem and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed point approach. This technique has been used to suggest and analyze several iterative methods for solving variational inequalities and related optimization problems, see [8-16,20] and the references therein. It is known that a substantial number of numerical methods can be obtained as special cases from this technique. The proposed algorithms include the projection-type algorithms of Byrne [2,3] and others as special cases. We also introduce the concept of the weakly relaxed monotonicity strongly monotonicity, which is a weaker condition than co-coercivity (inverse strongly monotonicity). We study the convergence criteria under this condition. This clearly improve the convergence of the previously known algorithms. In fact, we have shown that the variational inequality approach is more flexible and provides a natural and unified framework to suggest and analyze iterative methods for solving split feasibility problems. The main purpose of this paper is to demonstrate the close connection between the split feasibility problems and variational inequalities. This unified framework is of important and significant value, both as a means of summarizing existing techniques and to provide ideas and tools for explaining relationship and performing convergence analysis. The unified framework also allows a cross-fertilization among different areas where both the theory and computational techniques have been applied. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovative and potential applications in these areas. We have only given a very brief glimps of these fast growing fields. The interested reader is advised to explore these fields further

* Corresponding author: e-mail: noormaslam@hotmail.com, khalidanoor@hotmail.com

and discover novel and fascinating applications of these problems in other areas of pure and applied sciences.

2. Preliminaries

Let K and C be two nonempty, closed and convex sets in R^n and R^m respectively. Let A be an $n \times m$ be a matrix. The inner product and norm are denoted by $\langle ., . \rangle$. and $\|.\|$ respectively.

We consider the problem of finding $u \in K$ such that $Au \in C$ which is known as the split feasibility problem. It is well known that such type of problems arise in the image reconstruction and have applications in medical image and signal processing. It is known [2,3] that these problems are equivalent to finding $u \in K$ such that

$$u = P_K[u - \rho A^T(I - C_P)Au], \tag{1}$$

where P_K and C_P are projections of R^n and R^m on the closed convex sets K and C respectively. Here A^T denotes the transpose of the matrix A.

Related to the split feasibility problems, we consider another problem, which is known as the variational inequality problem. To be more precise, let K be a closed convex set in R^n. We consider the problem of finding $u \in K$ such that

$$\langle A^T(I - C_P)Au, v - u\rangle \geq 0, \quad \forall v \in K. \tag{2}$$

Problems of the type (2.2) are known as variational inequalities, which were introduced and studied by Stampacchia [17] in 1964. It can be shown that the minimum of a function

$$F(u) = \frac{1}{2}\|C_P Au - Au\|^2$$

on the convex set K can be characterized by the variational inequality of the type (2.2) with

$$F'(u) = A^T(I - C_P)Au, \tag{3}$$

where $F'(u)$ is the differential of the differentiable convex function $F(u)$ at $u \in K$. For the recent applications, generalizations, sensitivity analysis, dynamical systems and numerical methods of the variational inequalities, see [1, 3-20] and the references therein.

We now recall some well known results and concepts, which are needed.

Definition 2.1. The operator $T : K \longrightarrow R^n$ is said to be:

(i). *strongly monotone,* if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \quad \forall u, v \in K.$$

(ii). *co-coercive (inverse strongly monotone),* if there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \mu\|Tu - Tv\|^2, \quad \forall u, v \in K.$$

(iii). *weakly co-coercive,* if there exists a continuous function $g(u, v) > 0, \forall u, v \in K$ such that

$$\langle Tu - Tv, u - v\rangle \geq g(u, v)\|Tu - Tv\|^2, \quad \forall u, v \in K.$$

(iv). *partially relaxed strongly monotone,* if there exists a constant $\alpha_1 > 0$ such that

$$\langle Tu - Tv, z - v\rangle \geq -\alpha_1\|u - z\|^2, \quad \forall u, v, z \in K.$$

(v). *weakly partially relaxed strongly monotone,* if there exists a continuous function $g(u, v) > 0, \forall u, v \in K$ such that

$$\langle Tu - Tv, z - v\rangle \geq -g(u, v)\|u - z\|^2, \quad \forall u, v, z \in K.$$

(vi). *monotone,* if

$$\langle Tu - Tv, u - v\rangle \quad \geq 0, \quad \forall u, v \in K.$$

(vii). *pseudomonotone,* if

$$\langle Tu, v - u\rangle \geq 0 \quad \text{implies} \quad \langle Tv, v - u\rangle \geq 0, \quad \forall u, v \in K.$$

(viii). *Lipschitz continuous,* if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \quad \forall u, v \in K.$$

In particular, we note that, if $z = u$, then (weakly) partially relaxed strongly monotone operator is monotone. If $g(u, v)$ is a constant or has a minimum, then obviously weakly co-coercive and weakly partially relaxed strongly monotone operators are co-coercive and partially relaxed strongly monotone. However, if the convex set K is unbounded and $g(u, v)$ tends to zero as $\|u\|$ and $\|v\|$ approaches infinity, then T is neither co-coercive nor partially relaxed strongly monotone. This implies that weakly co-coercive and weakly partially relaxed strongly monotone operator T is not co-coercive and partially relaxed strongly monotone. Thus it is clear that weakly co-coercive and weakly partially relaxed strongly monotone are weaker conditions than co-coercivity and partially relaxed strongly monotonicity.

We now show that weakly co-coercivity implies weakly partially relaxed strongly monotonicity and this is the motivation of our next result.

Lemma 2.1. If T is weakly co-coercive with a continuous function g verifying $g(u, v) > 0, \quad \forall u, v \in K$, then T is weakly partially relaxed strongly monotone operator with $\frac{1}{4g(u,v)}$.

Proof. $\forall u, v, z \in H$, consider

$$\begin{aligned}
&\langle Tu - Tv, z - v\rangle \\
&= \langle Tu - Tv, u - v\rangle + \langle Tu - Tv, z - u\rangle \\
&\geq g(u, v)\|Tu - Tv\|^2 - g(u, v)\|Tu - Tv\|^2 \\
&\quad - \frac{1}{4g(u, v)}\|z - u\|^2 \\
&\geq \frac{-1}{4g(u, v)}\|z - u\|^2,
\end{aligned}$$

which shows that T is partially relaxed strongly monotone with $\frac{1}{4g(u,v)}$. □

For a constant function $g(u, v) = \eta$, we see that a co-coercive operator with a constant η is also a partially relaxed strongly monotone with a constant $\frac{1}{4\eta}$. This result is due to Noor [12-14].

Example 2.1. Consider the variational inequality problem (2.2) with $Tu = 1 - e^{-u}, \quad K = (-\infty, \infty)$. Then one can easily show that the operator T is weakly co-coercive. However this operator T is neither strongly monotone nor co-coercive, see [19].

We also need the following result, which establishes the relationship between the Lipschitz and co-coercivity properties of the differential operator $F'(u)$(the differential of a convex function $F(u)$.)

Lemma 2.2 [1]. For a differential operator F'(the differential of a convex function F), the following are equivalent.

(a). $\|F'(u) - F'(v)\| \leq L\|u - v\|, \quad \forall u, v \in R^n.$

(b). $\langle F'(u)-F'(v), u-v\rangle \geq \frac{1}{L}\|F'(u)-F'(v)\|^2, \quad \forall u, v \in R^n,$

where L is the Lipschitz constant of the operator F'.

From lemma 2.1 and Lemma 2.2, it follows that
Lipschitz continuity $\Longrightarrow$ co-coercivity
$\Longrightarrow$ weakly partially relaxed strongly monotonicity.

Lemma 2.3. For a given $z \in R^n, u \in K$ verifies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$u = P_K z,$$

where P_K is the projection of R^n onto the closed convex set K.

It is well known that the projection operator P_K is nonexpansive and firmly nonexpansive (co-coercive with a constant $\mu = 1$,) that is,

$$\|P_K u - P_K v\| \leq \|u - v\|, \quad \forall u, v \in R^n,$$

and

$$\langle P_K u - P_K v, u - v\rangle \geq \|P_K u - P_K v\|^2, \quad \forall u, v \in R^n.$$

One can easily show that the projection operator P is firmly nonexpansive if and only if its complement $I - P$ is firmly nonexpansive.

We also need the following, which is essentially due to Byrne [3]. We include its proof for the sake of completeness and to convey an idea.

Lemma 2.4. The differential operator $F'(u)$ defined by (2.3) is Lipschitz continuous with constant L, where L is the largest eigenvalue of $A^T A$. v

Proof. $\forall u, v \in R^n$, we have

$$\begin{aligned}
&\|F'(u) - F'(v)\|^2 \\
&= \|A^T(I - C_P)Au - A^T(I - C_P)Av\|^2 \\
&\leq L\|(I - C_P)Au - (I - C_P)Av\|^2 \\
&= L\{\|Au - Av\|^2 + \|C_P Au - C_P Av\|^2 \\
&\quad -2\langle C_P Au - C_P Av, Au - Av\rangle\} \\
&\leq L\{\|Au - Av\|^2 - \|C_P Au - C_P Av\|^2\} \\
&\leq L\|Au - Av\|^2 \leq L^2\|u - v\|^2,
\end{aligned}$$

from which the required result follows. □

Now we show that the problem (2.1) and problem (2.2) are equivalent by using Lemma 2.4. This is a well known result in variational inequalities theory.

Lemma 2.5. The function $u \in K$ is a solution of the variational inequality (2.2) if and only if $u \in K$ satisfies

$$u = P_K[u - \rho A^T(I - C_P)Au].$$

Here $\rho > 0$ is a constant and C_P is the projection of R^m onto the closed convex set C.

Lemma 2.5 implies that the split feasibility problem (2.1) and variational inequality problem (2.2) are equivalent. This alternative equivalent formulation has played an important and crucial part in the development of several projection-type iterative algorithms for solving variational inequalities problems and related optimization problems.

Lemma 2.6. $\forall u, v \in R^n$,

$$2\langle u, v\rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2. \tag{4}$$

Definition 2.2. A function f on a convex set K is said to be strongly convex function, if there exists a constant $\mu > 0$ such that

$$\begin{aligned}
&f((1 - t)u + tv) \leq (1 - t)f(u) + tf(v) \\
&-\mu t(1 - t)\|u - v\|^2, \forall u, v \in K, \quad t \in [0, 1].
\end{aligned}$$

For differentiable strongly convex functions, we have the following result.

Lemma 2.7. Let f be a differentiable function on the convex set K. Then the following are equivalent:
(i). f is strongly convex on the convex set K with a modulus $\mu > 0$.

(ii). $f(v) - f(u) \geq \langle f'(u), v - u\rangle + \mu\|u - v\|^2,$
$\forall u, v \in K.$

(iii). $\langle f'(u) - f'(v), u - v\rangle \geq 2\mu\|u - v\|^2,$
$\forall u, v \in K,$

that is, f' is strongly monotone with a constant $\mu > 0$.

3. Main Results

In this section, we use the auxiliary principle technique in conjunction with variational inequalities to suggest and analyze some iterative algorithms for solving split feasibility problems (2.1).

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\langle \rho A^T(I - C_P)Au + w - u, v - w\rangle \geq 0, \quad \forall v \in K, \tag{1}$$

where $\rho > 0$ is a constant. Note that if $w = u$, then w is a solution of the variational inequality (2.2). This fact allows us to suggest and analyze the following iterative algorithm for solving variational inequalities (2.2).

Algorithm 3.1. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$\langle \rho A^T(I - C_P)Au_n + u_{n+1} - u_n, \\ v - u_{n+1}\rangle \geq 0, \forall v \in K. \quad (2)$$

Using the projection operator technique Algorithm 3.1 can be written in the following equivalent form

Algorithm 3.2. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho A^T(I - C_P)Au_n], \quad n = 0, 1, \ldots,$$

Algorithm 3.2 is exactly the same Algorithm as suggested and analyzed in [2,3].

Note that, if $C = \{b\}$, then Algorithm 3.2 collapses to:

Algorithm 3.3. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative schemes

$$u_{n+1} = P_K[u_n - \rho A^T(Au_n - b)], \quad n = 0, 1, 2, \ldots,$$

which is known as the projected Landweber Algorithm, see [3].

We now study the convergence analysis of Algorithm 3.1.

Theorem 3.1. Let the operator $A^T(I - C_P)A$ be weakly partially relaxed strongly monotone with a continuous function $g(u_n, u)$. If $u \in K$ is a solution of (2.2) and u_{n+1} is the approximate solution obtained from Algorithm 3.1, then

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 \\ -(1 - 2\rho g(u_n, u))\|u_n - u_{n+1}\|^2. \quad (3)$$

Proof. Let $u \in K$ be a solution of (2.2). Then

$$\langle A^T(I - C_P)Au, v - u\rangle \geq 0, \quad \forall v \in K. \quad (4)$$

Now taking $v = u$ in (3.2) and $v = u_{n+1}$ in (3.4), we have

$$\langle A^T(I - C_P)Au, u_{n+1} - u\rangle \geq 0, \quad (5)$$

and

$$\langle \rho A^T(I - C_P)Au_n + u_{n+1} - u_n, u - u_{n+1}\rangle \geq 0. \quad (6)$$

Adding (3.5) and (3.6), we have

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \\ \geq \rho\{\langle A^T(I - C_P)Au_n - A^T(I - C_P)Au, u_{n+1} - u\rangle\} \\ \geq -\rho g(u_n, u)\|u_n - u_{n+1}\|^2, \quad (7)$$

where we have used the fact that $A^T(I - C_P)A$ is weakly partially strongly monotone with $g(u_n, u)$.

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (2.4), we have

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 \\ -\|u - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \quad (8)$$

Combining (3.7) and (3.8), we obtain (3.3), the required result. □

Theorem 3.2. Let $u \in K$ be a solution of (2.2) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $0 < \rho < \frac{1}{2g(u_n,u)}$, then

$$\lim_{n \longrightarrow \infty} \{u_n\} = u.$$

Proof. Let $u \in$ be a solution of (2.2). Since $0 < \rho < \frac{1}{2g(u_n,u)}$, it follows from (3.3) that the sequence $\{\|u - u_n\|\}$ is decreasing and consequently $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty}(1 - 2\rho g(u_n, u))\|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which implies

$$\lim_{n \longrightarrow \infty} \|u_n - u_{n+1}\| = 0. \quad (9)$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K$. replacing u_n by u_{n_j} in (3.2) and taking the limit $n_j \longrightarrow \infty$ and using (3.9), we have

$$\langle A^T(I - C_P)A\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall \quad v \in K,$$

which implies that $\hat{u}$ solves the variational inequality (2.2) and

$$\|u_n - u_{n+1}\|^2 \leq \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n \longrightarrow \infty} u_n = \hat{u}$. the required result. □

We now again use the auxiliary principle technique to suggest some proximal point algorithms for solving the variational inequalities (2.2) and split feasibility problems (2.1). These methods have been used and refined in recent years, see [7,11,12].

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\langle \rho A^T(I - P_C)Aw + w - u - \alpha(u - u), \\ v - w\rangle \geq 0 \forall v \in K, \quad (10)$$

which is known the auxiliary variational inequality associated with the variational inequality (2.2). Here $\alpha > 0$ is a parameter. Note that problems (3.1) and (3.10) are quite different problems. It is clear that, if $w = u$, then w is a solution of the variational inequality (2.2). This fact allows us to suggest the following iterative method for solving the variational inequality (2.2).

Algorithm 3.4. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$\langle \rho A^T(I - P_C)Au_{n+1} + u_{n+1} - u_n \\ -\alpha_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \quad (11)$$

where $\alpha_n > 0$ a constant. Algorithm 3.4 known as the inertial proximal method and can be written as

Algorithm 3.5. For a given $u_0 \in K$, compute the approximate solution $u_{n+1} \in K$ by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho A^T(I - C_P)Au_{n+1} \\ +\alpha_n(u_n - u_{n-1})], \quad n = 1, 2, \ldots,$$

For $\alpha_n = 0$, we obtain the original proximal method for solving variational inequalities. For the improved convergence analysis and applications of the proximal point algorithms, see [7,11-14] and the references therein.

We now consider the convergence analysis of Algorithm 3.4 using the technique of Theorem 3.1 and Theorem 3.2.

Theorem 3.3. Let $u \in K$ be a solution of the variational inequality (2.2) and let u_{n+1} be the approximate solution obtained from Algorithm 3.4. If $A^T(I - C_P)A$ is pseudomonotone, then

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 + \alpha_n\{\|u_n - u\|^2 - \|u - u_{n-1}\|^2 + 2\|u_n - u_{n-1}\|^2\} - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2. \quad (12)$$

Proof. Let $u \in K$ be a solution of (2.2). Then

$$\langle A^T(I - C_P)Au, v - u\rangle \geq 0, \quad \forall v \in K,$$

which implies that

$$\langle A^T(I - C_P)Av, v - u\rangle \geq 0, \quad \forall v \in K, \quad (13)$$

since $A^T(I - C_P)A$ is pseudomonotone.

Taking $v = u_{n+1}$ in (3.13) and $v = u$ in (3.11), we have

$$\langle A^T(I - C_P)Au_{n+1}, u_{n+1} - u\rangle \geq 0. \quad (14)$$

and

$$\langle \rho A^T(I - C_P)Au_{n+1} + u_{n+1} - u_n - \alpha_n\{u_n - u_{n-1}\}, u - u_{n+1}\rangle \geq 0. \quad (15)$$

Adding (3.14) and (3.15), we have

$$\langle u_{n+1} - u_n - \alpha_n\{u_n - u_{n-1}\}, u - u_{n+1}\rangle \geq 0,$$

which can be written as

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq \alpha_n\langle u_n - u_{n-1}, u - u_n + u_n - u_{n+1}\rangle. \quad (16)$$

Using Lemma 2.4 and rearranging the terms of (3.16), one can easily obtain the required result. □

Theorem 3.4. Let $u \in K$ be a solution of (2.2) and let u_{n+1} be the approximate solution obtained from algorithm 3.4. If there exist a $\alpha \in [0, 1)$ such that $0 \leq \alpha_n \leq \alpha$, for all $n \in N$ and

$$\sum_{n=1}^{\infty} \alpha_n\|u_n - u_{n-1}\|^2 \leq \infty,$$

then

$$\lim_{n\to\infty} u_n = u.$$

Proof. Its proof is similar to that of Theorem 3.2. See also [12]. □

We now suggest and analyze some more iterative methods using the auxiliary principle technique in conjunction with the Bregman function.

For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\langle \rho A^T(I - C_P)Au + E'(w) - E'(u), v - w\rangle \geq 0, \quad \forall v \in K, \quad (17)$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a differentiable strongly convex function $E(u)$ at $u \in K$. Due to the differentiable strongly convex function $E(u)$, problem (3.17) has a unique solution. Note that for $w = u$, w is a solution of the problem (2.2). This fact allows us to suggest the following algorithm for solving the variational inequality (2.2).

Algorithm 3.6. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho A^T(I - C_P)Au_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq \forall v \in K, \quad (18)$$

For $K = R^n$, and $A^T(I - C_P)Au = \bigtriangledown f(u)$, the differential of f at u, Algorithm 3.6 reduces to:

Algorithm 3.7. For a given $u_0 \in R^n$, find the approximate solution u_{n+1} by the iterative scheme

$$E'(u_{n+1}) = E'(u_n) - \rho \bigtriangledown f(u_n), \quad n = 0, 1, \ldots$$

which is known as the interior point algorithms. For the applications of the (projected) interior point algorithms in medical image, see [2,3].

Remark 3.1. The function $B(w, u) = E(w) - E(u) - \langle E'(u), w - u\rangle$ associated with the differentiable convex function $E(u)$ is known as the Bregman function, which plays a key part in the convergence analysis of Algorithms suggested by using the auxiliary principle technique, see [4,20].

We now consider the convergence analysis of Algorithm 3.6 and this is the main motivation of next result.

Theorem 3.5. Let $A^T(I - C_P)A$ be a weakly partially relaxed strongly monotone operator with a continuous function $g(u_n, u)$. Let $E(u)$ be a differentiable strongly convex function with modulus $\beta > 0$. If $0 < \rho < \frac{g(u_n,u)}{\beta}$, then the approximate solution u_{n+1} obtained from Algorithm 3.6 converges to a solution of the problem (2.2)(or 2.1).

Proof. Let $u \in K$ be a solution of (2.2). Then

$$\langle A^T(I - C_P)Au, v - u\rangle \geq 0, \quad \forall v \in K. \quad (19)$$

Taking $v = u_{n+1}$ in (3.19) and $v = u$ in (3.18), we have

$$\langle A^T(I - C_P)Au, u_{n+1} - u\rangle \geq 0. \quad (20)$$

and

$$\langle \rho A^T(I - C_P)Au_n + E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geq 0. \quad (21)$$

We now consider the function

$$B(u, w) = E(u) - E(w) - \langle E'(w), u - w\rangle \geq \beta\|u - w\|^2, \quad (22)$$

where we have used the fact that $E(u)$ is a differentiable strongly convex function with modulus $\beta > 0$.

Combining (3.20), (3.21) and (3.22), we have

$$\begin{aligned}
&B(u,u_n)-B(u,u_{n+1})\\
&=E(u_{n+1})-E(u_n)-\langle E'(u_n),u-u_n\rangle\\
&\quad\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle\\
&\geq \beta\|u_{n+1}-u_n\|^2+\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle\\
&\geq \beta\|u_{n+1}-u_n\|^2\\
&\quad+\rho\langle A^T(I-C_P)Au_n-A^T(I-C_P)Au,u_{n+1}-u\rangle\\
&\geq \{\beta-\rho g(u_n,u)\}\|u_{n+1}-u_n\|^2,
\end{aligned}$$

where we have used the fact that $A^T(I-C_P)A$ is weakly partially relaxed strongly monotone with a continuous function $g(u_n,u)$.

If $u_{n+1}=u_n$, then clearly u_n is a solution of the variational inequality problem (2.2). Otherwise, for $0<\rho<\frac{\beta}{g(u_n,u)}$, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the variational inequality problem (2.2). □

We again use the auxiliary principle technique to suggest the proximal method for solving the variational inequality (2.2).

For a given $u\in K$, consider the problem of finding $w\in K$ such that

$$\langle\rho A^T(I-C_P)Aw+E'(w)-E'(u),v-w\rangle\geq 0,\forall v\in K,\quad (23)$$

where $\rho>0$ is a constant and $E'(u)$ is the differential of a differentiable strongly convex function $E(u)$ at $u\in K$. Due to the differentiable strongly convex function $E(u)$, problem (3.1) has a unique solution. Note that for $w=u$, w is a solution of the problem (2.2). Note that the problems (3.17) and (3.23) are quite different. This fact allows us to suggest the following algorithm for solving the variational inequality (2.2).

Algorithm 3.8. For a given $u_0\in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}
&\langle\rho A^T(I-C_P)Au_{n+1}+E'(u_{n+1})-E'(u_n),\\
&v-u_{n+1}\rangle\geq\forall v\in K,
\end{aligned}\quad (24)$$

For $K=R^n$, and $A^T(I-C_P)Au=\bigtriangledown f(u)$, the differential of f at u, Algorithm 3.8 reduces to:

Algorithm 3.9. For a given $u_0\in R^n$, find the approximate solution u_{n+1} by the iterative scheme

$$E'(u_{n+1})=E'(u_n)-\rho\bigtriangledown f(u_{n+1}),\quad n=0,1,\ldots$$

which is known as the implicit interior point algorithms and appears to be a new one.

One can study the convergence analysis of Algorithm 3.8 using the technique of Theorem 3.5. However for the sake of completeness and to convey an idea of the techniques involved, we give its proof.

Theorem 3.6. Let $A^T(I-C_P)A$ be a weakly partially relaxed strongly monotone operator with $g(.,.)$. Let $E(u)$ be a differentiable strongly convex function with modulus $\beta>0$. Then the approximate solution u_{n+1} obtained from Algorithm 3.8 converges to a solution $u\in K$ of the problem (2.2).

Proof. Let $u\in K$ be a solution of (2.2). Then

$$\langle A^T(I-C_P)Av,v-u\rangle\geq 0,\quad\forall v\in K,\quad (25)$$

since $A^T(I-C_P)A$ is pseudomonotone.

Taking $v=u_{n+1}$ in (3.25) and $v=u$ in (3.24), we have

$$\langle A^T(I-C_P)Au_{n+1},u_{n+1}-u\rangle\geq 0.\quad (26)$$

and

$$\begin{aligned}
&\langle\rho A^T(I-C_P)Au_{n+1}+E'(u_{n+1})-E'(u_n),\\
&u-u_{n+1}\rangle\geq 0.
\end{aligned}\quad (27)$$

Combining (3.22), (3.26) and (3.27), we have

$$\begin{aligned}
&B(u,u_n)-B(u,u_{n+1})\\
&=E(u_{n+1})-E(u_n)-\langle E'(u_n),u-u_n\rangle\\
&\quad\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle\\
&\geq \beta\|u_{n+1}-u_n\|^2+\langle E'(u_{n+1})-E'(u_n),u-u_{n+1}\rangle\\
&\geq \beta\|u_{n+1}-u_n\|^2+\langle\rho A^T(I-C_P)Au_{n+1},u_{n+1}-u\rangle\\
&\geq \beta\|u_{n+1}-u_n\|^2,
\end{aligned}$$

using (3.26).

If $u_{n+1}=u_n$, then clearly u_n is a solution of the variational inequality problem (2.2). Otherwise, it follows that $B(u,u_n)-B(u,u_{n+1})$ is nonnegative , and we must have

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Now using the technique of Zhu and Marcotte [20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the variational inequality problem (2.2). □

Conclusion . In this paper, we have used the auxiliary principle technique to suggest and analyze several algorithms for solving the feasibility problems. Convergence analysis of these algorithms is analyzed under some weak and suitable conditions. Results proved in this paper can be viewed as an important and novel applications of the auxiliary principle technique in the convex feasibility problems.

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