

2021

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Recommended Citation

Z. Muhammed, Hiba; A. El-Sherpieny, El-Sayed; and M. Almetwally, Ehab (2021) "Dependency Measures For New Bivariate Models Based on Copula Function," *Information Sciences Letters*: Vol. 10 : Iss. 3 , Article 15.

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Dependency Measures For New Bivariate Models Based on Copula Function

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Received: 15 May 2021, Revised: 14 Jun. 2021, Accepted: 19 Aug. 2021

Published online: 1 Sep. 2021

Abstract: The first part of this paper reviews the properties of bivariate dependence measures as Spearman's rho and Kendall's tau under the different copula. The second part of this paper derives the bivariate inverted Topp-Leone (BITL) distribution based on Farlie-Gumbel-Morgenstern (FGM), Ali-Mikhail-Haq (AMH), Plackett, and Clayton copula. The reliability function obtained for bivariate ITL distributions based on copula. The maximum likelihood estimation method for the parameters of the four bivariate ITL distributions has been discussed. Asymptotic confidence intervals for the model parameters are also considered. To evaluate the performance of the models, a Monte Carlo simulation study is conducted to compare the efficiency between the four models. Also, a medical real data set of diabetic nephropathy is analyzed to investigate the models and useful results are obtained for illustrative purposes. Anderson-Darling-type statistic and Cramér-von Mises statistic for copula goodness-of-fit testing are obtained.

Keywords: Copula function, Spearman's rho, Kendall's tau, copula goodness-of-fit, Contour plot

1 Introduction

As we considering bivariate distribution, measuring the dependence between different bivariate data is of particular interest in a variety of applications. Spearman's rho, Kendall's tau, Blomqvist's beta, Gini's gamma, and a few other lesser-known measures are all classic bivariate dependence measures. These measures are designed to assess dependency in bivariate data based on copula function.

In fact, traditional 'dependence' measures such as Pearson's r, Spearman's rank correlation, or Kendall's are only useful in particular circumstances such as linearity, monotonicity, concordance, which are symmetric therefore undirected by design. Implying that the random variable X is equally dependent on the random variable Y and vice versa. The famous but often misunderstood measure of dependency is linear correlation. The popularity of linear correlation stems from its ease of calculation, and it is a natural scalar measure of dependency in elliptical distributions with well-known members like the multivariate-normal and multivariate-t distributions. In many real-life cases, the relationship between random variables is asymmetric. For example,

the linear correlation can result in incorrect risk assessments in stock market trades which was discussed by Okimoto [1], inaccurate gene network reconstructions which discussed by Wang and Huang [2]. Also see, Al-Sadoon [3], Bárdossy and Hörning [4], Coenen and Weitz [5] and Junker et al. [6].

In the literature review, we found additional more dependence measures including, e.g., dependence measures based on copula function which are discussed by Embrechts et al. [7], Smith et al. [8], distance correlation R which is discussed by Székely et al. [9], the maximal information coefficient (MIC) which is discussed by Reshef et al. [10,11], and robust copula dependence (RCD) discussed by Chang [12].

A copula is a simple way to describe a bivariate distribution with a dependency structure. A copula is a function that joins bivariate distribution functions with uniform [0, 1] margins, as defined by Nelsen [13].

Theorem 1(Sklar theorem). Assuming that the two random variables X and Y have distribution functions $F(x)$ and $F(Y)$. Let $C(u,v)$ is the copula cdf and $c(u,v)$ is the copula pdf. Then the cdf and pdf for bivariate

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distributions based on copula are given as

$$F(x, y) = C(F(x), F(y)), \quad (1)$$

and

$$f(x, y) = f(x)f(y)c(F(x), F(y)). \quad (2)$$

Many copulas based on Theorem 1 have been used to define the bivariate distributions. For more information see Nelsen [13]. For more examples see Elaal and Jarwan [14], Almetwally et al. [15], Almetwally and Muhammed [16] and El-Sherpieny et al. [17, 18, 19].

In the univariate cases, the inverted distributions are important in a variety of fields, including biological sciences, life test issues, medicine, and so on. The inverted conformation distributions have a different structure than non-inverted conformation distributions in terms of density and hazard ratio. The cdf and pdf of the inverted Topp-Leone distribution (ITL) with shape parameter $\lambda > 0$ were proposed by Hassan et al. [20] as follows:

$$F(x; \lambda) = 1 - \frac{(1+2x)^\lambda}{(1+x)^{2\lambda}}; \quad x \geq 0 > 0 \quad (3)$$

and,

$$f(x; \lambda) = 2\lambda x \frac{(1+2x)^{\lambda-1}}{(1+x)^{2\lambda+1}}; \quad x > 0 \quad (4)$$

Many authors introduced a new distribution based on ITL distribution such as Almetwally et al. [21] introduced modified Kies inverted Topp-Leone distribution. Muhammed [22] introduced the inverted Topp Leone distribution with different forms. Almetwally [23] introduced odd Weibull ITL distribution with applications of COVID-19 data. Abushal et al. [24] introduced power ITL distribution in acceptance sampling plans. Hassan et al. [25] introduced Kumaraswamy ITL distribution with applications of COVID-19 data. Ibrahim et al. [26] alpha power ITL distribution with different applications.

Kendall's tau and Spearman's rho are two essential indicators of dependency (concordance) that we will address. They are possibly the best alternatives to the linear correlation coefficient as a measure of dependency for non-elliptical distributions, in which the linear correlation coefficient is mostly ineffective and misleading.

In this paper, we study the bivariate extension of the ITL distribution based on different copula functions and obtain their statistical properties of dependence measure using Kendall's tau and Spearman's rho. Point and interval estimation of the parameters of BITL distribution are discussed using maximum likelihood estimation (MLE).

The present paper is organized as follows. After this introduction Section 2 reviews the original copula function with dependence measures. In Section 3, we introduced different bivariate extension of the ITL distribution based on different copula functions. In

Section 4 parameter estimation are obtained. Furthermore in Section 5, the potentiality of the four new models are illustrated by simulation study. In Section 6, application of a diabetic nephropathy real data set is discussed. Finally, Section 7 is devoted to the research perspective.

2 Copula Function

The family of Archimedean copulas is one of the most common families of copulas. If a copula has the functional form

$$C(u, v) = \varphi(\varphi^{-1}(u) + \varphi^{-1}(v))$$

for a suitable, decreasing function $\varphi: (0, \infty) \rightarrow [0, 1]$, with $\varphi(0) = 1$ and $\lim_{x \rightarrow \infty} \varphi(x) = 0$, it is called an Archimedean copula. For more information see Genest and Rivest [27] and Charpentier and Segers [28]. Popular completely monotone generators of Archimedean copulas as Clayton, Gumbel, and Ali-Mikhail-Haq (AMH) copula. Plackett [29] suggested the Plackett family of bivariate distributions, which were based on the principle of constant cross-product ratio (or odds ratio). Bekrizadeh et al. [30] discussed and developed Farlie Gumbel Morgenstern (FGM) copula.

Spearman's Rho

Let C be the copula of continuous random variables X and Y . The population variant of Spearman's rho for X and Y is then calculated as follows:

$$\rho_s = 12 \int_0^1 \int_0^1 C(u, v) dudv - 3 \quad (5)$$

It's worth noting that ρ_s corresponds to the correlation coefficient ρ between uniform marginal distributions.

Kendall's Tau

In terms of copula, Kendall's tau τ_k is defined as

$$\tau_k = 4 \int_0^1 \int_0^1 c(u, v) C(u, v) dudv - 1 \quad (6)$$

The AMH, Clayton, FGM, and Plackett copulas, which are used to form bivariate distributions, will be discussed.

2.1 FGM Copula

In FGM copula; the cdf of FGM copula with copula parameter $-1 < \theta < 1$ is given as following

$$C(u, v) = uv(\theta(1-u)(1-v)) \quad (7)$$

the pdf of FGM copula is as following

$$c(u, v) = 1 + \theta(1 - 2u)(1 - 2v) \quad (8)$$

The Spearman's rho of FGM is

$$\rho_s = \frac{\theta}{3}.$$

The Kendall's tau of FGM is

$$\tau_k = \frac{2\theta}{9}.$$

2.2 AMH Copula

In AMH copula; the cdf of AMH copula with copula parameter $-1 < \theta < 1$ is given as following

$$C(u, v) = \frac{uv}{1 - \theta(1 - u)(1 - v)} \quad (9)$$

the pdf of AMH copula is as following

$$c(u, v) = \frac{1 - \theta + 2\theta \frac{uv}{1 - \theta(1 - u)(1 - v)}}{[1 - \theta(1 - u)(1 - v)]^2} \quad (10)$$

For the AMH copula, Nelsen ([13], p. 172) provides the following formula for Spearman's rho

$$\rho_s = \frac{12(\theta + 1)}{\theta^2} \text{dilog}(1 - \theta) - \frac{24(1 - \theta)}{\theta^2} \ln(1 - \theta) - \frac{3(\theta + 12)}{\theta},$$

where his "dilogarithm" $\text{dilog}(x) = \text{polylog}(1 - x, 2)$, and it is the usual definition of the dilogarithm as a special function.

The Kendall's tau of AMH is

$$\tau_k = 1 - 2 \frac{(1 - \theta)^2 \ln(1 - \theta) + \theta}{3\theta^2}.$$

2.3 Clayton Copula

In Clayton copula; the cdf of Clayton copula with copula parameter $\theta > 0$ is as following

$$C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}} \quad (11)$$

the pdf of Clayton copula is given as following

$$c(u, v) = (1 + \theta)(uv)^{-1 - \theta} \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{2\theta - 1}{\theta}} \quad (12)$$

The Kendall's tau of Clayton is

$$\tau_k = \frac{\theta}{2 + \theta}.$$

2.4 Plackett Copula

In Plackett copula; the cdf of Plackett copula with copula parameter $\theta > 0$ is given as following

$$C(u, v) = \frac{1 + (\theta - 1)(u + v)}{2(\theta - 1)} - \frac{\sqrt{(1 + (\theta - 1)(u + v))^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)} \quad (13)$$

the pdf of Plackett copula is as following

$$c(u, v) = \frac{\theta(1 + (u - 2uv + v)(\theta - 1))}{[(1 + (\theta - 1)(u + v))^2 - 4uv\theta(\theta - 1)]^{1.5}} \quad (14)$$

For the Plackett copula the following formula for Spearman's rho is given as follows:

$$\rho_s = \frac{2\theta + \theta^2 - 2(\theta + 1)\ln(\theta + 1)}{\theta^2},$$

3 Bivariate ITL Distribution

In this section, we introduced four bivariate ITL distributions based on FGM, AMH, Clayton and Plackett copulas.

3.1 FGM Bivariate ITL Distribution

By using FGM copula Equations (7,8), Sklar Theorem 1 and ITL distribution in Equations (4,3), we get the FGM bivariate ITL distribution which can be denoted by FGMBITL with cdf, pdf and reliability function as follows in Equations (15, 16,17) respectively. Figure 1 shows the joint pdf of FGM bivariate ITL distribution with different shapes.

$$F_{FGMBITL}(x, y) = \left(1 - \frac{(1 + 2x)^{\lambda_1}}{(1 + x)^{2\lambda_1}}\right) \left(1 - \frac{(1 + 2y)^{\lambda_2}}{(1 + y)^{2\lambda_2}}\right) \left(1 + \theta \frac{(1 + 2x)^{\lambda_1} (1 + 2y)^{\lambda_2}}{(1 + x)^{2\lambda_1} (1 + y)^{2\lambda_2}}\right), \quad (15)$$

$$f_{FGMBITL}(x, y) = 2\lambda_1 x \frac{(1 + 2x)^{\lambda_1 - 1}}{(1 + x)^{2\lambda_1 + 1}} 2\lambda_2 y \frac{(1 + 2y)^{\lambda_2 - 1}}{(1 + y)^{2\lambda_2 + 1}} \left[1 + \theta \left(2 \frac{(1 + 2x)^{\lambda_1}}{(1 + x)^{2\lambda_1}} - 1\right) \left(2 \frac{(1 + 2y)^{\lambda_2}}{(1 + y)^{2\lambda_2}} - 1\right)\right], \quad (16)$$

and

$$R_{FGMBITL}(x, y) = \frac{(1 + 2x)^{\lambda_1} (1 + 2y)^{\lambda_2}}{(1 + x)^{2\lambda_1} (1 + y)^{2\lambda_2}} - 1 + \left(1 + \theta \left(1 - \frac{(1 + 2x)^{\lambda_1}}{(1 + x)^{2\lambda_1}}\right) \left(1 - \frac{(1 + 2y)^{\lambda_2}}{(1 + y)^{2\lambda_2}}\right)\right). \quad (17)$$

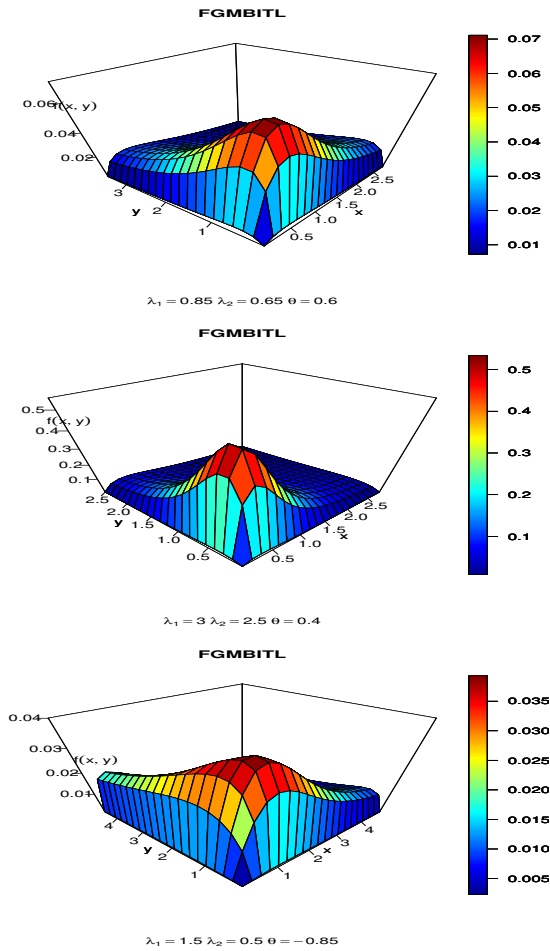


Fig. 1: Joint pdf of FGMBITL distribution

3.2 AMH Bivariate ITL Distribution

By using AMH copula Equations (9,10), Sklar theorem 1 and ITL distribution in Equations (4,3), we get the AMH bivariate ITL distribution which can be denoted by AMHBITL with cdf, pdf and reliability function given as follows in Equations (18, 19, 20) respectively. Figure 2 shows the joint pdf of AMH bivariate ITL distribution with different shapes.

$$F_{AMHBITL}(x,y) = \frac{\left(1 - \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right) \left(1 - \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1 - \theta \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}, \quad (18)$$

$$f_{AMHBITL}(x,y) = \frac{2\lambda_1 x \frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}} 2\lambda_2 y \frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}}}{1 - \theta + 2\theta \frac{\left(1 - \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right) \left(1 - \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1 - \theta \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}}, \quad (19)$$

and

$$R_{AMHBITL}(x,y) = \frac{\frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} + \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \frac{\left(1 - \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}}\right) \left(1 - \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}\right)}{1 - \theta \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}}{1 - \theta \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}}}}. \quad (20)$$

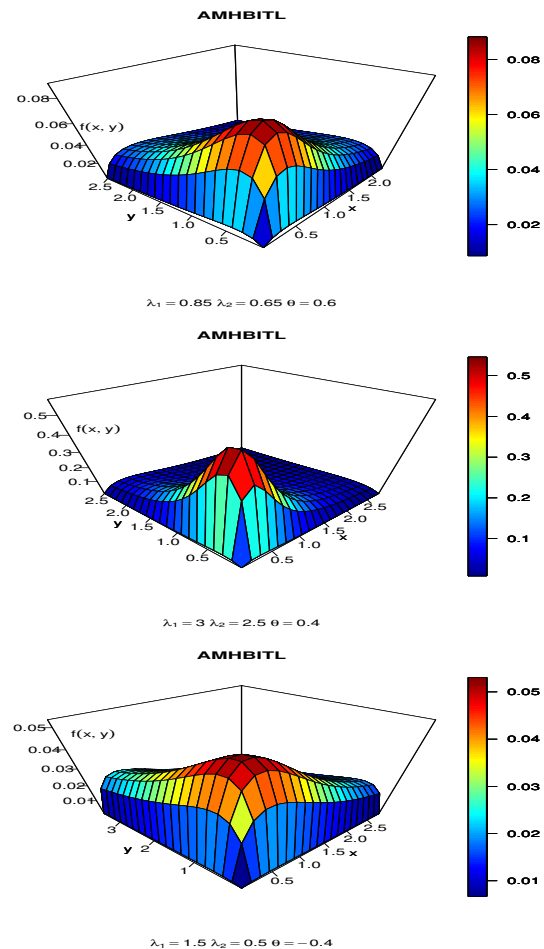


Fig. 2: Joint pdf of AMHBITL distribution

3.3 Clayton Bivariate ITL Distribution

By using Clayton copula Equations (11,12), Sklar theorem 1 and ITL distribution in Equations (4,3), to get the Clayton bivariate ITL distribution which can be denoted by CBITL with cdf, pdf and reliability function as follows in Equations (21, 22, 23) respectively. Figure 3 shows the joint pdf of Clayton Bivariate ITL distribution has different shapes.

3.4 Plackett Bivariate ITL Distribution

By using Plackett copula Equations (13,14), Sklar theorem 1 and ITL distribution in Equations (4,3), to get the Plackett bivariate ITL distribution which can be denoted by PBITL with cdf, pdf and reliability function as follows in Equations (25, 24, 26) respectively. Figure 4 shows the joint pdf of Plackett Bivariate ITL distribution with different shape parameters.

$$F_{CBITL}(x,y) = \left\{ \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right)^{-\theta} + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right)^{-\theta} - 1 \right\}^{-\frac{1}{\theta}}, \tag{21}$$

$$f_{CBITL}(x,y) = 2\lambda_1 x \frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}} 2\lambda_2 y \frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}} (1+\theta) \left[\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right]^{-1-\theta} \left\{ \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right)^{-\theta} + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right)^{-\theta} - 1 \right\}^{-\frac{2\theta-1}{\theta}}, \tag{22}$$

and

$$R_{CBITL}(x,y) = \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} + \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \left\{ \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right)^{-\theta} + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right)^{-\theta} - 1 \right\}^{-\frac{1}{\theta}}. \tag{23}$$

$$f_{PBITL}(x,y) = 2\lambda_1 x \frac{(1+2x)^{\lambda_1-1}}{(1+x)^{2\lambda_1+1}} 2\lambda_2 y \frac{(1+2y)^{\lambda_2-1}}{(1+y)^{2\lambda_2+1}} \frac{\theta \left(1 + \left[\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) - 2 \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right] (\theta - 1)}{\left[\left(1 + (\theta - 1) \left(\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right) \right]^2 - 4 \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \theta (\theta - 1)} \right]^{1.5}}, \tag{24}$$

$$F_{PBITL}(x,y) = \frac{1 + (\theta - 1) \left[\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right]}{2(\theta - 1)} \sqrt{\frac{\left(1 + (\theta - 1) \left[\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right] \right)^2 - 4 \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \theta (\theta - 1)}{2(\theta - 1)}}, \tag{25}$$

$$R_{PBITL}(x,y) = \frac{(1+2x)^{\lambda_1}}{(1+x)^{2\lambda_1}} + \frac{(1+2y)^{\lambda_2}}{(1+y)^{2\lambda_2}} - 1 + \sqrt{\frac{\left(1 + (\theta - 1) \left[\left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) + \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \right] \right)^2 - 4 \left(1 - \left(\frac{1+2x}{(1+x)^2} \right)^{\lambda_1} \right) \left(1 - \left(\frac{1+2y}{(1+y)^2} \right)^{\lambda_2} \right) \theta (\theta - 1)}{2(\theta - 1)}}. \tag{26}$$

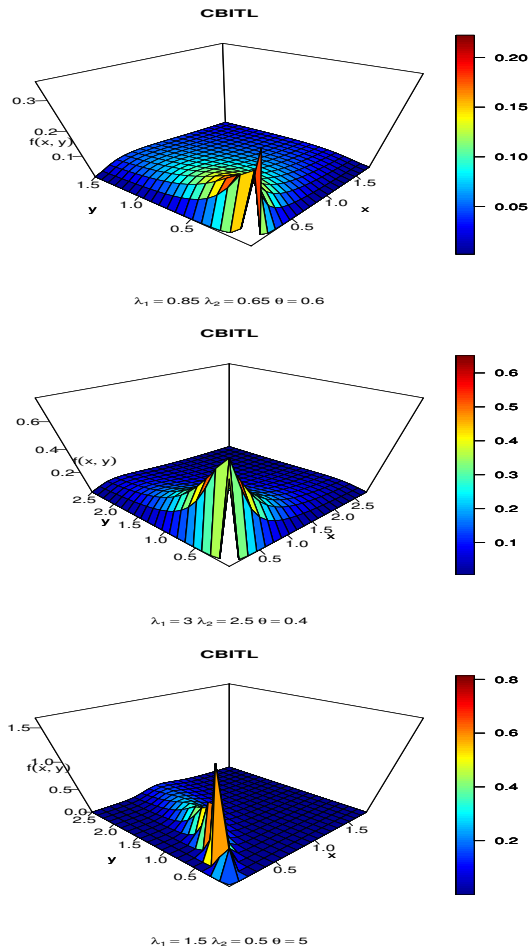


Fig. 3: Joint pdf of CBITL distribution

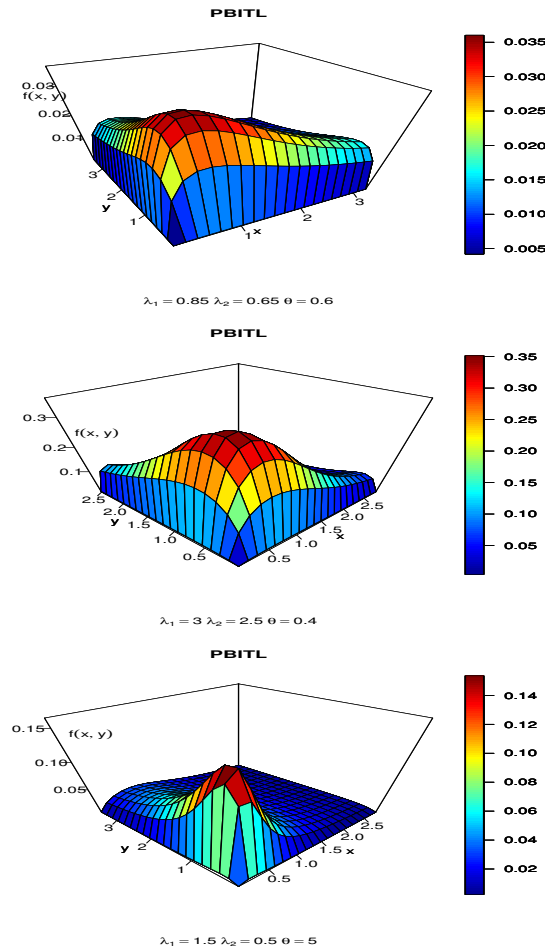


Fig. 4: Joint pdf of PBITL distribution

4 Maximum Likelihood Estimation (MLE)

In this section, we estimate the unknown parameters of the bivariate ITL distribution using the MLE method. Suppose that $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is a sample of size n . The likelihood function of bivariate ITL distribution with vector of parameters $\Theta = (\lambda_1, \lambda_2, \theta)$ based on different copulas are as follows:

The log-likelihood function of FGMBITL distribution can be written in Equation (27). The log-likelihood function of AMHBITL distribution can be written in Equation (28). The log-likelihood function of CBITL distribution can be written in Equation (29). The log-likelihood function of PBITL distribution can be written in Equation (30). By obtaining the first derivatives with respect to the unknown parameters for each log-likelihood function separately, and by equating them to zero, we can get the likelihood equations and hence solving them numerically to obtain the MLEs for each distribution.

In asymptotic confidence intervals (ACI), the asymptotic

normal distribution of the MLE is the most used to determine parameter confidence intervals of bivariate ITL distribution. Fisher information matrix $I(\Theta)$, which is composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at $\hat{\Theta} = (\hat{\lambda}_1, \hat{\lambda}_2, \hat{\theta})$, is related to the asymptotic variance-covariance matrix of the MLE of the parameters of bivariate ITL distribution. Suppose the asymptotic variance-covariance matrix of the parameter vector Θ is

$$I(\hat{\Theta}) = - \begin{bmatrix} I_{\hat{\lambda}_1 \hat{\lambda}_1} & & \\ I_{\hat{\lambda}_2 \hat{\lambda}_1} & I_{\hat{\lambda}_2 \hat{\lambda}_2} & \\ I_{\hat{\theta} \hat{\lambda}_1} & I_{\hat{\theta} \hat{\lambda}_2} & I_{\hat{\theta} \hat{\theta}} \end{bmatrix}$$

where $V(\hat{\Theta}) = I^{-1}(\hat{\Theta})$

A $100(1 - \alpha)\%$ confidence interval for the vector parameter Θ can be constructed based on the asymptotic normality of the MLE. $\hat{\lambda}_j \pm Z_{0.025} \sqrt{I_{\hat{\lambda}_j \hat{\lambda}_j}^{-1}}$; $j = 1, 2$

and $\hat{\theta}_j \pm Z_{0.025} \sqrt{I_{\hat{\theta}_j \hat{\theta}_j}^{-1}}$, where $Z_{0.025}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

$$l(\Theta) = n [\ln(4) + \ln(\lambda_1) + \ln(\lambda_2)] + \sum_{i=1}^n \ln(x_i) + \sum_{i=1}^n \ln(y_i) + \sum_{i=1}^n \ln \left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}} \right) + \sum_{i=1}^n \ln \left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}} \right) + \sum_{i=1}^n \ln \left[1 + \theta \left(2 \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}} - 1 \right) \left(2 \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}} - 1 \right) \right], \tag{27}$$

$$l(\Theta) = n [\ln(4) + \ln(\lambda_1) + \ln(\lambda_2)] + \sum_{i=1}^n \ln(x_i) + \sum_{i=1}^n \ln(y_i) + \sum_{i=1}^n \ln \left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}} \right) + \sum_{i=1}^n \ln \left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}} \right) + \sum_{i=1}^n \ln \left(\frac{1 - \theta + 2\theta \frac{\left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}}\right) \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}\right)}{1 - \theta \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}} \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}}}{\left[1 - \theta \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^{2\lambda_1}} \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^{2\lambda_2}}\right]^2} \right), \tag{28}$$

$$l(\Theta) = n [\ln(4) + \ln(\lambda_1) + \ln(\lambda_2)] + \sum_{i=1}^n \ln(x_i) + \sum_{i=1}^n \ln(y_i) + \sum_{i=1}^n \ln \left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}} \right) + \sum_{i=1}^n \ln \left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}} \right) + \sum_{i=1}^n \ln \left(\left\{ \left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^2}\right)^{-\theta} + \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^2}\right)^{-\theta} - 1 \right\}^{\frac{-2\theta-1}{\theta}} \right), \tag{29}$$

$$l(\Theta) = n [\ln(4) + \ln(\lambda_1) + \ln(\lambda_2)] + \sum_{i=1}^n \ln(x_i) + \sum_{i=1}^n \ln(y_i) + \sum_{i=1}^n \ln \left(\frac{(1+2x_i)^{\lambda_1-1}}{(1+x_i)^{2\lambda_1+1}} \right) + \sum_{i=1}^n \ln \left(\frac{(1+2y_i)^{\lambda_2-1}}{(1+y_i)^{2\lambda_2+1}} \right) + \sum_{i=1}^n \ln \left(\frac{\theta \left(1 + \left[\left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^2}\right) - 2 \left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^2}\right) \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^2}\right) + \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^2}\right) \right] (\theta - 1) \right)}{\left[\left(1 + (\theta - 1) \left[\left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^2}\right) + \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^2}\right) \right] \right)^2 - 4 \left(1 - \frac{(1+2x_i)^{\lambda_1}}{(1+x_i)^2}\right) \left(1 - \frac{(1+2y_i)^{\lambda_2}}{(1+y_i)^2}\right) \theta (\theta - 1) \right]^{1.5}} \right) \tag{30}$$

5 Simulation and Generation of Bivariate ITL Sample

In this section; a Monte Carlo simulation is done for comparison between bivariate ITL under different copulas by using point, interval estimation and dependence measures. To generate random variables: Nelsen [13] discussed generating a sample from a specified joint distribution by using the conditional distribution method. we will discuss the generation of bivariate ITL under different copulas, using the conditional distribution and iterative algorithms.

Firstly: We generate Q and W independently from a uniform distribution with lower is 0 and upper is 1.

Secondly: We obtain the conditional distribution of X given Y for each distribution based on copulas:

In FGM copula

$$C(v|u) = v(1 + \theta(1 - v)(1 - 2u)), \quad (31)$$

and

$$C(u|v) = u(1 + \theta(1 - u)(1 - 2v)). \quad (32)$$

In Clayton copula

$$C(v|u) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{1}{\theta}-1} u^{-\theta-1}, \quad (33)$$

and

$$C(u|v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{\frac{1}{\theta}-1} v^{-\theta-1}. \quad (34)$$

In AMH copula

$$C(v|u) = \frac{v}{1 - \theta(1 - u)(1 - v)} - \frac{\theta uv(1 - v)}{(1 - \theta(1 - u)(1 - v))^2}, \quad (35)$$

and

$$C(u|v) = \frac{u}{1 - \theta(1 - u)(1 - v)} - \frac{\theta uv(1 - u)}{(1 - \theta(1 - u)(1 - v))^2}. \quad (36)$$

In Plackett copula

$$C(v|u) = 0.5 \left[1 - (1 + (\theta - 1)(u + v)) - v\theta \frac{((1 + (\theta - 1)(u + v))^2 - 4uv\theta(\theta - 1))^{-0.5}}{4(\theta - 1)} \right], \quad (37)$$

and

$$C(u|v) = 0.5 \left[1 - (1 + (\theta - 1)(u + v)) - u\theta \frac{((1 + (\theta - 1)(u + v))^2 - 4uv\theta(\theta - 1))^{-0.5}}{4(\theta - 1)} \right]. \quad (38)$$

where u is a cdf of ITL distribution with parameter λ_1 and v is a cdf of ITL distribution with parameter λ_2 .

A simulation algorithm: Simulation experiments were carried out based on the following data generated from

bivariate ITL distributions under different copulas, where the values of the parameters λ_1, λ_2 , and θ are chosen as in the following cases for the random variables generation:

Case 1: ($\lambda_2 = 2.5, \theta = 0.4$) and λ_1 are change from 0.75 to 3.

Case 2: ($\lambda_2 = 0.5, \theta = 0.4$) and λ_1 are change from 0.75 to 3.

Case 3: ($\lambda_1 = 0.75, \lambda_2 = 0.5$) and θ are change from 0.1 to 0.9.

For different sample size $n = 35, 70$ and 150. The simulation results are compared using parameter estimation criteria, which include measuring the Bias, MSE, length of ACI, and dependence measure for each model as follows:

$$Bias = (\hat{\Theta} - \Theta)$$

Where $\hat{\Theta}$ is the estimated value of Θ .

$$MSE = Mean(\hat{\Theta} - \Theta)^2.$$

$$\text{and L.CI} = \text{Upper.CI}(\Theta) - \text{Lower.CI}(\Theta)$$

The number of repeated samples in simulation was limited to 5000.

Figure 5 Kendall's Tau, and Spearman's Rho coefficients limiting for BITL distribution. The simulation results of the methods discussed in this paper for point estimation, interval estimation, and dependency calculation are summarized in Tables 1, 2, and 3. The Bias, MSE, and L.CI for parameter of bivariate ITL distributions under different copula, Kendall's Tau, and Spearman's Rho values are used to perform the requisite comparison between different point estimation methods. The following conclusions can be drawn from these tables:

1. The Bias, MSE, and L.CI decrease as n increases for actual parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients.
2. For fixed sample size and case of actual values, we note that the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients increase when λ_1 increases.
3. In case 1 and fixed sample size, we note that the PBITL has the smallest Bias and MSE for $\lambda_1, \lambda_2, \theta$, while the shortest CI in case of AMHBITL distribution.
4. For fixed sample size and case of actual values, we note the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients increase when λ_2 increases.
5. For fixed sample size and case of actual values, we note that the Bias, MSE, and L.CI for parameters of bivariate ITL distributions under different copulas, Kendall's Tau, and Spearman's Rho coefficients

increase when θ increases.

6 Application

In this section, we used medical data related to diabetic nephropathy. We focused at the duration of diabetes and serum creatinine levels (SrCr). We are measuring the complications that may arise because of diabetics using SrCr values and non-diabetic nephropathy (SrCr < 1.4mg/dl). Pathological reports for these patients were collected from Dr. Lal's path lab database between January 2012 and August 2013. Grover et al. [31] looked at these findings, which included the average diabetes cycle of 132 patient Type-II diabetic nephropathy patients over time intervals. The parameters estimation for each distribution have been done, see Table 4. And it is through that we gain access to a model that is well-suited to the study of fragile relationships and the degree to which they have an influence and effectiveness. Table 5 shows the model criterion selection as Akaike information criterion (AIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC) and goodness of fit test of copulas as Anderson-Darling-type statistics (ADTS) with p-value and Cramer-von Mises functional (CVMF) with p-value for FGMBITL, AMHBITL, CBITL and PBITLF distributions. According to Tables 4 and 5, we can conclude that the CBITL model is better than FGMBITL, AMHBITL and PBITL distributions. For more information of goodness of fit test of copula see Genest et al. [32].

The plot of the bivariate distribution's density and cumulative is more interesting. However, keep in mind that the scatter plot and density plot may be deceiving. This is one of the reasons why, if contour plots are open, we tend to look at them in most cases. As shown in Figures 6, 7, these four copulas have correlation properties of various characteristics. The Clayton copula correlates better than the others copulas. The correlation structure of the FGM copula is weaker than the others copulas.

7 Conclusion

In this paper, we have proposed the use of the bivariate models based on copula functions for medical data related to diabetic nephropathy. We showed that the duration of diabetes and SrCr levels are closely related in a probabilistic way to each other by analyzing medical data. These empirical facts authorize our endeavor to bivariate model diabetic nephropathy based on different copulas. Four copula models were employed and their accuracy was examined through fitting to the real data. We have

proposed a new class of bivariate ITL distributions based on FGM, AMH, Plackett, and Clayton copulas. Moreover, we obtained the reliability functions for bivariate ITL distributions based on different copulas under study, therefore, it can be used quite effectively in life testing data. We studied dependence measures as Kendall's Tau and Spearman's Rho. Monte Carlo simulation is used to compare distributions. The CBITL distribution is the best model to fit this data.

Table 1: Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 1

$\lambda_2 = 2.5, \theta = 0.4$		FGM			Clayton			AMH			Plackett			
λ_1	n		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
3	35	λ_1	0.0900	0.2782	2.0384	-0.2574	0.4908	2.5554	0.0434	0.2709	2.0343	0.0017	0.2412	2.0218
		λ_2	0.0719	0.2056	1.7557	-0.1581	0.2815	1.9862	0.0298	0.1902	1.7063	0.0042	0.1886	1.7304
		θ	0.0040	0.2210	1.8437	0.1486	0.1508	1.4070	0.2264	0.1308	1.1060	-0.1802	0.0426	1.1566
		τ_k	0.0009	0.0109	0.4097	0.0339	0.0124	0.4162	0.0815	0.0148	0.3534	-0.1421	0.0278	0.8729
		ρ_s	0.0013	0.0246	0.6146	0.0448	0.0245	0.5880	0.1179	0.0309	0.5118	-0.1899	0.0489	1.2319
	70	λ_1	0.0506	0.1302	1.4014	-0.3834	0.4689	2.2251	0.0090	0.1230	1.3753	-0.0312	0.1190	1.4015
		λ_2	0.0505	0.1041	1.2499	-0.2498	0.2425	1.6641	0.0151	0.0987	1.2308	-0.0128	0.0960	1.2503
		θ	-0.0053	0.1283	1.4049	0.0939	0.0653	0.9321	0.2406	0.0950	0.7556	-0.1941	0.0417	0.9371
		τ_k	-0.0012	0.0063	0.3122	0.0244	0.0061	0.2918	0.0804	0.0106	0.2514	-0.1463	0.0250	0.7705
		ρ_s	-0.0018	0.0143	0.4683	0.0332	0.0123	0.4156	0.1175	0.0225	0.3653	-0.1976	0.0450	1.0949
	150	λ_1	0.0178	0.0595	0.9541	-0.4029	0.3272	1.5923	-0.0240	0.0587	0.9454	-0.0490	0.0556	0.9498
		λ_2	0.0155	0.0450	0.8298	-0.2751	0.1670	1.1849	-0.0196	0.0438	0.8167	-0.0394	0.0414	0.8199
		θ	-0.0012	0.0632	0.9857	0.0688	0.0287	0.6070	0.2521	0.0804	0.5095	-0.1958	0.0402	0.7872
		τ_k	-0.0003	0.0031	0.2190	0.0200	0.0030	0.2007	0.0812	0.0085	0.1729	-0.1441	0.0225	0.6957
		ρ_s	-0.0004	0.0070	0.3286	0.0280	0.0062	0.2885	0.1194	0.0184	0.2519	-0.1956	0.0412	0.9936
0.75	35	λ_1	0.0287	0.0203	0.5468	-0.0204	0.0162	0.4925	0.0178	0.0196	0.5453	-0.0124	0.0167	0.5554
		λ_2	0.1075	0.2270	1.8203	-0.0698	0.2170	1.8063	0.0634	0.2129	1.7925	0.0878	0.2115	1.7570
		θ	-0.0141	0.2431	1.9328	0.0817	0.1121	1.2732	0.2152	0.1406	1.2042	-0.1792	0.0433	1.2035
		τ_k	-0.0031	0.0120	0.4295	0.0146	0.0101	0.3909	0.0791	0.0154	0.3740	-0.1419	0.0280	0.8829
		ρ_s	-0.0047	0.0270	0.6443	0.0177	0.0204	0.5557	0.1142	0.0322	0.5424	-0.1896	0.0492	1.2468
	70	λ_1	0.0129	0.0088	0.3646	-0.0359	0.0084	0.3304	0.0029	0.0084	0.3595	-0.0251	0.0084	0.3805
		λ_2	0.0408	0.0913	1.1744	-0.1381	0.1112	1.1907	0.0055	0.0862	1.1510	0.0309	0.0866	1.1238
		θ	-0.0079	0.1188	1.3516	0.0272	0.0442	0.8173	0.2366	0.0914	0.7377	-0.1900	0.0405	0.9256
		τ_k	-0.0018	0.0059	0.3004	0.0036	0.0046	0.2643	0.0788	0.0102	0.2468	-0.1426	0.0240	0.7648
		ρ_s	-0.0026	0.0132	0.4505	0.0036	0.0094	0.3790	0.1152	0.0216	0.3583	-0.1927	0.0433	1.0873
	150	λ_1	0.0046	0.0040	0.2473	-0.0446	0.0052	0.2230	-0.0061	0.0038	0.2415	-0.0314	0.0045	0.2617
		λ_2	0.0185	0.0413	0.7941	-0.1541	0.0701	0.8447	-0.0176	0.0403	0.7840	0.0202	0.0405	0.7467
		θ	0.0122	0.0622	0.9770	0.0145	0.0221	0.5805	0.2564	0.0825	0.5080	-0.1876	0.0372	0.7862
		τ_k	0.0027	0.0031	0.2171	0.0020	0.0024	0.1929	0.0828	0.0088	0.1735	-0.1361	0.0203	0.6898
		ρ_s	0.0041	0.0069	0.3257	0.0021	0.0050	0.2778	0.1216	0.0189	0.2526	-0.1852	0.0373	0.9869



Table 2: Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 2

$\lambda_2 = 0.5, \theta = 0.4$		FGM			Clayton			AMH			Plackett			
λ_1	n		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.75	35	λ_1	0.0289	0.0203	0.5470	-0.0158	0.0159	0.4903	0.0195	0.0206	0.5571	0.0094	0.0165	0.5399
		λ_2	0.0207	0.0090	0.3637	-0.0118	0.0071	0.3273	0.0128	0.0087	0.3621	-0.0082	0.0071	0.3668
		θ	-0.0183	0.2434	1.9338	-0.0031	0.0941	1.2029	0.2126	0.1404	1.2103	-0.1436	0.0451	1.2682
		τ_k	-0.0041	0.0120	0.4297	-0.0138	0.0104	0.3953	0.0783	0.0152	0.3740	-0.1182	0.0244	0.8855
		ρ_s	-0.0061	0.0270	0.6446	-0.0230	0.0215	0.5679	0.1130	0.0319	0.5428	-0.1576	0.0433	1.2550
	70	λ_1	0.0129	0.0088	0.3644	-0.0300	0.0083	0.3374	0.0025	0.0086	0.3630	-0.0037	0.0079	0.3618
		λ_2	0.0073	0.0036	0.2349	-0.0249	0.0035	0.2108	0.0000	0.0035	0.2314	-0.0180	0.0033	0.2412
		θ	-0.0135	0.1187	1.3503	-0.0418	0.0415	0.7819	0.2350	0.0914	0.7458	-0.1622	0.0338	0.9403
		τ_k	-0.0030	0.0059	0.3001	-0.0204	0.0050	0.2652	0.0784	0.0102	0.2490	-0.1199	0.0192	0.7600
		ρ_s	-0.0045	0.0132	0.4501	-0.0312	0.0105	0.3835	0.1145	0.0216	0.3616	-0.1626	0.0349	1.0841
	150	λ_1	0.0046	0.0040	0.2472	-0.0397	0.0048	0.2228	-0.0063	0.0038	0.2414	-0.0091	0.0037	0.2406
		λ_2	0.0030	0.0016	0.1586	-0.0285	0.0022	0.1437	-0.0046	0.0016	0.1566	-0.0199	0.0018	0.1673
		θ	0.0068	0.0614	0.9712	-0.0571	0.0214	0.5288	0.2552	0.0822	0.5122	-0.1627	0.0298	0.7865
		τ_k	0.0015	0.0030	0.2158	-0.0230	0.0028	0.1869	0.0824	0.0088	0.1746	-0.1148	0.0155	0.6809
		ρ_s	0.0023	0.0068	0.3237	-0.0343	0.0060	0.2721	0.1210	0.0188	0.2542	-0.1571	0.0288	0.9771
3	35	λ_1	0.0864	0.2876	2.0757	-0.2209	0.4631	2.5244	0.0392	0.2746	2.0494	0.0935	0.2566	1.9465
		λ_2	0.0129	0.0087	0.3618	-0.0272	0.0076	0.3258	0.0045	0.0079	0.3485	-0.0204	0.0073	0.3621
		θ	-0.0008	0.2209	1.8433	0.0395	0.0997	1.2286	0.2250	0.1326	1.1228	-0.1566	0.0388	1.1771
		τ_k	-0.0002	0.0109	0.4096	0.0010	0.0098	0.3879	0.0814	0.0150	0.3583	-0.1239	0.0241	0.8695
		ρ_s	-0.0003	0.0245	0.6144	-0.0016	0.0199	0.5539	0.1176	0.0313	0.5186	-0.1658	0.0427	1.2306
	70	λ_1	0.0499	0.1307	1.4041	-0.3231	0.3992	2.1295	0.0066	0.1235	1.3778	0.0578	0.1257	1.3236
		λ_2	0.0093	0.0042	0.2504	-0.0340	0.0048	0.2372	0.0022	0.0040	0.2491	-0.0227	0.0040	0.2656
		θ	-0.0102	0.1282	1.4038	-0.0035	0.0425	0.8087	0.2391	0.0951	0.7638	-0.1739	0.0358	0.9412
		τ_k	-0.0023	0.0063	0.3120	-0.0071	0.0048	0.2703	0.0800	0.0106	0.2535	-0.1285	0.0206	0.7614
		ρ_s	-0.0034	0.0142	0.4679	-0.0119	0.0100	0.3896	0.1169	0.0225	0.3683	-0.1742	0.0373	1.0853
	150	λ_1	0.0178	0.0595	0.9539	-0.3832	0.3181	1.6233	-0.0249	0.0587	0.9450	0.0383	0.0563	0.8687
		λ_2	0.0024	0.0018	0.1660	-0.0410	0.0033	0.1600	-0.0050	0.0018	0.1632	-0.0277	0.0023	0.1798
		θ	-0.0062	0.0629	0.9833	-0.0143	0.0170	0.5088	0.2508	0.0801	0.5142	-0.1761	0.0335	0.7826
		τ_k	-0.0014	0.0031	0.2185	-0.0075	0.0021	0.1763	0.0809	0.0085	0.1742	-0.1259	0.0179	0.6841
		ρ_s	-0.0021	0.0070	0.3278	-0.0115	0.0044	0.2558	0.1188	0.0183	0.2538	-0.1719	0.0331	0.9801



Table 3: Point, interval and dependence measure of the bivariate ITL distributions under different copulas: Case 3

$\lambda_1 = 0.75, \lambda_2 = 0.5$		FGM			Clayton			AMH			Plackett			
θ	n		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.9	35	λ_1	0.0306	0.0208	0.5526	-0.0785	0.0181	0.4291	-0.0695	0.0180	0.4495	0.0276	0.0201	0.4002
		λ_2	-0.0084	0.0082	0.3527	-0.0545	0.0082	0.2823	-0.0487	0.0078	0.2896	0.0200	0.0090	0.2579
		θ	-0.3228	0.2799	1.6442	-0.2531	0.1881	1.3811	-0.1645	0.0549	0.6543	0.0505	0.3162	1.2103
		τ_k	-0.0717	0.0138	0.3654	-0.0784	0.0152	0.3726	-0.0640	0.0078	0.2401	-0.0207	0.0151	0.6160
		ρ_s	-0.1076	0.0311	0.5481	-0.1109	0.0297	0.5170	-0.0914	0.0160	0.3427	-0.0300	0.0324	0.9000
	70	λ_1	0.0137	0.0086	0.3599	-0.0924	0.0138	0.2842	-0.0823	0.0125	0.2958	0.0113	0.0088	0.2370
		λ_2	-0.0212	0.0036	0.2188	-0.0655	0.0064	0.1796	-0.0591	0.0058	0.1866	0.0071	0.0037	0.1449
		θ	-0.2986	0.1804	1.1846	-0.3040	0.1543	0.9755	-0.1543	0.0353	0.4203	-0.0453	0.1179	0.7686
		τ_k	-0.0664	0.0089	0.2633	-0.0876	0.0129	0.2830	-0.0633	0.0057	0.1630	-0.0262	0.0072	0.5046
		ρ_s	-0.0995	0.0200	0.3949	-0.1223	0.0252	0.3972	-0.0896	0.0115	0.2322	-0.0386	0.0156	0.7405
	150	λ_1	0.0049	0.0039	0.2442	-0.1004	0.0125	0.1926	-0.0896	0.0106	0.1983	0.0036	0.0040	0.1297
		λ_2	-0.0250	0.0020	0.1476	-0.0693	0.0058	0.1249	-0.0630	0.0050	0.1281	0.0035	0.0017	0.0773
		θ	-0.2755	0.1237	0.8575	-0.3260	0.1362	0.6786	-0.1477	0.0273	0.2905	-0.0517	0.0520	0.4847
		τ_k	-0.0612	0.0061	0.1906	-0.0908	0.0109	0.2012	-0.0622	0.0047	0.1115	-0.0204	0.0037	0.4278
		ρ_s	-0.0918	0.0137	0.2858	-0.1258	0.0210	0.2830	-0.0876	0.0093	0.1598	-0.0302	0.0081	0.6304
0.1	35	λ_1	0.0220	0.0189	0.5326	0.0163	0.0156	0.4858	0.0232	0.0185	0.5255	-0.1549	0.0326	0.6230
		λ_2	0.0140	0.0086	0.3592	0.0117	0.0080	0.3468	0.0135	0.0082	0.3511	-0.1089	0.0158	0.4215
		θ	0.0242	0.2622	2.0062	0.1487	0.0902	1.0232	0.0565	0.2250	1.8473	0.1121	0.0203	1.0404
		τ_k	0.0054	0.0130	0.4458	0.0516	0.0127	0.3921	0.0290	0.0141	0.4520	0.1289	0.0230	0.7829
		ρ_s	0.0081	0.0291	0.6687	0.0755	0.0270	0.5722	0.0424	0.0309	0.6687	0.1603	0.0363	1.1064
	70	λ_1	0.0119	0.0082	0.3512	0.0091	0.0076	0.3395	0.0116	0.0081	0.3499	-0.1599	0.0298	0.4737
		λ_2	0.0098	0.0042	0.2524	0.0089	0.0043	0.2534	0.0096	0.0042	0.2522	-0.1114	0.0145	0.3376
		θ	-0.0049	0.1436	1.4861	0.1229	0.0421	0.6439	0.0558	0.1251	1.3700	0.1017	0.0135	0.7486
		τ_k	-0.0011	0.0071	0.3303	0.0478	0.0067	0.2605	0.0213	0.0073	0.3241	0.1263	0.0189	0.6643
		ρ_s	-0.0016	0.0160	0.4954	0.0707	0.0147	0.3852	0.0316	0.0162	0.4833	0.1552	0.0290	0.9437
	150	λ_1	0.0043	0.0037	0.2390	0.0027	0.0038	0.2416	0.0038	0.0037	0.2391	-0.1640	0.0288	0.3729
		λ_2	0.0030	0.0018	0.1667	0.0015	0.0019	0.1690	0.0027	0.0018	0.1666	-0.1150	0.0141	0.2595
		θ	-0.0004	0.0671	1.0158	0.1136	0.0261	0.4504	0.0767	0.0603	0.9146	0.1022	0.0119	0.5051
		τ_k	-0.0001	0.0033	0.2257	0.0464	0.0044	0.1853	0.0221	0.0037	0.2214	0.1302	0.0183	0.5752
		ρ_s	-0.0001	0.0075	0.3386	0.0691	0.0097	0.2752	0.0329	0.0082	0.3309	0.1594	0.0277	0.8207



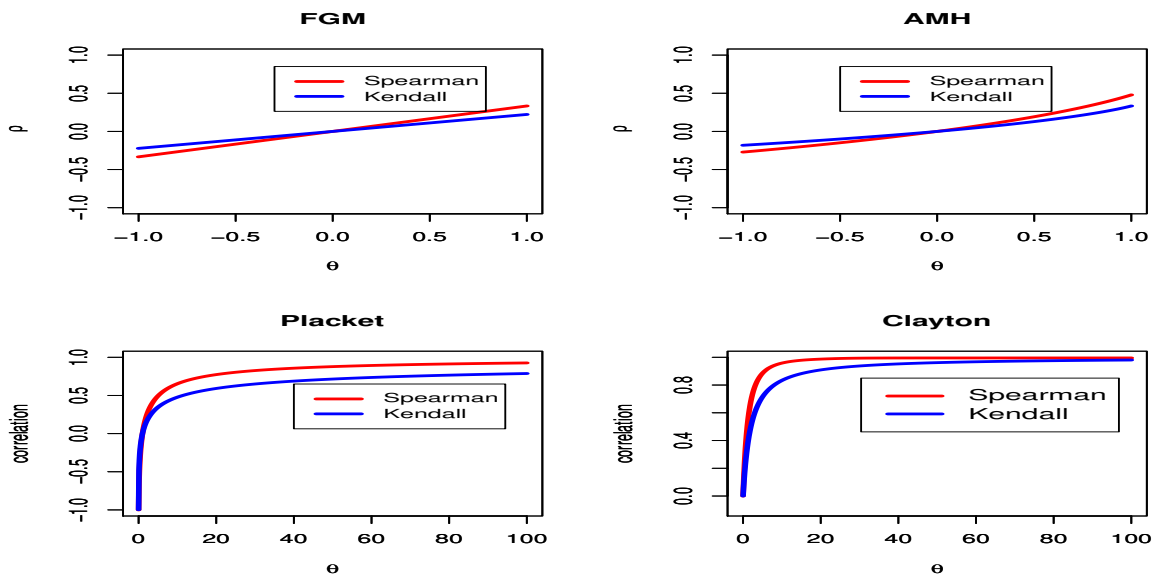


Fig. 5: Kendall's Tau, and Spearman's Rho coefficients limiting for BITL

Table 4: MLE and dependance measure of FGMBITL, AMHBITL, CBITL and PBITLF distributions for medical data

	λ_1	λ_2	θ	τ_k	ρ_s
FGMBITL	0.5121	0.3981	0.4670	0.1038	0.1557
AMHBITL	0.4083	0.2407	0.4975	0.1280	0.1913
CBITL	0.3605	0.3409	3.0826	0.1317	0.1964
PBITL	0.3359	0.3290	1.6633	0.1138	0.1682

Table 5: The model criterion selection and goodness of fit for FGMBITL, AMHBITL, CBITL and PBITLF distributions for medical data

	AIC	CAIC	BIC	HQIC	AD	p-value	CVM	p-value
FGMBITL	754.0737	754.9968	758.2773	755.4185	0.2903	0.4000	0.02503	0.2532
AMHBITL	749.8751	750.7982	754.0787	751.2199	0.2114	0.5509	0.020513	0.4363
CBITL	731.5039	732.4270	735.7075	732.8487	0.1442	0.7494	0.020806	0.4744
PBITL	734.7395	735.6626	738.9431	736.0843	0.2623	0.4434	0.023196	0.2972

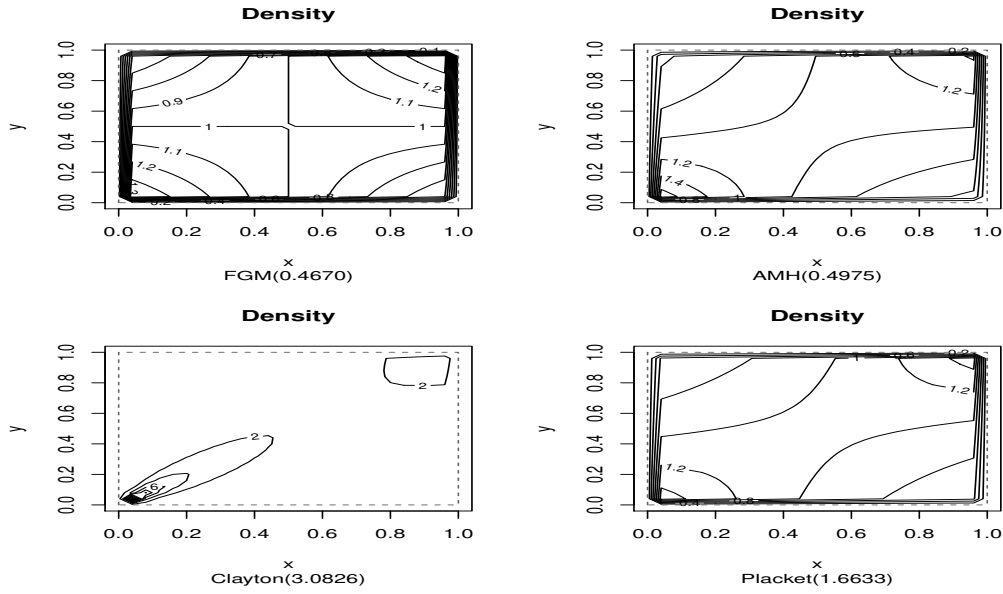


Fig. 6: Density contour plot for BITL distribution

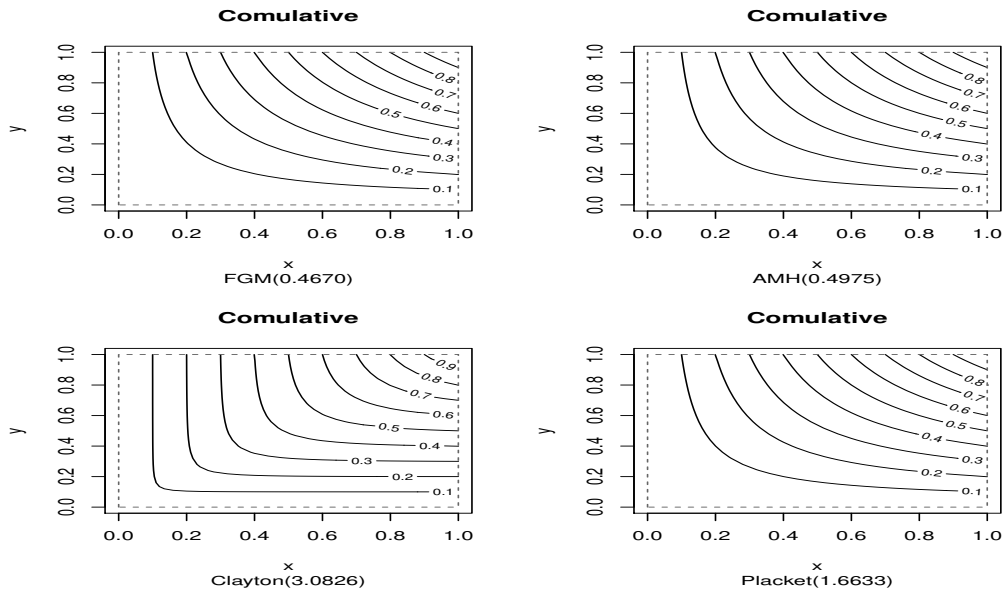


Fig. 7: Comulative contour plot for BITL distribution

Acknowledgment

All thanks and appreciation to everyone who helped and encouraged us in completing this work.

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