Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/060113

Statistical Inference under Unified Hybrid Censoring Scheme

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Received: 4 Oct. 2016, Revised: 13 Jan. 2017, Accepted: 18 Jan. 2017
Published online: 1 Mar. 2017

Abstract: In this paper, a general exponential form of the underlying distribution and a general conjugate prior are used to discuss the maximum likelihood and Bayesian estimation based on an unified hybrid censored sample. A general procedure for deriving the point and interval Bayesian prediction of the future order statistics from the same sample as well as that from an unobserved future sample is also developed. The exponential and Pareto distributions are then used as illustrative examples. Finally, two numerical examples are presented for illustrating all the inferential procedures developed here.

Keywords: Bayesian estimation; Bayesian prediction; Exponential distribution; Maximum likelihood estimation; Pareto distribution; Order statistics; Unified hybrid censored sample.

1 Introduction

In life-testing experiments, the experimenter may stop the experiment before all the units on the test have failed due to some considerations such as time and cost. In such cases, the obtained data is called censored data. The most two common forms of censoring are Type-I and Type-II censoring schemes. Type-I hybrid censoring scheme is introduced by Epstein in [1] as a mixture of Type-I and Type-II censoring schemes. Type-II hybrid censoring scheme (Type-II HCS) is proposed by Childs et al. in [2] to fix the disadvantages inherent in Type-I hybrid censoring scheme. Chandrasekar et al. in [3] introduced generalized Type-I hybrid and generalized Type-II HCS as mixtures of Type-I hybrid and Type-II HCS. For more details about HCS, one may refer to [4].

Recently, Balakrishnan et al. in [5] proposed the unified HCS to fix the disadvantages inherent in the generalized Type-I hybrid and generalized Type-II HCS, suggested by Chandrasekar et al. in [3]. This censoring scheme can be described as follows. Consider a life-testing experiment in which n identical units are placed on a life-test. Fix integers $k, r \in \{0,\dots,n\}$ and $T_1, T_2 \in (0,\infty)$ such that $k < r$ and $T_1 < T_2$. If the k^{th} failure occurs before time T_1, the experiment is terminated at $\min\{\max(X_{r:n}, T_1), T_2\}$. If the k^{th} failure occurs between T_1 and T_2, the experiment is terminated at $\min(X_{r:n}, T_2)$ and if the k^{th} failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. Under this censoring scheme, we can guarantee that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures. The described unified HCS and inferential methods based on such a scheme have been discussed earlier in the literature; see, for example; [4], [6], [7], [8], and [9].

AL-Hussaini [10] suggested a general exponential form of the underlying distribution to develop a general procedure for the Bayesian inference. This general form can described as follows; Motivated by the fact that the survival function (SF) $\bar{F}(x|\theta) = 1 - F(x|\theta)$ corresponding to any cumulative distribution function (CDF) $F(x|\theta)$ can be written in the form

$$\bar{F}(x|\theta) = \exp[-\psi(x;\theta)], \tag{1}$$

where $\psi(x;\theta) = -\ln \bar{F}(x|\theta)$ is monotone increasing, continuous and differentiable function, with $\psi(x;\theta) \to 0$ as $x \to -\infty$ and $\psi(x;\theta) \to \infty$ as $x \to \infty$.

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The probability density function (PDF) corresponding to (1) is given by

$$f(x|\theta) = \psi'(x;\theta)\exp[-\psi(x;\theta)], \tag{2}$$

where $\psi'(x;\theta)$ is the first derivative of $\psi(x;\theta)$ with respect to x.

Several distributions that are used in reliability studies, such as exponential, Pareto, Weibull and Burr Type-XII distributions, can be obtained as special cases from the general exponential form (1) by using an appropriate choice of $\psi(x;\theta)$. Many authors considered this general exponential form to develop a general procedure of the statistical inference based on different forms of censored data, see, for example; [11], [12], [13], [14], [15], [16], [17], and [18]. Recently, Mohie El-Din et al. in [19] have considered the inverse general exponential form and developed a general procedures for Bayesian estimation and two-sample prediction using the unified hybrid censoring schemes. We discuss in this paper the same problem based on the unified hybrid censoring scheme which involves some additional complications.

The rest of this paper is organized as follows. In Section 2, a general procedure of deriving the maximum likelihood (ML) and Bayesian estimators is presented. A general procedure of predicting the future order statistics from the same sample is discussed in Section 3. In Section 4, a general procedure of predicting the future order statistics from an unobserved future sample is then developed. The exponential and Pareto distributions are presented in Section 5 as special cases from the general exponential form (1). Finally, in Section 6, some computational results for the exponential and Pareto distributions are presented for illustrating all the inferential methods developed here.

2 The ML and Bayesian estimation

Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the failure times of n independent and identical units are placed on a life-test with an absolutely continuous CDF $F(x) \equiv F(x|\theta)$ and PDF $f(x) \equiv f(x|\theta)$ where the parameter $\theta \in \Theta$ may be a real vector. Let D_j denote the number of $X_{i:n}$'s that are at most T_j, $j = 1,2$. Then, D_j is a discrete random variable has the binomial distribution $B(n, F(T_j))$, $j = 1,2$, with support $\{0,1,\ldots,n\}$. Therefore, we observe one of the following six cases of observations under the unified hybrid censoring scheme:

1. If $0 < X_{k:n} < X_{r:n} \le T_1 < T_2$, then the experiment is terminated at T_1 and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{r:n} < \ldots < X_{D_1:n}$.
2. If $0 < X_{k:n} \le T_1 < X_{r:n} \le T_2$, then the experiment is terminated at $X_{r:n}$ and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{D_1:n} < \ldots < X_{r:n}$.
3. If $0 < X_{k:n} \le T_1 < T_2 < X_{r:n}$, then the experiment is terminated at T_2 and we will observe $X_{1:n} < \ldots < X_{k:n} < \ldots < X_{D_1:n} < \ldots < X_{D_2:n}$.
4. If $0 < T_1 < X_{k:n} < X_{r:n} \le T_2$, then the experiment is terminated at $X_{r:n}$ and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{k:n} < \ldots < X_{r:n}$.
5. If $0 < T_1 < X_{k:n} \le T_2 < X_{r:n}$, then the experiment is terminated at T_2 and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{k:n} < \ldots < X_{D_2:n}$.
6. If $0 < T_1 < T_2 < X_{k:n} < X_{r:n}$, then the experiment is terminated at $X_{k:n}$ and we will observe $X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{D_2:n} < \ldots < X_{k:n}$.

Thus, the joint density function of the unified hybrid censored sample $\underline{\mathbf{X}} = (X_{1:n}, X_{2:n}, \ldots, X_{D:n})$ is as follows:

$$f_{\underline{\mathbf{X}}}(\underline{\mathbf{x}}) = \frac{n!}{(n-D)!}\prod_{i=1}^{D} f(x_i)\{1-F(T)\}^{n-D}, \tag{3}$$

where

$$(D,T) = \begin{cases} (D_1,T_1), & \text{in Case 1}, \\ (r, X_{r:n}), & \text{in Cases 2 and 4}, \\ (D_2,T_2), & \text{in Cases 3 and 5}, \\ (k, X_{k:n}), & \text{in Case 6}, \end{cases} \tag{4}$$

and $\underline{\mathbf{x}} = (x_1, x_2, \ldots, x_D)$ is a vector of realizations.

Upon substituting (1) and (2) in (3), the likelihood function of θ, given the observed unified hybrid censored sample, is obtained as

$$L(\theta;\underline{\mathbf{x}}) = \frac{n!}{(n-D)!}\left(\prod_{i=1}^{D}\psi'(x_{i:n};\theta)\right)\exp\left[-\left(\sum_{i=1}^{D}\psi(x_i;\theta) - (n-D)\psi(T;\theta)\right)\right], \tag{5}$$

The log-likelihood function of θ, given the observed unified hybrid censored sample, is then given by

$$\log L(\theta;\underline{\mathbf{x}}) = \log n! - \log(n-D)! + \sum_{i=1}^{D} \log \psi'(x_i;\theta) - \sum_{i=1}^{D} \psi(x_i;\theta) - (n-D)\psi(T;\theta). \tag{6}$$

By differentiating (6) with respect to θ and equating the result to zero, we can obtain the ML estimator of θ by solving the following equation

$$\frac{d\log L(\theta|\underline{\mathbf{x}})}{d\theta} = 0.$$

This equation is appropriate for a single value θ, but for a vector θ of course, the partial derivatives produce a system of equations that are solved simultaneously.

For the Bayesian method, we consider here a general conjugate prior, suggested by AL-Hussaini [10], that is given as

$$\pi(\theta;\delta) \propto A(\theta;\delta)\exp[-B(\theta;\delta)], \tag{7}$$

where $\theta \in \Theta$ is the vector of parameters of the distribution in (1) and δ is the vector of prior parameters. The prior family in (7) includes several priors used in the literature as special cases.

Upon combining (3) and (7), we obtain the posterior density function of θ, given the observed unified hybrid censored sample, as

$$\begin{aligned}\pi^*(\theta;\underline{\mathbf{x}}) &= L(\theta;\underline{\mathbf{x}})\pi(\theta;\delta) \Big/ \int_{\theta\in\theta} L(\theta;\underline{\mathbf{x}})\pi(\theta;\delta)d\theta \\ &= I^{-1}\eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{\mathbf{x}})], \end{aligned} \tag{8}$$

with

$$\eta(\theta;\underline{\mathbf{x}}) = A(\theta;\delta)\prod_{i=1}^{D}\psi'(x_i;\theta),$$

$$\zeta(\theta;\underline{\mathbf{x}}) = \sum_{i=1}^{D}\psi(x_i;\theta) + (n-D)\psi(T;\theta) + B(\theta;\delta),$$

and

$$I = \int_{\theta\in\theta} \eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{\mathbf{x}})]d\theta.$$

By using the squared error loss function, the Bayesian estimator of θ is obtained as the mean of the posterior density function, which is given by

$$\widehat{\theta} = I^{-1}\int_{\theta\in\theta} \theta\eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{\mathbf{x}})]d\theta. \tag{9}$$

3 One-Sample Bayesian Prediction

In one-sample prediction, we use the observed unified hybrid censored sample $\underline{\mathbf{X}} = (X_{1:n}, X_{2:n}, \ldots, X_{D:n})$ to develop a general procedure for deriving the point and interval prediction for the future order statistic $X_{s:n}$, $D < s \le n$, from the same sample.

The conditional density function of $X_{s:n}$, $D < s \le n$, given the observed unified hybrid censored sample $\underline{\mathbf{X}} = (X_{1:n}, X_{2:n}, \ldots, X_{D:n})$, is given by:

$$f(x_s|\underline{\mathbf{x}}) = \begin{cases} f_1(x_s|\underline{\mathbf{x}}), & \text{if } (D,T) = (D_1,T_1), \\ f_2(x_s|\underline{\mathbf{x}}), & \text{if } (D,T) = (r,X_{r:n}), \\ f_3(x_s|\underline{\mathbf{x}}), & \text{if } (D,T) = (D_2,T_2), \\ f_4(x_s|\underline{\mathbf{x}}), & \text{if } (D,T) = (k,X_{k:n}), \end{cases} \tag{10}$$

where

$$\begin{aligned} f_1(x_s|\underline{\mathbf{x}}) &= \frac{1}{P(r \le D_1 \le s-1)} \sum_{d_1=r}^{s-1} f(x_s|\underline{\mathbf{x}}, D_1 = d_1)P(D_1 = d_1) \\ &= \sum_{d_1=r}^{s-1} \frac{(n-d_1)!\phi_{d_1}(T_1)}{(s-d_1-1)!(n-s)!} \frac{[F(x_s)-F(T_1)]^{s-d_1-1}[1-F(x_s)]^{n-s} f(x_s)}{[1-F(T_1)]^{n-d_1}}, \quad x_s > T_1, \end{aligned} \tag{11}$$

with $\underline{\mathbf{x}} = (x_1, x_2, ..., x_{D_1})$ and $\phi_{d_1}(T_1) = \frac{P(D_1=d_1)}{P(r \le D_1 \le s-1)}$,

$$\begin{aligned} f_2(x_s|\underline{\mathbf{x}}) &= f(x_s|x_r) \\ &= \frac{(n-r)!}{(s-r-1)!(n-s)!} \frac{[F(x_s) - F(x_r)]^{s-r-1} [1 - F(x_s)]^{n-s} f(x_s)}{[1 - F(x_r)]^{n-r}}, \quad x_s > x_r, \end{aligned} \tag{12}$$

with $\underline{\mathbf{x}} = (x_1, x_2, ..., x_r)$,

$$\begin{aligned} f_3(x_s|\underline{\mathbf{x}}) &= \frac{1}{P(k \le D_2 \le r^* - 1)} \sum_{d_2=k}^{r^*-1} f(x_s|\underline{\mathbf{x}}, D_2 = d_2) P(D_2 = d_2) \\ &= \sum_{d_2=k}^{r^*-1} \frac{(n-d_2)! \gamma_{d_2}(T_2)}{(s-d_2-1)!(n-s)!} \frac{[F(x_s) - F(T_2)]^{s-d_2-1} [1 - F(x_s)]^{n-s} f(x_s)}{[1 - F(T_2)]^{n-d_2}}, \quad x_s > T_2, \end{aligned} \tag{13}$$

with $\underline{\mathbf{x}} = (x_1, x_2, ..., x_{D_2})$, $\gamma_{d_2}(T_2) = \frac{P(D_2=d_2)}{P(k \le D_2 \le r^*-1)}$ and $r^* = \min(r, s)$,

$$\begin{aligned} f_4(x_s|\underline{\mathbf{x}}) &= f(x_s|x_k) \\ &= \frac{(n-k)!}{(s-k-1)!(n-s)!} \frac{[F(x_s) - F(x_k)]^{s-k-1} [1 - F(x_s)]^{n-s} f(x_s)}{[1 - F(x_k)]^{n-k}}, \quad x_s > x_k, \end{aligned} \tag{14}$$

with $\underline{\mathbf{x}} = (x_1, x_2, ..., x_k)$.

Upon substituting (1) and (2) in (11) - (13), $f_1(x_s|\underline{\mathbf{x}})$, $f_2(x_s|\underline{\mathbf{x}})$, $f_3(x_s|\underline{\mathbf{x}})$ and $f_4(x_s|\underline{\mathbf{x}})$ will become:

$$f_1(x_s|\underline{\mathbf{x}}) = \sum_{d_1=r}^{s-1} \sum_{w=0}^{s-d_1-1} C_{1w} \phi_{d_1}(T_1;\theta) \psi'(x_s;\theta) \exp\left\{-(n-s+w+1)\left[\psi(x_s;\theta) - \psi'(T_1;\theta)\right]\right\}, x_s > T_1, \tag{15}$$

where $\underline{\mathbf{x}} = (x_1, x_2, ..., x_{D_2})$, $C_{1w} = \frac{(-1)^w \binom{s-d_1-1}{w} (n-d_1)!}{(s-d_1-1)!(n-s)!}$ and

$$\phi_{d_1}(T_1;\theta) = \frac{\binom{n}{d_1} \exp\left[-(n-d_1)\psi(T_1;\theta)\right] \left\{1 - \exp\left[-\psi(T_1;\theta)\right]\right\}^{d_1}}{\sum_{d_1=r}^{s-1} \binom{n}{d_1} \exp\left[-(n-d_1)\psi(T_1;\theta)\right] \left\{1 - \exp\left[-\psi(T_1;\theta)\right]\right\}^{d_1}},$$

$$f_2(x_s|\underline{\mathbf{x}}) = \sum_{w=0}^{s-r-1} C_{2w} \psi'(x_s;\theta) \exp\left\{-(n-s+w+1)\left[\psi(x_s;\theta) - \psi'(x_r;\theta)\right]\right\}, \quad x_s > x_r, \tag{16}$$

where $\underline{\mathbf{x}} = (x_1, x_2, ..., x_r)$ and $C_{2w} = \frac{(-1)^w \binom{s-r-1}{w} (n-r)!}{(s-r-1)!(n-s)!}$.

$$f_3(x_s|\underline{\mathbf{x}}) = \sum_{d_2=k}^{r^*-1} \sum_{w=0}^{s-d_2-1} C_{3w} \gamma_{d_2}(T_2;\theta) \psi'(x_s;\theta) \exp\left\{-(n-s+w+1)\left[\psi(x_s;\theta) - \psi'(T_2;\theta)\right]\right\}, \quad x_s > T_2, \tag{17}$$

where $\underline{\mathbf{x}} = (x_1, x_2, ..., x_{D_2})$, $C_{3w} = \frac{(-1)^w \binom{s-d_2-1}{w} (n-d)!}{(s-d_2-1)!(n-s)!}$ and

$$\gamma_{d_2}(T_2;\theta) = \frac{\binom{n}{d_2} \exp\left[-(n-d_2)\psi(T_2;\theta)\right] \left\{1 - \exp\left[-\psi(T_2;\theta)\right]\right\}^{d_2}}{\sum_{d_2=k}^{r^*-1} \binom{n}{d_2} \exp\left[-(n-d_2)\psi(T_2;\theta)\right] \left\{1 - \exp\left[-\psi(T_2;\theta)\right]\right\}^{d_2}}$$

$$f_4(x_s|\underline{\mathbf{x}}) = \sum_{w=0}^{s-k-1} C_{4w} \psi'(x_s;\theta) \exp\left\{-(n-s+w+1)\left[\psi(x_s;\theta) - \psi'(x_k;\theta)\right]\right\}, \quad x_s > x_k, \tag{18}$$

where $\underline{\mathbf{x}} = (x_1, x_2, ..., x_k)$ and $C_{4w} = \frac{(-1)^w \binom{s-k-1}{w} (n-k)!}{(s-k-1)!(n-s)!}$.

Upon combining (8) and (10), the Bayesian predictive density function of $X_{s:n}$, given the unified hybrid censored sample, is obtained as

$$f^*(x_s|\underline{\mathbf{x}}) = \begin{cases} f_1^*(x_s|\underline{\mathbf{x}}), & \text{if } (D,T)=(D_1,T_1), \\ f_2^*(x_s|\underline{\mathbf{x}}), & \text{if } (D,T)=(r,X_{r:n}), \\ f_3^*(x_s|\underline{\mathbf{x}}), & \text{if } (D,T)=(D_2,T_2), \\ f_4^*(x_s|\underline{\mathbf{x}}), & \text{if } (D,T)=(k,X_{k:n}), \end{cases} \tag{19}$$

where

$$\begin{aligned} f_1^*(x_s|\underline{\mathbf{x}}) &= \int_{\theta\in\theta} f_1(x_s|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= I^{-1}\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1} C_{1w}\int_{\theta\in\theta}\psi'(x_s;\theta)\phi_{d_1}(T_1;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(x_s;\theta)-\psi(T_1;\theta)]\}d\theta,\ \ x_s>T_1, \end{aligned} \tag{20}$$

$$\begin{aligned} f_2^*(x_s|\underline{\mathbf{x}}) &= \int_{\theta\in\theta} f_2(x_s|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= I^{-1}\sum_{w=0}^{s-r-1} C_{2w}\int_{\theta\in\theta}\psi'(x_s;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(x_s;\theta)-\psi(x_r;\theta)]\}d\theta,\ \ x_s>x_r, \end{aligned} \tag{21}$$

$$\begin{aligned} f_3^*(x_s|\underline{\mathbf{x}}) &= \int_{\theta\in\theta} f_3(x_s|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= I^{-1}\sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1} C_{3w}\int_{\theta\in\theta}\psi'(x_s;\theta)\gamma_{d_2}(T_2;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(x_s;\theta)-\psi(T_2;\theta)]\}d\theta,\ \ x_s>T_2, \end{aligned} \tag{22}$$

and

$$\begin{aligned} f_4^*(x_s|\underline{\mathbf{x}}) &= \int_{\theta\in\theta} f_4(x_s|\underline{\mathbf{x}})\pi^*(\theta|\underline{\mathbf{x}})d\theta \\ &= I^{-1}\sum_{w=0}^{s-k-1} C_{4w}\int_{\theta\in\theta}\psi'(x_s;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(x_s;\theta)-\psi(x_k;\theta)]\}d\theta,\ \ x_s>x_k. \end{aligned} \tag{23}$$

The predictive survival function $\bar{F}^*(t|\underline{\mathbf{x}})$, given the unified hybrid censored sample, can be obtained from (19) as

$$\bar{F}^*(t|\underline{\mathbf{x}}) = \begin{cases} \bar{F}_1^*(t|\underline{\mathbf{x}}), & \text{if } (D,T)=(D_1,T_1), \\ \bar{F}_2^*(t|\underline{\mathbf{x}}), & \text{if } (D,T)=(r,X_{r:n}), \\ \bar{F}_3^*(t|\underline{\mathbf{x}}), & \text{if } (D,T)=(D_2,T_2), \\ \bar{F}_4^*(t|\underline{\mathbf{x}}), & \text{if } (D,T)=(k,X_{k:n}), \end{cases} \tag{24}$$

where $t \geq 0$ and

$$\begin{aligned} \bar{F}_1^*(t|\underline{\mathbf{x}}) &= \int_t^\infty f_1^*(x_s|\underline{\mathbf{x}})dx_s \\ &= I^{-1}\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1}\frac{C_{1w}}{n-s+w+1}\int_{\theta\in\theta}\phi_{d_1}(T_1;\theta)\eta(\theta;\underline{\mathbf{x}}) \\ &\quad\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(t;\theta)-\psi(T_1;\theta)]\}d\theta, \end{aligned} \tag{25}$$

$$\bar{F}_2^*(t|\underline{\mathbf{x}}) = \int_t^\infty f_2^*(x_s|\underline{\mathbf{x}})dx_s$$
$$= I^{-1}\sum_{w=0}^{s-r-1}\frac{C_{2w}}{n-s+w+1}\int_{\theta\in\theta}\eta(\theta;\underline{\mathbf{x}})\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(t;\theta)-\psi(x_r;\theta)]\}d\theta, \tag{26}$$

$$\bar{F}_3^*(t|\underline{\mathbf{x}}) = \int_t^\infty f_3^*(x_s|\underline{\mathbf{x}})dx_s$$
$$= I^{-1}\sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1}\frac{C_{3w}}{n-s+w+1}\int_{\theta\in\theta}\gamma_{d_2}(T_2;\theta)\eta(\theta;\underline{\mathbf{x}})$$
$$\times\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(t;\theta)-\psi(T_2;\theta)]\}d\theta, \tag{27}$$

and

$$\bar{F}_4^*(t|\underline{\mathbf{x}}) = \int_t^\infty f_4^*(x_s|\underline{\mathbf{x}})dx_s$$
$$= I^{-1}\sum_{w=0}^{s-k-1}\frac{C_{4w}}{n-s+w+1}\int_{\theta\in\theta}\eta(\theta;\underline{\mathbf{x}})\exp\{-\zeta(\theta;\underline{\mathbf{x}})-(n-s+w+1)[\psi(t;\theta)-\psi(x_k;\theta)]\}d\theta. \tag{28}$$

As in the case of estimation, prediction can be either a point or an interval prediction. The Bayesian point predictor of $X_{s:n}$ can be obtained as the mean of the predictive density function in (19) and given by

$$\widehat{X}_{s:n} = \int_0^\infty x_s f^*(x_s|\underline{\mathbf{x}})dx_s, \tag{29}$$

Therefore, the Bayesian predictive bounds of $100(1-\gamma)\%$ two-sided equi-tailed (ET) interval for $X_{s:n}$ can be obtained by solving the following two equations:

$$\bar{F}^*(L_{ET}|\underline{\mathbf{x}}) = \frac{\gamma}{2} \quad \text{and} \quad \bar{F}^*(U_{ET}|\underline{\mathbf{x}}) = 1-\frac{\gamma}{2}, \tag{30}$$

where L_{ET} and U_{ET} denote the lower and upper bounds of the ET interval, respectively.

For the highest posterior density (HPD) interval method, we need to solve the following two equations:

$$\bar{F}^*(L_{HPD}|\underline{\mathbf{x}}) - \bar{F}^*(U_{HPD}|\underline{\mathbf{x}}) = 1-\gamma$$

and

$$f^*(L_{HPD}|\underline{\mathbf{x}}) - f^*(U_{HPD}|\underline{\mathbf{x}}) = 0,$$

where L_{HPD} and U_{HPD} denote the lower and upper bounds of the HPD interval, respectively.

4 Two-Sample Bayesian Prediction

In two-sample Bayesian prediction, we use the observed unified hybrid censored sample to develop a general procedure for deriving the point and interval prediction for the order statistics from an unobserved future sample from the same distribution.

Let $Y_{1:m} \le Y_{2:m} \le \ldots \le Y_{m:m}$ be the order statistics from a future random sample of size m from the same population. Then, the marginal density function of the order statistic $Y_{q:m}$, $1 \le q \le m$, see [20], is given by

$$f_{Y_{s:m}}(y_s|\theta) = \frac{m!}{(s-1)!(m-s)!}[F(y_s)]^{s-1}[1-F(y_s)]^{m-s}f(y_s), \quad y_s > 0. \tag{31}$$

Upon substituting (1) and (2) in (31), the marginal density function of $Y_{s:m}$ becomes

$$f_{Y_{s:m}}(y_s|\theta) = \sum_{w=0}^{s-1} C_{5w}\psi'(y_s;\theta)\exp[-(m-s+w+1)\psi(y_s;\theta)], \quad y_s > 0. \tag{32}$$

where $C_{5w} = \frac{(-1)^w \binom{s-1}{w} m!}{(s-1)!(m-s)!}$.

From (8) and (32), the Bayesian predictive density function of $Y_{s:m}$, given the unified hybrid censored sample, is obtained as

$$f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}}) = \int\limits_{\theta\in\theta} f_{Y_{s:m}}(y_s|\theta)\pi^*(\theta|\underline{\mathbf{x}})d\theta$$
$$= I^{-1}\sum_{w=0}^{s-1} C_{5w} \int\limits_{\theta\in\theta} \psi'(y_s;\theta)\eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{\mathbf{x}}) - (m-s+w+1)\psi(y_s;\theta)]d\theta, \;\; y_s > 0. \tag{33}$$

From (33), the predictive survival function $\bar{F}^*_{Y_{s:m}}(t|\underline{\mathbf{x}})$, for $t \geq 0$, is obtained as

$$\bar{F}^*_{Y_{s:m}}(t|\underline{\mathbf{x}}) = \int\limits_t^\infty f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s$$
$$= I^{-1}\sum_{w=0}^{s-1} \frac{C_{5w}}{m-s+w+1} \int\limits_{\theta\in\theta} \eta(\theta;\underline{\mathbf{x}})\exp[-\zeta(\theta;\underline{\mathbf{x}}) - (m-s+w+1)\psi(t;\theta)]d\theta. \tag{34}$$

The Bayesian point predictor of $Y_{s:m}$, $1 \leq s \leq m$, can be obtained as the mean of the predictive density function in (33) and given by

$$\widehat{Y}_{s:m} = \int\limits_0^\infty y_s f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s. \tag{35}$$

Therefore, the Bayesian predictive bounds of $100(1-\gamma)\%$ ET interval for $Y_{s:m}$, $1 \leq s \leq m$, can be obtained by solving the following two equations:

$$\bar{F}^*_{Y_{s:m}}(L_{ET}|\underline{\mathbf{x}}) = \frac{\gamma}{2} \qquad \text{and} \qquad \bar{F}^*_{Y_{s:m}}(U_{ET}|\underline{\mathbf{x}}) = 1 - \frac{\gamma}{2}, \tag{36}$$

where L_{ET} and U_{ET} denote the lower and upper bounds of the ET interval, respectively.

For the HPD interval method, also we need solve the following equations:

$$\bar{F}^*_{Y_{s:m}}(L_{HPD}|\underline{\mathbf{x}}) - \bar{F}^*_{Y_{s:m}}(U_{HPD}|\underline{\mathbf{x}}) = 1-\gamma$$

and

$$f^*_{Y_{s:m}}(L_{HPD}|\underline{\mathbf{x}}) - f^*_{Y_{s:m}}(U_{HPD}|\underline{\mathbf{x}}) = 0,$$

where L_{HPD} and U_{HPD} denote the lower and upper bounds of the HPD interval, respectively.

5 Examples

In this section, we apply the procedure derived in the preceding sections for the the exponential and Pareto distributions as special cases from the general exponential form given in (1).

5.1 The Exponential Distribution

The survival function of the exponential distribution is

$$\bar{F}(x|\lambda) = \exp[-\lambda x], \quad x > 0, \tag{37}$$

where $\lambda > 0$, and then we have

$$\psi(x;\lambda) = \lambda x \quad \text{and} \quad \psi'(x;\lambda) = \lambda.$$

Thus, the likelihood function of λ using the unified hybrid censored sample, is given by

$$L(\lambda;\underline{\mathbf{x}}) = \frac{n!}{(n-D)!}\lambda^D \exp\left[-\lambda\left(\sum_{i=1}^D x_i + (n-D)T\right)\right], \tag{38}$$

where D and T are given as in (4). So, the ML estimator of λ is

$$\widehat{\lambda}_{ML} = \frac{D}{\sum\limits_{i=1}^{D} x_i + (n-D)T}.$$

For the Bayesian inference, we use the conjugate gamma prior for λ with density function

$$\pi(\lambda;\delta) = \frac{b^a}{\Gamma(a)}\,\lambda^{a-1}\exp[-b\lambda], \quad \lambda > 0, \tag{39}$$

where $\delta = (a,b)$, a and b are positive constants and $\Gamma(\cdot)$ denotes the complete gamma function, and so we have

$$A(\lambda;\delta) = \lambda^{a-1} \text{ and } B(\lambda;\delta) = \lambda b.$$

Then, the posterior density function is given as in (8) where

$$\eta(\lambda;\underline{\mathbf{x}}) = \lambda^{D+a-1}, \quad \zeta(\lambda;\underline{\mathbf{x}}) = \lambda\left[\sum_{i=1}^{D} x_i + (n-D)T + b\right]$$

and

$$I = \Gamma(D+a)\left[\sum_{i=1}^{D} x_i + (n-D)T + b\right]^{-(D+a)}.$$

Therefor, the Bayesian estimator of λ under the squared error loss function is

$$\widehat{\lambda}_B = \frac{D+a}{\sum\limits_{i=1}^{D} x_i + (n-D)T + b}. \tag{40}$$

5.1.1 One-sample Bayesian prediction

The Bayesian predictive density function of $X_{s:n}$, given the unified hybrid censored sample, in this case is given as in (19), where

$$f_1^*(x_s|\underline{\mathbf{x}}) = I^{-1}\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1} C_{1w}\int_0^\infty \lambda^{d_1+a}\phi_{d_1}(T_1,\lambda) \times \exp\left\{-\lambda\left[\sum_{i=1}^{d_1} x_i + (n-d_1)T_1 + (n-s+w+1)(x_s-T_1) + b\right]\right\} d\lambda, \tag{41}$$

with

$$\phi_{d_1}(T_1,\lambda) = \frac{\binom{n}{d_1}\exp[-(n-d_1)\lambda T_1](1-\exp[-\lambda T_1])^{d_1}}{\sum\limits_{d_1=r}^{s-1}\binom{n}{d_1}\exp[-(n-d_1)\lambda T_1](1-\exp[-\lambda T_1])^{d_1}},$$

$$f_2^*(x_s|\underline{\mathbf{x}}) = I^{-1}\Gamma(r+a+1)\sum_{w=0}^{s-r-1} C_{2w}\left[\sum_{i=1}^{r} x_i + (n-r)x_r + (n-s+w+1)(x_s-x_r) + b\right]^{-(r+a+1)}, \tag{42}$$

$$f_3^*(x_s|\underline{\mathbf{x}}) = I^{-1}\sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1} C_{3w}\int_0^\infty \lambda^{d_2+a}\gamma_{d_2}(T_2,\lambda) \times \exp\left\{-\lambda\left[\sum_{i=1}^{d_2} x_i + (n-d_2)T_2 + (n-s+w+1)(x_s-T_2) + b\right]\right\} d\lambda, \tag{43}$$

with

$$\gamma_{d_2}(T_2,\lambda)=\frac{\binom{n}{d_2}\exp[-(n-d_2)\lambda T_2](1-\exp[-\lambda T_2])^{d_2}}{\sum_{d_2=k}^{r^*-1}\binom{n}{d_2}\exp[-(n-d_2)\lambda T_2](1-\exp[-\lambda T_2])^{d_2}},$$

and

$$f_4^*(x_s|\underline{\mathbf{x}})=I^{-1}\Gamma(k+a+1)\sum_{w=0}^{s-k-1}C_{4w}\left[\sum_{i=1}^{k}x_i+(n-r)x_k+(n-s+w+1)(x_s-x_k)+b\right]^{-(k+a+1)}. \quad (44)$$

The Bayesian predictive survival function of $X_{s:n}$, given the unified hybrid censored sample , is then given as in (24), where

$$\bar{F}_1^*(t|\underline{\mathbf{x}})=I^{-1}\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1}\frac{C_{1w}}{(n-s+w+1)}\int_0^\infty\lambda^{d_1+a}\phi_d(T_1,\lambda)$$
$$\times\exp\left\{-\lambda\left[\sum_{i=1}^{d_1}x_i+(n-d_1)T_1+(n-s+w+1)(t-T_1)+b\right]\right\}d\lambda, \quad (45)$$

$$\bar{F}_2^*(t|\underline{\mathbf{x}})=I^{-1}\Gamma(r+a)\sum_{w=0}^{s-r-1}\frac{C_{2w}}{(n-s+w+1)}\left[\sum_{i=1}^{r}x_i+(n-r)x_r+(n-s+w+1)(t-x_r)+b\right]^{-(r+a)}, \quad (46)$$

$$\bar{F}_3^*(t|\underline{\mathbf{x}})=I^{-1}\sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1}\frac{C_{3w}}{(n-s+w+1)}\int_0^\infty\lambda^{d_2+a}\gamma_d(T_2,\lambda)$$
$$\times\exp\left\{-\lambda\left[\sum_{i=1}^{d_2}x_i+(n-d_2)T_2+(n-s+w+1)(t-T_2)+b\right]\right\}d\lambda, \quad (47)$$

and

$$\bar{F}_4^*(t|\underline{\mathbf{x}})=I^{-1}\Gamma(k+a)\sum_{w=0}^{s-k-1}\frac{C_{4w}}{(n-s+w+1)}\left[\sum_{i=1}^{k}x_i+(n-r)x_k+(n-s+w+1)(t-x_k)+b\right]^{-(k+a)}. \quad (48)$$

5.1.2 Two-sample Bayesian prediction

The Bayesian predictive density function of $Y_{s:m}$, given the unified hybrid censored sample, is given by

$$f_{Y_{s:m}}^*(y_s|\underline{\mathbf{x}})=I^{-1}\Gamma(D+a+1)\sum_{w=0}^{s-1}C_{5w}\left[\sum_{i=1}^{D}x_i+(n-D)T+(m-s+w+1)y_s+b\right]^{-(D+a+1)}, \quad (49)$$

and the Bayesian predictive survival function of $Y_{s:m}$, given the unified hybrid censored sample, is then given by

$$\bar{F}_{Y_{s:m}}^*(t|\underline{\mathbf{x}})=I^{-1}\Gamma(D+a)\sum_{w=0}^{s-1}\frac{C_{5w}}{(m-s+w+1)}\left[\sum_{i=1}^{D}x_i+(n-D)T+(m-s+w+1)t+b\right]^{-(D+a)} \quad (50)$$

5.2 The Pareto distribution

The survival function for Pareto distribution is given by

$$\bar{F}(x|\alpha,\beta)=\left(\frac{\beta}{x}\right)^{\alpha},x\geq\beta, \quad (51)$$

where $\alpha > 0$ and $\beta > 0$, and so we have

$$\psi(x;\alpha,\beta) = \alpha\log\left(\frac{x}{\beta}\right) \text{ and } \psi'(x;\alpha,\beta) = \frac{\alpha}{x}.$$

Thus, the likelihood function of α and β using the unified hybrid censored sample, is

$$L(\alpha,\beta;\underline{\mathbf{x}}) = \frac{n!}{(n-D)!}\,\alpha^D\left(\prod_{i=1}^{D}\frac{1}{x_i}\right)\exp\left\{-\alpha\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - n\log\beta\right]\right\}, \tag{52}$$

where D and T are given as in (4).

It is clear that the likelihood function is monotone increasing function in β, so its maximum value $\widehat{\beta}_{ML}$ will be attained at the minimum value x_1 of β. So, the ML estimator of α is obtained as

$$\widehat{\alpha}_{ML} = \frac{D}{\sum_{i=1}^{D}\log x_i + (n-D)\log T - n\log x_1}. \tag{53}$$

For the Bayesian estimation and prediction, we consider here the joint prior density function of α and β which was suggested by Lwin in [21] and generalized by Arnold and Press in [22], and given by

$$\pi(\alpha,\beta;\delta) \propto \alpha^a\beta^{-1}\exp[-\alpha(\log c - b\log\beta)],\ \ \alpha>0,\ 0<\beta<d, \tag{54}$$

where $a,\ b,\ c,\ d$ are positive constants and $d^b < c$. Then, we have

$$A(\alpha,\beta;\delta) = \alpha^a\beta^{-1} \text{ and } B(\alpha,\beta;\delta) = \alpha(\log c - b\log\beta),$$

where $\delta = (a,b,c,d)$. The posterior density function of α and β is then given by (8), where

$$\eta(\alpha,\beta;\underline{\mathbf{x}}) = \alpha^{D+a}\beta^{-1},$$

$$\zeta(\alpha,\beta;\underline{\mathbf{x}}) = \alpha\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log\beta + \log c\right],$$

and

$$I = \frac{\Gamma(D+a)}{n+b}\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c\right]^{-(D+a)},$$

with $x_0 = \min(x_1,d)$.

Hence, under the squared error loss function, the Bayesian estimator of α is obtained as

$$\widehat{\alpha}_B = \frac{D+a}{\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c}, \tag{55}$$

and the Bayesian estimator of β is obtained as

$$\begin{aligned}
\widehat{\beta}_B &= I^{-1}x_0\int_0^\infty \frac{\alpha^{D+a}}{\alpha(n+b)+1}\exp\left\{-\alpha\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c\right]\right\}d\alpha \\
&= \frac{I^{-1}x_0}{(n+b)}\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c\right]^{-(D+a)} \\
&\quad\times\int_0^\infty \frac{t^{D+a}e^{-t}}{t + \left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c\right]/(n+b)}dt \\
&= \frac{x_0}{\Gamma(m+a)}\Phi\left(D+a, \frac{\left[\sum_{i=1}^{D}\log x_i + (n-D)\log T - (n+b)\log x_0 + \log c\right]}{(n+b)}\right),
\end{aligned} \tag{56}$$

where

$$\Phi(x,y)=\int_0^\infty \frac{t^x e^{-t}}{t+y}dt.$$

A partial tabulation of $\psi(x,y)=(y/\Gamma(x))\Phi(x-1,y)$ has been provided by Arnold and Press in [22].

5.2.1 One-sample Bayesian prediction

The Bayesian predictive density function of $X_{s:n}$, given the unified hybrid censored sample, is given as in (19), where

$$f_1^*(x_s|\underline{\mathbf{x}})=\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1}\frac{C_{1w}}{x_s}\int_0^{x_0}\int_0^\infty \alpha^{d_1+a+1}\beta^{-1}\phi_{d_1}(T_1;\alpha,\beta)$$
$$\times\exp\left\{-\alpha\left[\sum_{i=1}^{d_1}\log x_i+(n-d_1)\log T_1-(n+b)\log\beta\right.\right.$$
$$\left.\left.+(n-s+w+1)(\log x_s-\log T_1)+\log c\right]\right\}d\alpha d\beta, \tag{57}$$

with

$$\phi_{d_1}(T_1;\alpha,\beta)=\frac{\binom{n}{d_1}(\beta/T_1)^{(n-d_1)\alpha}(1-\beta/T_1)^{d_1}}{\sum_{d_1=r}^{s-1}\binom{n}{d_1}(\beta/T_1)^{(n-d_1)\alpha}(1-\beta/T_1)^{d_1}},$$

$$f_2^*(x_s|\underline{\mathbf{x}})=\frac{I^{-1}\Gamma(r+a+1)}{n+b}\sum_{w=0}^{s-r-1}\frac{C_{2w}}{x_s}\left[\sum_{i=1}^{r}\log x_i+(n-r)\log x_r-(n+b)\log x_0\right.$$
$$\left.+(n-s+w+1)(\log x_s-\log x_r)+\log c\right]^{-(r+a+1)}, \tag{58}$$

$$f_3^*(x_s|\underline{\mathbf{x}})=\sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1}\frac{C_{3w}}{x_s}\int_0^{x_0}\int_0^\infty \alpha^{d_2+a+1}\beta^{-1}\gamma_{d_2}(T_2;\alpha,\beta)$$
$$\times\exp\left\{-\alpha\left[\sum_{i=1}^{d_2}\log x_i+(n-d_2)\log T_2-(n+b)\log\beta\right.\right.$$
$$\left.\left.+(n-s+w+1)(\log x_s-\log T_2)+\log c\right]\right\}d\alpha d\beta, \tag{59}$$

with

$$\gamma_{d_2}(T_2;\alpha,\beta)=\frac{\binom{n}{d_2}(\beta/T_2)^{(n-d_2)\alpha}(1-\beta/T_2)^{d_2}}{\sum_{d_2=k}^{r^*-1}\binom{n}{d_2}(\beta/T_2)^{(n-d_2)\alpha}(1-\beta/T_2)^{d_2}},$$

and

$$f_4^*(x_s|\underline{\mathbf{x}})=\frac{I^{-1}\Gamma(k+a+1)}{n+b}\sum_{w=0}^{s-k-1}\frac{C_{4w}}{x_s}\left[\sum_{i=1}^{k}\log x_i+(n-k)\log x_k-(n+b)\log x_0\right.$$
$$\left.+(n-s+w+1)(\log x_s-\log x_k)+\log c\right]^{-(k+a+1)}. \tag{60}$$

Therefore, the Bayesian predictive survival function of $X_{s:n}$, given the unified hybrid censored sample, is given as in (24), where

$$\bar{F}_1^*(t|\underline{\mathbf{x}})=\sum_{d_1=r}^{s-1}\sum_{w=0}^{s-d_1-1}\frac{C_{1w}}{n-s+w+1}\int_0^{x_0}\int_0^\infty \alpha^{d_1+a}\beta^{-1}\phi_{d_1}(T_1;\alpha,\beta)$$
$$\times\exp\left\{-\alpha\left[\sum_{i=1}^{d_1}\log x_i+(n-d_1)\log T_1-(n+b)\log\beta\right.\right.$$
$$\left.\left.+(n-s+w+1)(\log t-\log T_1)+\log c\right]\right\}d\alpha d\beta, \tag{61}$$

$$\bar{F}_2^*(t|\underline{\mathbf{x}}) = \frac{I^{-1}\Gamma(r+a)}{n+b}\sum_{w=0}^{s-r-1}\frac{C_{2w}}{n-s+w+1}\left[\sum_{i=1}^{r}\log x_i+(n-r)\log x_r-(n+b)\log x_0\right.$$
$$\left.+(n-s+w+1)(\log t-\log x_r)+\log c\right]^{-(r+a)}, \tag{62}$$

$$\bar{F}_3^*(t|\underline{\mathbf{x}}) = \sum_{d_2=k}^{r^*-1}\sum_{w=0}^{s-d_2-1}\frac{C_{3w}}{n-s+w+1}\int_0^{x_0}\int_0^{\infty}\alpha^{d_2+a}\beta^{-1}\gamma_{d_2}(T_2;\alpha,\beta)$$
$$\times\exp\left\{-\alpha\left[\sum_{i=1}^{d_2}\log x_i+(n-d_2)\log T_2-(n+b)\log\beta\right.\right.$$
$$\left.\left.+(n-s+w+1)(\log t-\log T_2)+\log c\right]\right\}d\alpha d\beta, \tag{63}$$

and

$$\bar{F}_4^*(t|\underline{\mathbf{x}}) = \frac{I^{-1}\Gamma(k+a)}{n+b}\sum_{w=0}^{s-k-1}\frac{C_{4w}}{n-s+w+1}\left[\sum_{i=1}^{k}\log x_i+(n-k)\log x_k-(n+b)\log x_0\right.$$
$$\left.+(n-s+w+1)(\log t-\log x_k)+\log c\right]^{-(k+a+1)}. \tag{64}$$

5.2.2 Two-sample Bayesian prediction

The Bayesian predictive density function of $Y_{s:m}$, given the unified hybrid censored sample, is then given by

$$f^*_{Y_{s:m}}(y_s|\underline{\mathbf{x}}) = \begin{cases} f^*_{1Y_{s:m}}(y_s|\underline{\mathbf{x}}), & 0<y_s\le x_0,\\ f^*_{2Y_{s:m}}(y_s|\underline{\mathbf{x}}), & y_s>x_0,\end{cases} \tag{65}$$

where

$$f^*_{1Y_{s:m}}(y_s|\underline{\mathbf{x}}) = \int_0^{y_s}\int_0^{\infty} f_{Y_{s:m}}(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\beta$$
$$= I^{-1}\Gamma(D+a+1)\sum_{w=0}^{s-1}\frac{C_{5w}y_s^{-1}}{n+b+m-s+w+1}$$
$$\times\left[\sum_{i=1}^{D}\log x_i+(n-D)\log T-(n+b)\log y_s+\log c\right]^{-(D+a+1)}$$

and

$$f^*_{2Y_{s:m}}(y_s|\underline{\mathbf{x}}) = \int_0^{x_0}\int_0^{\infty} f_{Y_{s:m}}(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\beta|\underline{\mathbf{x}})d\alpha d\beta$$
$$= I^{-1}\Gamma(D+a+1)\sum_{w=0}^{s-1}\frac{C_{5w}y_s^{-1}}{n+b+m-s+w+1}\left[\sum_{i=1}^{D}\log x_i+(n-D)\log T\right.$$
$$\left.-(n+b+m-s+w+1)\log x_0+(m-s+w+1)\log y_s+\log c\right]^{-(D+a+1)}.$$

Therefore, we can obtain the predictive survival function of $Y_{s:m}$, given the unified hybrid censored sample, as

$$\bar{F}^*_{Y_{s:m}}(t|\underline{\mathbf{x}}) = \begin{cases} \bar{F}^*_{1Y_{s:m}}(t|\underline{\mathbf{x}}), & 0<t\le x_0,\\ \bar{F}^*_{2Y_{s:m}}(t|\underline{\mathbf{x}}), & t>x_0,\end{cases} \tag{66}$$

where

$$\bar{F}^*_{1Y_{s:m}}(t|\underline{\mathbf{x}}) = \int_t^{x_0} f^*_{1Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s + \int_{x_0}^{\infty} f^*_{2Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s$$
$$= I^{-1}\Gamma(D+a)\sum_{w=0}^{s-1}\frac{C_{5w}}{(n+b+m-s+w+1)(n+b)(m-s+w+1)}$$
$$\times\Bigg\{(n+b+m-s+w+1)\left[\sum_{i=1}^{D}\log x_i+(n-D)\log T-(n+b)\log x_0+\log c\right]^{-(D+a)}$$
$$-(m-s+w+1)\left[\sum_{i=1}^{D}\log x_i+(n-D)\log T-(n+b)\log t+\log c\right]^{-(D+a)}\Bigg\}, \tag{67}$$

and

$$\bar{F}^*_{2Y_{s:m}}(t|\underline{\mathbf{x}}) = \int_t^{\infty} f^*_{2Y_{s:m}}(y_s|\underline{\mathbf{x}})dy_s$$
$$= I^{-1}\Gamma(D+a)\sum_{w=0}^{s-1}\frac{C_{5w}}{(n+b+m-s+w+1)(m-s+w+1)}\left[\sum_{i=1}^{D}\log x_i+(n-D)\log T\right.$$
$$\left.-(n+b+m-s+w+1)\log x_0+(m-s+w+1)\log t+\log c\right]^{-(D+a)}. \tag{68}$$

6 Numerical Results

In this section, we present two numerical examples to illustrate the inferential procedures developed in the preceding sections.

6.1 Numerical example for the exponential distribution

In order to illustrate all the inferential results established for the exponential distribution, we consider here the following real data: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71 and 72.89, which consisting of the times to breakdown on insulating fluids tested at 34 kilovolts produced by Viveros and Balakrishnan in [23] (1994) from Table 6.1 of ([24], p.228). We assume the exponential distribution for these times to breakdown and consider the following four unified hybrid censoring schemes:

1. Scheme 1: Suppose $k=4$, $r=6$, $T_1=4$ and $T_2=8$, then $x_{4:19}<x_{6:19}<T_1$ and the experiment would have terminated at $T_1=4$. Therefore, we would have the following data: 0.19, 0.78, 0.96, 1.31, 2.78 and 3.16;
2. Scheme 2: Suppose $k=6$, $r=9$, $T_1=4$ and $T_2=8$, then $x_{6:19}<T_1<x_{9:19}<T_2$ and the experiment would have terminated at $x_{9:20}=4.85$. Therefore, we would have the following data: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67 and 4.85;
3. Scheme 3: Suppose $k=6$, $r=12$, $T_1=4$ and $T_2=8$, then $x_{6:19}<T_1<T_2<x_{12:19}$ and the experiment would have terminated at $T_2=8$. Therefore, we would have the following data: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.5 and 7.35;
4. Scheme 4: Suppose $k=14$, $r=15$, $T_1=4$ and $T_2=8$, then $T_1<T_2<x_{14:19}<x_{15:19}$ and the experiment would have terminated at $x_{14:19}=12.06$. Therefore, we would have the following data: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27 and 12.06.

Based on the above four unified hybrid censoring schemes, we used the results presented in Subsection 5.1 to calculate the ML and Bayesian estimates of the unknown parameter λ. Also, we calculated the point predictor and 95% two-sided ET and HPD prediction intervals for the future order statistics $X_{s:19}$, where $15\le s\le 19$, from the same sample and that for the order statistics $Y_{s:10}$, where1 $\le s\le 10$, from a future unobserved sample with size $m=10$. All obtained results for the Bayesian estimation and prediction, presented in Tables 1-3, are computed based on two different choices of the hyperparameters a and b, namely,

Table 1: The ML and Bayesian estimates of λ.

	$\hat{\lambda}_{ML}$	$\hat{\lambda}_B$	
		GP	*JP*
Scheme1	0.0981	0.0857	0.0817
Scheme2	0.1261	0.1119	0.1121
Scheme3	0.1092	0.1003	0.0993
Scheme4	0.1117	0.1042	0.1037

Table 2: Bayesian point predictor and 95% ET and HPD prediction intervals for $X_{s:19}$ for $s = 15,\ldots,19$.

		GP			*JP*		
Scheme	s	$\hat{X}_{s:19}$	ET interval	HPD interval	$\hat{X}_{s:19}$	ET interval	HPD interval
Scheme1	15	19.308	(8.529,43.908)	(6.851,37.456)	20.776	(8.577,50.829)	(6.705,42.346)
	16	22.797	(2.128,52.681)	(7.657,44.848)	24.599	(9.754,61.165)	(7.471,50.848)
	17	27.449	(11.203,64.626)	(8.670,54.863)	29.698	(11.267,75.199)	(8.434,62.345)
	18	34.428	(1.546,83.231)	(10.023,70.323)	37.345	(13.383,96.950)	(9.723,80.031)
	19	48.385	(16.853,123.775)	(12.055,103.324)	52.640	(16.980,143.852)	(11.672,117.498)
Scheme2	15	13.343	(7.242,25.941)	(6.318,22.972)	13.469	(7.201,26.873)	(6.248,23.591)
	16	15.854	(8.163,31.623)	(6.989,27.939)	16.018	(8.102,32.833)	(6.888,28.746)
	17	19.201	(9.356,39.387)	(7.848,34.679)	19.415	(9.270,40.966)	(7.707,35.735)
	18	24.223	(11.028,51.585)	(9.003,45.138)	24.512	(10.907,53.706)	(8.810,46.561)
	19	34.266	(13.841,78.677)	(10.720,67.758)	34.705	(13.666,81.838)	(10.456,69.879)
Scheme3	15	14.955	(9.576,26.072)	(8.796,23.453)	15.100	(9.579,26.759)	(8.778,23.947)
	16	17.695	(10.550,32.106)	(7.087,28.285)	17.897	(10.552,33.054)	(9.457,29.490)
	17	21.348	(11.850,40.327)	(10.425,36.033)	21.626	(11.848,41.623)	(10.374,36.974)
	18	26.828	(13.697,53.257)	(11.715,47.244)	27.221	(13.690,55.076)	(11.638,48.571)
	19	37.789	(16.823,82.156)	(13.647,71.579)	38.410	(16.810,85.023)	(13.538,73.670)
Scheme4	15	14.126	(12.109,20.154)	(12.060,18.468)	14.149	(12.109,20.285)	(12.060,18.557)
	16	16.709	(12.567,26.300)	(12.122,23.839)	16.760	(12.568,26.565)	(12.121,24.025)
	17	20.153	(13.514,34.158)	(12.610,30.805)	20.242	(13.515,34.602)	(12.600,31.122)
	18	25.318	(15.071,46.263)	(13.594,41.415)	25.464	(15.070,46.978)	(13.571,41.929)
	19	35.650	(17.876,73.194)	(10.136,63.163)	35.909	(17.870,74.452)	(10.166,64.086)

1.: $a = 0.1$ and $b = 10$ Gamma informative prior (GP) by letting the mean of the prior distribution of λ is 0.01 and its variance is 0.001).
2.: $a = 0$ and $b = 0$ Jeffreys non-informative prior (JP).

From the results in Tables 5 and 6, we notice that, The point predictor of mean is between the upper and lower bounds of the prediction intervals. Also, a comparison of the results for the Gamma informative prior with the corresponding ones for Jeffreys non-informative prior reveals that the former produce more precise results, as we would expect. Moreover, the HPD prediction intervals seem to be more precise than the ET prediction intervals, Finally when we use the same value of T_1 and T_2 but increasing k and r, the Bayesian prediction bounds become tighter as expected since the duration of the life-testing experiment is longer in this case.

6.2 Numerical example for the Pareto distribution

In order to illustrate all the inferential results established for the Pareto distribution, we generated order statistics from a sample of size $n = 20$ from the Pareto distribution with $\alpha = 5$ and $\beta = 7$. The generated order statistics as follows: 7.032, 7.159, 7.307, 7.340, 7.583, 7.718, 7.742, 7.744, 7.785, 7.826, 7.903, 8.874, 9.100, 9.141, 9.237, 9.417, 9.660, 10.132, 12.499 and 21.590. We will apply the following four unified hybrid censoring schemes:

1.Scheme 1: Suppose $k = 2$, $r = 4$, $T_1 = 7.650$ and $T_2 = 7.800$, then $x_{2:20} < x_{4:20} < T_1$ and the experiment would have terminated at $T_1 = 7.65$. Therefore, we would have the following data: 7.032, 7.159, 7.307, 7.340, and 7.583;

Table 3: Bayesian point predictor and 95% ET and HPD prediction intervals for $Y_{s:10}$ for $s = 1,\dots,10$.

		IP			*NIP*		
Scheme	*s*	$\widehat{Y}_{s:10}$	ET interval	HPD interval	$\widehat{Y}_{s:10}$	ET interval	HPD interval
Scheme1	1	1.396	(0.030,5.913)	(0.000,4.514)	1.530	(0.031,6.676)	(0.000,5.020)
	2	2.946	(0.281,10.093)	(0.023,8.002)	3.229	(0.292,11.550)	(0.020,8.999)
	3	4.691	(0.729,14.546)	(0.213,11.774)	5.141	(0.751,16.776)	(0.198,13.318)
	4	6.685	(1.325,19.548)	(0.560,16.024)	7.326	(1.356,22.665)	(0.526,18.194)
	5	9.011	(2.063,25.363)	(1.031,20.954)	9.875	(2.103,29.516)	(0.974,23.860)
	6	11.802	(2.967,32.370)	(1.629,26.878)	12.934	(3.016,37.769)	(1.542,30.668)
	7	15.292	(4.092,41.223)	(2.378,34.335)	16.758	(4.151,48.185)	(2.254,39.234)
	8	19.944	(5.550,53.264)	(3.337,44.423)	21.856	(5.621,62.318)	(3.167,50.804)
	9	26.922	(7.597,72.012)	(4.631,59.984)	29.504	(7.687,84.219)	(4.400,68.593)
	10	40.879	(11.065,112.863)	(6.580,93.191)	44.799	(11.202,131.456)	(6.268,106.278)
Scheme2	1	1.004	(0.023,4.066)	(0.000,3.171)	1.019	(0.023,4.180)	(0.000,3.241)
	2	2.120	(0.219,6.796)	(0.023,5.527)	2.152	(0.218,7.028)	(0.021,5.676)
	3	3.376	(0.576,9.669)	(0.198,8.059)	3.426	(0.570,10.036)	(0.187,8.297)
	4	4.810	(1.055,12.880)	(0.510,10.896)	4.882	(1.041,13.404)	(0.484,11.239)
	5	6.484	(1.655,16.604)	(0.932,14.180)	6.581	(1.629,17.313)	(0.887,14.648)
	6	8.493	(2.392,21.091)	(1.468,18.122)	8.619	(2.351,22.023)	(1.398,18.741)
	7	11.004	(3.313,26.773)	(2.139,23.089)	11.168	(3.252,27.983)	(2.038,23.896)
	8	14.351	(4.506,34.537)	(2.998,29.829)	14.565	(4.420,36.116)	(2.857,30.885)
	9	19.373	(6.178,46.735)	(4.153,40.288)	19.662	(6.057,48.856)	(3.960,41.711)
	10	29.416	(8.991,73.827)	(5.870,62.908)	29.855	(8.816,76.988)	(5.606,65.029)
Scheme3	1	1.096	(0.025,4.364)	(0.000,3.430)	1.119	(0.026,4.492)	(0.000,3.517)
	2	2.314	(0.246,7.236)	(0.028,5.939)	2.362	(0.248,7.479)	(0.027,6.110)
	3	3.684	(0.650,10.242)	(0.237,8.628)	3.761	(0.653,10.614)	(0.231,8.893)
	4	5.250	(1.196,13.594)	(0.607,11.636)	5.359	(1.198,14.113)	(0.593,12.009)
	5	7.076	(1.880,17.477)	(1.108,15.113)	7.224	(1.881,18.169)	(1.084,15.613)
	6	9.268	(2.724,22.156)	(1.743,19.284)	9.462	(2.722,23.056)	(1.706,19.938)
	7	12.009	(3.779,28.086)	(2.540,24.542)	12.259	(3.773,29.248)	(2.485,25.389)
	8	15.662	(5.147,36.208)	(3.558,31.684)	15.989	(5.135,37.718)	(3.482,32.789)
	9	21.142	(7.062,49.018)	(4.924,42.799)	21.583	(7.043,51.051)	(4.822,44.289)
	10	32.103	(10.273,77.712)	(6.943,66.981)	32.772	(10.248,80.785)	(6.807,69.232)
Scheme4	1	1.033	(0.024,4.047)	(0.000,3.204)	1.045	(0.024,4.113)	(0.000,3.248)
	2	2.181	(0.239,6.658)	(0.030,5.513)	2.205	(0.239,6.782)	(0.029,5.601)
	3	3.472	(0.633,9.377)	(0.245,7.981)	3.511	(0.634,9.567)	(0.241,8.117)
	4	4.948	(1.169,12.399)	(0.624,10.735)	5.003	(1.168,12.666)	(0.614,10.927)
	5	6.670	(1.843,15.896)	(1.136,13.914)	6.744	(1.840,16.253)	(1.119,14.172)
	6	8.737	(2.677,20.109)	(1.786,17.724)	8.833	(2.671,20.575)	(1.759,18.063)
	7	11.319	(3.720,25.454)	(2.601,22.529)	11.444	(3.710,26.056)	(2.562,22.969)
	8	14.763	(5.074,32.791)	(3.642,29.066)	14.926	(5.058,33.574)	(3.588,29.640)
	9	19.929	(6.968,44.418)	(5.036,39.269)	20.148	(6.944,45.470)	(4.962,40.042)
	10	30.260	(10.133,70.698)	(7.082,61.608)	30.593	(10.099,72.278)	(6.984,62.770)

2.Scheme 2: Suppose $k = 4$, $r = 7$, $T_1 = 7.650$ and $T_2 = 7.800$, then $x_{4:20} < T_1 < x_{7:20} < T_2$ and the experiment would have terminated at $x_{7:20} = 45$. Therefore, we would have the following data: 7.032, 7.159, 7.307, 7.340, 7.583, 7.718, and 7.742;

3.Scheme 3: Suppose $k = 6$, $r = 12$, $T_1 = 7.650$ and $T_2 = 7.800$, then $T_1 < x_{6:20} < T_2 < x_{14:20}$ and the experiment would have terminated at $T_2 = 7.800$. Therefore, we would have the following data: 7.032, 7.159, 7.307, 7.340, 7.583, 7.718, 7.742, and 7.744;

4.Scheme 4: Suppose $k = 11$, $r = 13$, $T_1 = 7.650$ and $T_2 = 7.800$, then $T_1 < T_2 < x_{11:20} < x_{13:20}$ and the experiment would have terminated at $x_{11:20} = 7.903$. Therefore, we would have the following data: 7.032, 7.159, 7.307, 7.340, 7.583, 7.718, 7.742, 7.744, 7.785, 7.826, and 7.903.

Table 4: The ML and Bayesian estimates of α and β.

	$\hat{\alpha}_{ML}$	$\hat{\alpha}_B$		$\hat{\beta}_{ML}$	$\hat{\beta}_B$	
		IP	NIP		IP	NIP
Scheme1	3.476	3.837	2.780	7.032	6.931	6.869
Scheme2	4.335	4.387	3.716	-	6.945	6.921
Scheme3	5.286	5.033	4.698	-	6.958	6.947
Scheme4	5.988	5.536	5.444	-	6.965	6.961

Table 5: Bayesian point predictor and 95% ET and HPD prediction intervals for $X_{s:20}$ for $s = 16,\ldots,20$.

s		IP			NIP		
		$\widehat{X}_{s:20}$	ET interval	HPD interval	$\widehat{X}_{s:20}$	ET interval	HPD interval
Scheme1	16	13.265	(8.226,16.791)	(7.618,15.054)	13.379	(5.648,38.244)	(8.067,27.318)
	17	14.985	(7.658,19.650)	(8.191,17.157)	15.219	(6.374,52.802)	(7.393,34.803)
	18	15.096	(8.597,24.366)	(7.687,20.644)	16.609	(8.582,81.710)	(7.709,49.503)
	19	21.318	(9.079,34.161)	(8.667,27.384)	22.079	(8.881,160.138)	(7.290,81.055)
	20	29.648	(9.447,72.251)	(8.878,50.664)	31.679	(9.409,671.114)	(7.591,252.658)
Scheme2	16	10.968	(8.268,14.468)	(8.084,13.348)	11.020	(6.526,19.590)	(8.080,16.699)
	17	13.451	(8.491,16.577)	(8.234,15.006)	13.689	(6.639,24.027)	(8.163,19.828)
	18	14.458	(8.732,19.980)	(8.382,17.635)	15.986	(8.739,31.724)	(8.300,24.877)
	19	16.008	(9.441,26.827)	(8.738,22.643)	17.761	(9.086,48.899)	(8.441,35.470)
	20	22.124	(9.688,51.876)	(8.966,39.084)	23.145	(9.703,125.436)	(8.490,75.559)
Scheme3	16	9.615	(8.310,12.297)	(8.196,11.521)	10.539	(8.592,13.647)	(8.092,12.536)
	17	10.965	(8.741,13.762)	(8.312,12.819)	11.179	(6.810,15.695)	(8.187,14.122)
	18	13.984	(9.027,16.061)	(8.484,14.663)	14.484	(9.137,18.998)	(8.318,16.600)
	19	15.457	(9.534,20.511)	(8.502,18.082)	16.219	(9.588,25.628)	(8.478,21.219)
	20	19.968	(10.177,35.580)	(8.791,28.600)	20.989	(10.404,49.642)	(8.664,36.597)
Scheme4	16	9.219	(8.523,11.154)	(8.240,10.667)	10.303	(8.711,11.599)	(8.181,10.967)
	17	9.456	(8.789,12.336)	(8.371,11.666)	10.693	(8.980,12.996)	(8.308,12.106)
	18	11.847	(9.092,14.162)	(8.539,13.060)	13.359	(9.334,15.180)	(8.433,13.738)
	19	14.959	(8.905,17.621)	(8.569,15.903)	15.725	(9.856,19.378)	(8.602,17.023)
	20	18.254	(10.324,28.878)	(8.718,24.035)	19.270	(10.816,33.393)	(8.761,26.527)

Based on the above four unified hybrid censoring schemes, we used the results presented in Subsection 5.2 to calculate the ML and Bayesian estimates of the unknown parameters α and β. Also, we calculate the point predictor and 95% ET and HPD prediction intervals for the future order statistics $X_{s:20}$, where $16 \le s \le 20$, from the same sample and that for the order statistics $Y_{s:10}$, where $1 \le s \le 10$, from a future unobserved sample with size $m = 10$. All obtained results for the Bayesian estimation and prediction, presented in Tables 4-6, are computed based on two different choices of the hyperparameters (a,b,c,d), namely,

1.: $a = 3.57$, $b = 0.20$, $c = 3.27$ and $d = 10.83$: informative prior (IP) (by letting the mean of the marginal prior distribution of α is 5 and its variance is 7, and the median of the marginal prior distribution of β is 0.5 and its third quartile is 0.25).
2.: $a = -1$, $b = 0$, $c = 1$ and $d = \infty$: noninformative prior (NIP).

From the results in Tables 5 and 6, we notice that, The point predictor of mean is between the upper and lower bounds of the prediction intervals. Also, a comparison of the results for the informative prior with the corresponding ones for non-informative prior reveals that the former produce more precise results, as we would expect. Moreover, the HPD prediction intervals seem to be more precise than the ET prediction intervals, Finally when we use the same value of T_1 and T_2 but increasing k and r, the Bayesian prediction bounds become tighter as expected since the duration of the life-testing experiment is longer in this case.

Table 6: Bayesian point predictor and 95% ET and HPD prediction intervals for $Y_{s:10}$ for $s = 1,\ldots,10$.

s		IP			NIP		
		$\widehat{Y}_{s:10}$	ET interval	HPD interval	$\widehat{Y}_{s:10}$	ET interval	HPD interval
Scheme1	1	7.224	(6.586,8.444)	(6.457,8.218)	7.142	(6.764,7.806)	(6.703,7.709)
	2	7.671	(6.819,9.892)	(6.604,9.280)	7.386	(6.895,8.447)	(6.799,8.240)
	3	8.269	(7.010,11.778)	(6.750,10.606)	7.673	(7.018,9.187)	(6.897,8.844)
	4	9.136	(7.171,14.375)	(6.887,12.363)	8.018	(7.139 10.099)	(6.995,9.577)
	5	10.506	(7.343,18.158)	(7.012,14.814)	8.446	(7.276,11.275)	(7.092,10.508)
	6	12.880	(7.547,24.083)	(7.122,18.458)	9.000	(7.442,12.880)	(7.186,11.756)
	7	17.447	(7.800 ,34.411)	(7.219,24.394)	9.763	(7.649,15.249)	(7.277,13.555)
	8	27.511	(8.134,55.817)	(7.301,35.613)	10.931	(6.640,19.210)	(7.364,16.464)
	9	55.218	(8.624,117.759)	(7.367,63.752)	13.150	(8.319,27.611)	(7.447,22.301)
	10	182.722	(9.524,579.130)	(7.569,217.816)	21.649	(9.027,61.720)	(7.518,43.155)
Scheme2	1	7.153	(6.732,7.910)	(6.660,7.792)	7.125	(6.803,7.678)	(6.755,7.602)
	2	7.423	(6.882,8.687)	(6.764,8.411)	7.331	(6.913,8.193)	(6.837,8.037)
	3	7.744	(7.016,9.613)	(6.869,9.132)	7.572	(7.020,8.775)	(6.923,8.523)
	4	8.136	(7.139,10.781)	(6.971,10.024)	7.858	(7.127,9.476)	(7.010,9.102)
	5	8.632	(7.275,12.328)	(7.067,11.182)	8.209	(7.248,10.363)	(7.097,9.826)
	6	9.293	(7.438,14.497)	(7.157,12.766)	8.657	(7.396,11.546)	(7.184,10.778)
	7	10.253	(7.640,17.799)	(7.240,15.103)	9.260	(7.581,13.243)	(7.270,12.118)
	8	11.869	(7.905,23.541)	(7.316,18.988)	10.154	(7.825,15.984)	(7.354,14.225)
	9	15.554	(8.290,36.395)	(7.383,27.086)	11.742	(8.176,21.505)	(7.436,18.281)
	10	34.592	(8.981,94.068)	(7.437,58.055)	16.574	(8.798,41.769)	(7.511,31.675)
Scheme3	1	7.121	(6.807,7.666)	(6.758,7.589)	7.111	(6.836,7.577)	(6.796,7.515)
	2	7.320	(6.916,8.189)	(6.838,8.022)	7.287	(6.930,7.999)	(6.868,7.877)
	3	7.553	(7.019,8.787)	(6.920,8.509)	7.490	(7.022,8.469)	(6.943,8.276)
	4	7.830	(7.118,9.512)	(7.001,9.093)	7.731	(7.115,9.026)	(7.020,8.745)
	5	8.170	(7.229,10.434)	(7.081,9.826)	8.023	(7.222,9.720)	(7.098,9.324)
	6	8.605	(7.362,11.668)	(7.157,10.792)	8.392	(7.353,10.630)	(7.176,10.073)
	7	9.193	(7.527, 13.445)	(7.231,12.155)	8.882	(7.516,11.909)	(7.255,11.110)
	8	10.071	(6.717,16.319)	(7.300,14.297)	9.595	(7.731,13.919)	(7.334,12.704)
	9	11.660	(8.056,22.113)	(7.365,18.413)	10.817	(8.039,17.819)	(7.413,15.678)
	10	16.949	(8.606,43.285)	(7.422,31.888)	14.135	(8.580,31.082)	(7.488,24.918)
Scheme4	1	7.106	(6.843,7.552)	(6.805,7.493)	7.103	(6.856,7.516)	(6.821,7.463)
	2	7.272	(6.934,7.964)	(6.873,7.841)	7.260	(6.940,7.884)	(6.886,7.781)
	3	7.464	(7.021,8.425)	(6.943,8.226)	7.441	(7.023,8.289)	(6.954,8.129)
	4	7.690	(7.107,8.97)	(7.014,8.680)	7.654	(7.108,8.766)	(7.025,8.536)
	5	7.966	(7.204,9.657)	(7.085,9.241)	7.913	(7.206,9.353)	(7.097,9.032)
	6	8.313	(7.322,10.552)	(7.155,9.967)	8.236	(7.326,10.115)	(7.171,9.669)
	7	8.774	(7.468, 11.810)	(7.223,10.970)	8.664	(7.476,11.173)	(7.245,10.542)
	8	9.443	(7.660, 13.779)	(7.290,12.506)	9.278	(7.673,12.810)	(7.321,11.866)
	9	10.589	(7.934,17.571)	(7.355,15.350)	10.311	(7.955,15.917)	(7.398,14.290)
	10	13.734	(8.413,30.230)	(7.415,24.031)	12.977	(8.447,26.027)	(7.474,21.557)

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