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# Bayesian Analysis of a Scale Parameter of a New Class of Generalized Inverse Weibull Distribution Using Different Loss Functions

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**Abstract:** In this paper, we propose to obtain the Bayesian estimators of unknown parameter of a three parameter gamma inverse Weibull distribution, based on non-informative and informative priors using different loss functions. A real life example has been used to compare the performance of the estimates under different loss functions.

**Keywords:** Inverse Weibull distribution, informative and non- informative priors, loss functions.

## 1 Introduction

The Weibull distribution is one of the most popular distributions in the lifetime data analyzing because a wide variety of shapes with varying levels of its parameters can be created. During the past decades, extensive work has been done on this distribution in both the frequentist and Bayesian points of view, like, Johnson et al. (1995) and Kundu (2008). Moreover, the Weibull probability density function can be decreasing (or increasing) or unimodal, depending on the shape of distribution parameters. The inverse Weibull distribution (IW) is usually used in reliability and biological studies. The inverse Weibull distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. It can also be used to determine the cost effectiveness, maintenance periods of reliability centered maintenance activities and applications in medicine, reliability and ecology. The inverse Weibull distribution provides a good fit to several data such as the times to breakdown of an insulating fluid, subject to the action of a constant tension, see Nelson (1982).The inverse Weibull distribution has initiated a large volume of research .For example, Calabria and Pulcini (1990) have discussed the maximum likelihood and least square estimations of its parameters, and Calabria and Pulcini (1994) have considered Bayes 2-sample prediction of the distribution. Keller (1985) obtained the inverse Weibull model by investigating failures of mechanical components subject to degradation. The three- parameter generalized inverse Weibull (GIW) distribution,which extends to several distributions, and commonly used in the lifetime-literature, is more flexible than the inverse Weibull distribution. Mudholkar et al. (1994) and De Gusmao et al. (2011) introduced and discussed the three-parameter GIW distribution. Additional results on the generalizations of the inverse Weibull and related distributions with applicationsare given by Oluyede and Yang (2014).and Afaq Ahmad, S.P Ahmad and A.Ahmed (2015), disscued Bayesian Estimation of Exponentiated Inverted Weibull Distribution under Asymmetric Loss Functions. A new three-parameter distribution, called the new class of Generalized Inverse Weibull distribution (NGIWD) has been introduced recently by M.Pararai, Warahena Liyanage and B. O. Oluyede (2014).

$$f(x;\delta,\beta,\lambda)=\frac{\beta}{\Gamma(\delta)}\lambda^{\delta}\;x^{-(\beta\delta+1)}\;e^{-\lambda x^{-\beta}}\qquad o<x<\infty;\;\delta,\beta>0\;\;and\;\lambda>0,\tag{1.1}$$

where $\delta,\beta$ are the shape parameters, and $\lambda$ is the scale parameter.

## 2 Maximum Likelihood Estimation for the Scale Parameter λ Of NGIW Assuming Shape Parameters β And δ Are To Be Known

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Let us consider a random sample $\underline{x} = (x_1, x_2, ..., x_n)$ of size n from the New Generalized Inverse Weibull Distribution NGIWD. Then the likelihood function for the given sample observation is

$$L(\underline{x}/\lambda) = \prod_{i=1}^{n} \frac{\beta}{\Gamma(\delta)} \lambda^{\delta} x^{-(\beta\delta+1)} e^{-\lambda x^{-\beta}}$$

$$L(\underline{x}/\lambda) = \frac{\beta^n}{(\Gamma(\delta))^n} \lambda^{n\delta} \prod_{i=1}^{n} x_i^{-(\beta\delta+1)} e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}} \tag{2.1}$$

The log-likelihood function is

$$\ln L(\underline{x}/\lambda) = n\ln\beta - n\ln\Gamma(\delta) + n\delta\ln\lambda - (\beta\delta+1)\sum_{i=1}^{n}\ln(x_i) - \lambda\sum_{i=1}^{n} x_i^{-\beta} \tag{2.2}$$

As both shape parameters $\delta\, and\, \beta$ are assumed to be known, the ML estimator of scale parameter $\lambda$ is obtained by solving the

$$\frac{\partial \ln L(\underline{x}/\lambda)}{\partial\lambda} = \frac{n\delta}{\lambda} - \sum_{i=1}^{n} x_i^{-\beta} = 0$$

$$\hat{\lambda} = \frac{n\delta}{\sum_{i=1}^{n}\left(x_i^{-\beta}\right)} \tag{2.3}$$

## 3 Bayesian Inference Using Different Priors

The Bayesian inference requires appropriate choice of prior(s) for the parameter(s). From the Bayesian viewpoint, there is no clear cut way from which one can conclude that one prior is better than the other. Nevertheless, very often priors are chosen according to one’s subjective knowledge and beliefs. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s); otherwise, it is preferable to use non-informative prior(s). In this paper we utilize two non-informative (the Uniform and the Jeffrey’s) priors along with two informative (the Gamma and the exponential) priors for a New class of Generalized Inverse Weibull distribution.

The standard Uniform distribution is assumed as non-informative prior for the parameter $\lambda$. The

Uniform prior for $\lambda$ is

$$p_1(\lambda) \propto 1, \qquad 0 < \lambda < \infty \tag{3.1}$$

The Jeffrey’s prior proposed by Jeffrey, H.(1964), is given as:

$$P_2(\lambda) \propto \frac{1}{\lambda}, \qquad \lambda > 0 \tag{3.2}$$

The exponential prior, and the prior distribution is taken as

$$p_3(\lambda) = c_1 e^{-\lambda c_1}, \qquad c_1, \lambda > 0 \tag{3.3}$$

The gamma prior, and the prior distribution is taken as

$$p_4(\lambda) = \frac{b^a}{\Gamma a} e^{-\lambda b} \lambda^{a-1}, \qquad a, b, \lambda > 0 \tag{3.4}$$

With the above priors, we use three different loss functions for the model (1.1).

## 4 Bayesian Method of Estimation

In this section Bayesian estimation of the scale parameter of gamma inverse Weibull distribution is obtained by using various priors under different symmetric and asymmetric loss functions.

### *4.1 Posterior density under the Assumption of Uniform Prior*

Combining the prior distribution (3.1) and the likelihood function (2.1), the posterior density of $\lambda$ is derived as follows:

$$\pi_1(\lambda \mid \underline{x}) \propto \frac{\beta^n}{(\Gamma(\delta))^n} \lambda^{\delta n} \prod_{i=1}^{n} x_i^{-(\beta\delta+1)} \, e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}}$$

$$\pi_1(\lambda \mid \underline{x}) = K\lambda^{\delta n} e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}}$$

$$\pi_1(\lambda \mid \underline{x}) = K\lambda^{\delta n} e^{-\lambda T}$$

Where k is independent of $\lambda$ , $T = \sum_{i=1}^{n} x_i^{-\beta}$

$$\text{and } K^{-1} = \int_0^\infty \lambda^{n\delta} e^{-T\lambda} \, d\lambda$$

$$\Rightarrow K^{-1} = \frac{\Gamma(n\delta+1)}{T^{n\delta+1}}$$

Hence the posterior density of $\lambda$ is given as

$$\pi_1(\lambda \mid \underline{x}) = \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \lambda^{n\delta} \, e^{-\lambda T} \qquad , \lambda > 0 \qquad (4.1)$$

Where $T = \sum_{i=1}^{n} x_i^{-\beta}$ and $(n\delta+1)$ are the parameters of the posterior distribution similar to the gamma distribution $G(T, n\delta+1)$

### *4.2 Posterior density under the Assumption of Jeffrey's prior*

Combining the prior distribution (3.2) and the likelihood function (2.1), the posterior density of $\lambda$ is derived as follows:

$$\pi_2(\lambda \mid \underline{x}) \propto \frac{\beta^n}{(\Gamma(\delta))^n} \lambda^{\delta n} \prod_{i=1}^{n} x_i^{-(\beta\delta+1)} \, e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}} \, \frac{1}{\lambda}$$

$$\pi_2(\lambda \mid \underline{x}) = K\lambda^{n\delta-1} e^{-\lambda \sum_{i=1}^{n} x_i^{-\beta}}$$

$$\pi_2(\lambda \mid \underline{x}) = K\lambda^{n\delta-1} e^{-\lambda T}$$

Where k is independent of $\lambda$ , $T = \sum_{i=1}^{n} x_i^{-\beta}$

$$\text{and } K^{-1} = \int_0^\infty \lambda^{n\delta-1} e^{-T\lambda} \, d\lambda \Rightarrow K^{-1} = \frac{\Gamma(n\delta)}{T^{n\delta}}$$

Hence the posterior density of $\lambda$ is given as

$$\pi_2(\lambda\,|\,\underline{x}) = \frac{T^{n\delta}}{\Gamma(n\delta)}\lambda^{n\delta-1}\; e^{-\lambda T} \quad, \lambda > 0 \tag{4.2}$$

Where $T = \sum_{i=1}^{n} x_i^{-\beta}$ and $(n\delta)$ are the parameters of the posterior distribution similar to the gamma distribution $G(T, n\delta)$

### *4.3 Posterior density under the Assumption of of exponential Prior*

Combining the prior distribution (3.3) and the likelihood function (2.1), the posterior density of $\lambda$ is derived as follows:

$$\pi_3(\lambda\,|\,\underline{x}) \propto \frac{\beta^n}{(\Gamma(\delta))^n}\lambda^{\delta n}\prod_{i=1}^{n} x_i^{-(\beta\delta+1)}\; e^{-\lambda\sum_{i=1}^{n} x_i^{-\beta}}\; c_1 e^{-c_1\lambda}$$

$$\pi_3(\lambda\,|\,\underline{x}) = K\lambda^{n\delta}\, e^{-\lambda\left(\sum_{i=1}^{n} x_i^{-\beta} + c_1\right)}$$

$$\pi_3(\lambda\,|\,\underline{x}) = K\lambda^{n\delta}\, e^{-\lambda T}$$

Where k is independent of $\lambda$ , $T = \sum_{i=1}^{n} x_i^{-\beta} + c_1$

$$\text{and}\quad K^{-1} = \int_0^\infty \lambda^{n\delta+1} e^{-T\lambda}\, d\lambda \quad \Rightarrow K^{-1} = \frac{\Gamma(n\delta+1)}{T^{n\delta+1}}$$

Hence the posterior density of $\lambda$ is given as

$$\pi_3(\lambda\,|\,\underline{x}) = \frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\lambda^{n\delta}\; e^{-\lambda T} \quad, \lambda > 0 \tag{4.3}$$

Where $T = \sum_{i=1}^{n} x_i^{-\beta} + c_1$ and $(n\delta+1)$ are the parameters of the posterior distribution similar to the gamma distribution $G(T, n\delta+1)$

### *4.4 Posterior density under the Assumption of Gamma Prior*

Combining the prior distribution (3.4) and the likelihood function (2.1), the posterior density of $\lambda$ is derived as follows:

$$\pi_4(\lambda\,|\,\underline{x}) \propto \frac{\beta^n}{(\Gamma(\delta))^n}\lambda^{\delta n}\prod_{i=1}^{n} x_i^{-(\beta\delta+1)}\; e^{-\lambda\sum_{i=1}^{n} x_i^{-\beta}}\; \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}$$

$$\pi_4(\lambda\,|\,\underline{x}) = K\lambda^{n\delta+a-1}\, e^{-\lambda\left(\sum_{i=1}^{n} x_i^{-\beta} + b\right)}$$

$$\pi_4(\lambda\,|\,\underline{x}) = K\lambda^{n\delta+a-1}\, e^{-\lambda T}$$

Where k is independent of $\lambda$ , $T = \sum_{i=1}^{n} x_i^{-\beta} + b$

$$\text{and}\quad K^{-1} = \int_0^\infty \lambda^{n\delta+a-1} e^{-T\lambda}\, d\lambda \Rightarrow K^{-1} = \frac{\Gamma(n\delta+a)}{T^{n\delta+a}}$$

Hence the posterior density of $\lambda$ is given as

$$\pi_4(\lambda \mid \underline{x}) = \frac{T^{n\delta+a}}{\Gamma(n\delta+a)} \lambda^{n\delta+a-1} \; e^{-\lambda T} \quad , \quad \lambda > 0 \tag{4.4}$$

Where $T = \sum_{i=1}^{n} x_i^{-\beta} + b$ and $(n\delta + a)$ are the parameters of the posterior distribution similar to the gamma distribution $G(T, n\delta + a)$

## 5 Bayesian estimation by using Uniform prior under different Loss Functions

**Theorem 5.1**:- Assuming the loss function $l_s(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_S = \frac{(n\delta+1)}{T}, \; T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the squared error loss function $l(\hat{\lambda}, \lambda) = c(\hat{\lambda} - \lambda)^2$ is given by the formula

$$R(\hat{\lambda}, \lambda) = \int_0^\infty c(\hat{\lambda} - \lambda)^2 \pi_1(\lambda / \underline{x})\, d\lambda \tag{5.1}$$

On substituting (4.1) in (5.1.1), we have

$$R(\hat{\lambda}, \lambda) = \int_0^\infty c(\hat{\lambda} - \lambda)^2 \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \lambda^{n\delta} \; e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda}, \lambda) = c \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \left[ \hat{\lambda}^2 \int_0^\infty \lambda^{n\delta+1-1} \; e^{-\lambda T}\, d\lambda + \int_0^\infty \lambda^{n\delta+3-1} \; e^{-\lambda T}\, d\lambda - 2\hat{\lambda} \int_0^\infty \lambda^{n\delta+2-1} \; e^{-\lambda T}\, d\lambda \right] \tag{5.2}$$

On solving (5.2), we get

$$R(\hat{\lambda}, \lambda) = c \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \left[ \hat{\lambda}^2 \frac{\Gamma(n\delta+1)}{T^{n\delta+1}} + \frac{\Gamma(n\delta+3)}{T^{n\delta+3}} - 2\hat{\lambda} \frac{\Gamma(n\delta+2)}{T^{n\delta+2}} \right]$$

$$R(\hat{\lambda}, \lambda) = c \left[ \hat{\lambda}^2 + \frac{(n\delta+2)(n\delta+1)}{T^2} - 2\hat{\lambda} \frac{(n\delta+1)}{T} \right] \tag{5.3}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_S = \frac{(n\delta+1)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} \tag{5.4}$$

**Theorem 5.2**:- Assuming the loss function $l_A(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_A = \frac{(n\delta + c_2 + 1)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the Al-Bayyati's loss function $l(\hat{\lambda}, \lambda) = \lambda^{c_2}(\hat{\lambda} - \lambda)^2$ is given by the formula

$$R(\hat{\lambda}, \lambda) = \int_0^\infty \lambda^{c_2}(\hat{\lambda} - \lambda)^2 \pi_1(\lambda / \underline{x})\, d\lambda \tag{5.5}$$

On substituting (4.1) in (5.5), we have

$$R(\hat{\lambda}, \lambda) = \int_0^\infty \lambda^{c_2}(\hat{\lambda} - \lambda)^2 \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \lambda^{n\delta}\ e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda}, \lambda) = \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \left[ \hat{\lambda}^2 \int_0^\infty \lambda^{n\delta+c_2+1-1}\ e^{-\lambda T}\, d\lambda + \int_0^\infty \lambda^{n\delta+c_2+3-1}\ e^{-\lambda T}\, d\lambda - 2\hat{\lambda}\int_0^\infty \lambda^{n\delta+c_2+2-1}\ e^{-\lambda T}\, d\lambda \right] \tag{5.6}$$

On solving (5.6), we get

$$R(\hat{\lambda}, \lambda) = \frac{1}{\Gamma(n\delta+1)} \left[ \hat{\lambda}^2 \frac{\Gamma(n\delta + c_2 + 1)}{T^{c_2}} + \frac{\Gamma(n\delta + c_2 + 3)}{T^{c_2+2}} - 2\hat{\lambda} \frac{\Gamma(n\delta + c_2 + 2)}{T^{c_2+1}} \right]$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_A = \frac{(n\delta + c_2 + 1)}{T} \tag{5.7}$$

**Theorem 5.3**:- Assuming the loss function $l_E(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_E = \frac{(n\delta)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the Entropy loss function $l(\hat{\lambda}, \lambda) = b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right)$ is given by the formula

$$R(\hat{\lambda}, \lambda) = \int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right) \pi_1(\lambda \mid \underline{x})\, d\lambda \tag{5.8}$$

On substituting (4.1) in (5.8), we have

$$R(\hat{\lambda}, \lambda) = \int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right) \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \lambda^{n\delta}\ e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda},\lambda)=b_1\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\begin{array}{l}\hat{\lambda}\int_0^\infty \lambda^{n\delta-1}\, e^{-\lambda T}\,d\lambda-\log(\hat{\lambda})\int_0^\infty \lambda^{n\delta+1-1}\, e^{-\lambda T}d\lambda \\ \qquad +\int_0^\infty \log(\lambda)\,\lambda^{n\delta+1-1}\, e^{-\lambda T}d\lambda-\int_0^\infty \lambda^{n\delta+1-1}\, e^{-\lambda T}d\lambda\end{array}\right] \quad (5.9)$$

On solving (5.9), we get

$$R(\hat{\lambda},\lambda)=b_1\left[\hat{\lambda}\frac{T}{n\delta}-\log(\hat{\lambda})+\frac{\Gamma'(n\delta+1)}{\Gamma(n\delta+1)}-1\right]$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_{1E}=\frac{(n\delta)}{T} \quad (5.10)$$

**Theorem 5.4**:- Assuming the loss function $l_p(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_P=\frac{[(n\delta+2)(n\delta+1)]^{1/2}}{T}, \quad T=\sum_{i=1}^n x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the precautionary loss function $l(\hat{\lambda},\lambda)=\frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}$, is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}\pi_1(\lambda/\underline{x})\,d\lambda \quad (5.11)$$

On substituting (4.1) in (5.11), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\lambda^{n\delta}\, e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=\frac{T^{n\delta+1}}{\hat{\lambda}\,\Gamma(n\delta+1)}\left[\begin{array}{l}\hat{\lambda}^2\int_0^\infty \lambda^{n\delta+1-1}\, e^{-\lambda T}\,d\lambda+\int_0^\infty \lambda^{n\delta+3-1}\, e^{-\lambda T}\,d\lambda \\ \qquad -2\hat{\lambda}\int_0^\infty \lambda^{n\delta+2-1}\, e^{-\lambda T}\,d\lambda\end{array}\right] \quad (5.12)$$

On solving (5.12), we get

$$R(\hat{\lambda},\lambda)=\hat{\lambda}+\frac{(n\delta+2)(n\delta+1)}{\hat{\lambda}T^2}-2\frac{(n\delta+1)}{T}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_P=\frac{[(n\delta+2)(n\delta+1)]^{1/2}}{T}, \quad T=\sum_{i=1}^n x_i^{-\beta} \quad (5.13)$$

## 6 Bayesian Estimation of λ under the Assumption of Jeffrey's Prior

**Theorem 6.1**:- Assuming the loss function $l_s(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_S = \frac{(n\delta)}{T}, \quad T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the squared error loss function $l(\hat{\lambda},\lambda) = c(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda) = \int_0^\infty c(\hat{\lambda}-\lambda)^2 \pi_2(\lambda/\underline{x})\, d\lambda \tag{6.1}$$

On substituting (4.2) in (6.1), we have

$$R(\hat{\lambda},\lambda) = \int_0^\infty c(\hat{\lambda}-\lambda)^2 \frac{T^{n\delta}}{\Gamma(n\delta)} \lambda^{n\delta-1} e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda},\lambda) = c\frac{T^{n\delta}}{\Gamma(n\delta)}\left[\hat{\lambda}^2\int_0^\infty \lambda^{n\delta-1} e^{-\lambda T}\, d\lambda + \int_0^\infty \lambda^{n\delta+2-1} e^{-\lambda T}\, d\lambda - 2\hat{\lambda}\int_0^\infty \lambda^{n\delta+1-1} e^{-\lambda T}\, d\lambda\right] \tag{6.2}$$

On solving (6.2), we get

$$R(\hat{\lambda},\lambda) = c\frac{T^{n\delta}}{\Gamma(n\delta)}\left[\hat{\lambda}^2\frac{\Gamma(n\delta)}{T^{n\delta}} + \frac{\Gamma(n\delta+2)}{T^{n\delta+2}} - 2\hat{\lambda}\frac{\Gamma(n\delta+1)}{T^{n\delta+1}}\right]$$

$$R(\hat{\lambda},\lambda) = c\left[\hat{\lambda}^2 + \frac{n\delta(n\delta+1)}{T^2} - 2\hat{\lambda}\frac{(n\delta)}{T}\right] \tag{6.3}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_S = \frac{(n\delta)}{T}, \quad T = \sum_{i=1}^{n} x_i^{-\beta} \tag{6.4}$$

**Theorem 6.2**:- Assuming the loss function $l_A(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_A = \frac{(n\delta + c_2)}{T}, \quad T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the Al-Bayyati's loss function $l(\hat{\lambda},\lambda) = \lambda^{c_2}(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda) = \int_0^\infty \lambda^{c_2}(\hat{\lambda}-\lambda)^2 \pi_2(\lambda/\underline{x})\, d\lambda \tag{6.5}$$

On substituting (4.2) in (6.5), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty \lambda^{c_2}(\hat{\lambda}-\lambda)^2\frac{T^{n\delta}}{\Gamma(n\delta)}\lambda^{n\delta-1}e^{-\lambda T}d\lambda$$

$$R(\hat{\lambda},\lambda)=c\frac{T^{n\delta}}{\Gamma(n\delta)}\left[\hat{\lambda}^2\int_0^\infty \lambda^{n\delta-1}e^{-\lambda T}d\lambda+\int_0^\infty \lambda^{n\delta+2-1}e^{-\lambda T}d\lambda-2\hat{\lambda}\int_0^\infty \lambda^{n\delta+1-1}e^{-\lambda T}d\lambda\right] \quad (6.6)$$

On solving (6.6), we get

$$R(\hat{\lambda},\lambda)=\frac{1}{\Gamma(n\delta)}\left[\hat{\lambda}^2\frac{\Gamma(n\delta+c_2)}{T^{c_2}}+\frac{\Gamma(n\delta+c_2+2)}{T^{c_2+2}}-2\hat{\lambda}\frac{\Gamma(n\delta+c_2+1)}{T^{c_2+1}}\right] \quad (6.7)$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_A=\frac{(n\delta+c_2)}{T} \quad , T=\sum_{i=1}^n x_i^{-\beta} \quad (6.8)$$

**Theorem 6.3**:- Assuming the loss function $l_E(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_E=\frac{(n\delta-1)}{T} \quad , T=\sum_{i=1}^n x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the Entropy loss function $l(\hat{\lambda},\lambda)=b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)$ is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)\pi_2(\lambda\mid \underline{x})\,d\lambda \quad (6.9)$$

On substituting (4.2) in (6.9), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)\frac{T^{n\delta}}{\Gamma(n\delta)}\lambda^{n\delta-1}e^{-\lambda T}d\lambda$$

$$R(\hat{\lambda},\lambda)=b_1\frac{T^{n\delta}}{\Gamma(n\delta)}\left[\hat{\lambda}\int_0^\infty \lambda^{n\delta-1-1}e^{-\lambda T}d\lambda-\log(\hat{\lambda})\int_0^\infty \lambda^{n\delta-1}e^{-\lambda T}d\lambda+\int_0^\infty \log(\lambda)\lambda^{n\delta-1}e^{-\lambda T}d\lambda-\int_0^\infty \lambda^{n\delta-1}e^{-\lambda T}d\lambda\right] \quad (6.10)$$

On solving (6.10), we get

$$R(\hat{\lambda},\lambda)=b_1\left[\hat{\lambda}\frac{T}{(n\delta-1)}-\log(\hat{\lambda})+\frac{\Gamma'(n\delta)}{\Gamma(n\delta)}-1\right] \quad (6.11)$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_E = \frac{(n\delta - 1)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} \tag{6.12}$$

**Theorem 6.4**:- Assuming the loss function $l_p(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_P = \frac{[(n\delta+1)(n\delta)]^{1/2}}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta}$$

**Proof: -** The risk function of the estimator $\lambda$ under the precautionary loss function $l(\hat{\lambda}, \lambda) = \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}$, is given by the formula

$$R(\hat{\lambda}, \lambda) = \int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \pi_2(\lambda / \underline{x})\, d\lambda \tag{6.13}$$

On substituting (4.2) in (6.13), we have

$$R(\hat{\lambda}, \lambda) = \int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \frac{T^{n\delta}}{\Gamma(n\delta)} \lambda^{n\delta-1}\, e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda}, \lambda) = \frac{T^{n\delta}}{\hat{\lambda}\,\Gamma(n\delta)} \left[ \hat{\lambda}^2 \int_0^\infty \lambda^{n\delta-1}\, e^{-\lambda T}\, d\lambda + \int_0^\infty \lambda^{n\delta+2-1}\, e^{-\lambda T}\, d\lambda - 2\hat{\lambda} \int_0^\infty \lambda^{n\delta+1-1}\, e^{-\lambda T}\, d\lambda \right] \tag{6.14}$$

On solving (6.14), we get

$$R(\hat{\lambda}, \lambda) = \hat{\lambda} + \frac{(n\delta+1)(n\delta)}{\hat{\lambda} T^2} - 2\frac{(n\delta)}{T} \tag{6.15}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_P = \frac{[(n\delta+1)(n\delta)]^{1/2}}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} \tag{6.16}$$

## 7 Bayesian Estimation of λ under the Assumption of exponential Prior

**Theorem 7.1**:- Assuming the loss function $l_s(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_S = \frac{(n\delta+1)}{T}, \quad T = \sum_{i=1}^{n} x_i^{-\beta} + c_1$$

**Proof: -** The risk function of the estimator $\lambda$ under the squared error loss function $l(\hat{\lambda}, \lambda) = c(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty c(\hat{\lambda}-\lambda)^2\pi_3(\lambda/\underline{x})\,d\lambda \tag{7.1}$$

On substituting (4.3) in (7.1), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty c(\hat{\lambda}-\lambda)^2\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\lambda^{n\delta}\,e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=c\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\hat{\lambda}^2\int_0^\infty\lambda^{n\delta}\,e^{-\lambda T}\,d\lambda+\int_0^\infty\lambda^{n\delta+3-1}\,e^{-\lambda T}\,d\lambda-2\hat{\lambda}\int_0^\infty\lambda^{n\delta+2-1}\,e^{-\lambda T}\,d\lambda\right] \tag{7.2}$$

On solving (7.2), we get

$$R(\hat{\lambda},\lambda)=c\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\hat{\lambda}^2\frac{\Gamma(n\delta+1)}{T^{n\delta+1}}+\frac{\Gamma(n\delta+3)}{T^{n\delta+3}}-2\hat{\lambda}\frac{\Gamma(n\delta+2)}{T^{n\delta+2}}\right]$$

$$R(\hat{\lambda},\lambda)=c\left[\hat{\lambda}^2+\frac{(n\delta+2)(n\delta+1)}{T^2}-2\hat{\lambda}\frac{(n\delta+1)}{T}\right] \tag{7.3}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_S=\frac{(n\delta+1)}{T}\;,\quad T=\sum_{i=1}^n x_i^{-\beta}+c_1 \tag{7.4}$$

**Theorem 7.2**:- Assuming the loss function $l_A(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_A=\frac{(n\delta+c_2+1)}{T}\qquad ,\;T=\sum_{i=1}^n x_i^{-\beta}+c_1$$

**Proof: -** The risk function of the estimator $\lambda$ under the Al-Bayyati's loss function $l(\hat{\lambda},\lambda)=\lambda^{c_2}(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty\lambda^{c_2}(\hat{\lambda}-\lambda)^2\pi_3(\lambda/\underline{x})\,d\lambda \tag{7.5}$$

On substituting (4.3) in (7.5), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty\lambda^{c_2}(\hat{\lambda}-\lambda)^2\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\lambda^{n\delta}\,e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\hat{\lambda}^2\int_0^\infty\lambda^{n\delta+c_2+1-1}\,e^{-\lambda T}\,d\lambda+\int_0^\infty\lambda^{n\delta+c_2+3-1}\,e^{-\lambda T}\,d\lambda-2\hat{\lambda}\int_0^\infty\lambda^{n\delta+c_2+2-1}\,e^{-\lambda T}\,d\lambda\right] \tag{7.6}$$

On solving (7.6), we get

$$R(\hat{\lambda},\lambda)=\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\hat{\lambda}^{2}\frac{\Gamma(n\delta+c_2+1)}{T^{n\delta+c_2+1}}+\frac{\Gamma(n\delta+c_2+3)}{T^{n\delta+c_2+3}}-2\hat{\lambda}\frac{\Gamma(n\delta+c_2+2)}{T^{n\delta+c_2+2}}\right]$$

$$R(\hat{\lambda},\lambda)=\frac{1}{\Gamma(n\delta+1)}\left[\hat{\lambda}^{2}\frac{\Gamma(n\delta+c_2+1)}{T^{c_2}}+\frac{\Gamma(n\delta+c_2+3)}{T^{c_2+2}}-2\hat{\lambda}\frac{\Gamma(n\delta+c_2+2)}{T^{c_2+1}}\right] \quad (7.7)$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_A=\frac{(n\delta+c_2+1)}{T} \quad (7.8)$$

**Theorem 7.3**:- Assuming the loss function $l_E(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_E=\frac{(n\delta)}{T} \quad , \; T=\sum_{i=1}^{n}x_i^{-\beta}+c_1$$

**Proof: -** The risk function of the estimator $\lambda$ under the Entropy loss function $l(\hat{\lambda},\lambda)=b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)$ is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^{\infty}b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)\pi_3(\lambda\,|\,\underline{x})\,d\lambda \quad (7.9)$$

On substituting (4.3) in (7.9), we have

$$R(\hat{\lambda},\lambda)=\int_0^{\infty}b_1\left(\frac{\hat{\lambda}}{\lambda}-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1\right)\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\lambda^{n\delta}\;e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=b_1\frac{T^{n\delta+1}}{\Gamma(n\delta+1)}\left[\begin{array}{l}\hat{\lambda}\int_0^{\infty}\lambda^{n\delta-1}\;e^{-\lambda T}\,d\lambda-\log(\hat{\lambda})\int_0^{\infty}\lambda^{n\delta+1-1}\;e^{-\lambda T}d\lambda\\ \qquad+\int_0^{\infty}\log(\lambda)\,\lambda^{n\delta+1-1}\;e^{-\lambda T}d\lambda-\int_0^{\infty}\lambda^{n\delta+1-1}\;e^{-\lambda T}d\lambda\end{array}\right] \quad (7.10)$$

On solving (7.10), we get

$$R(\hat{\lambda},\lambda)=b_1\left[\hat{\lambda}\frac{T}{(n\delta)}-\log(\hat{\lambda})+\frac{\Gamma'(n\delta+1)}{\Gamma(n\delta+1)}-1\right] \quad (7.11)$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_E=\frac{(n\delta)}{T} \quad , \; T=\sum_{i=1}^{n}x_i^{-\beta}+c_1 \quad (7.12)$$

**Theorem 7.4**:- Assuming the loss function $l_p(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_P = \frac{\left[(n\delta+2)(n\delta+1)\right]^{1/2}}{T} \quad , \quad T = \sum_{i=1}^{n} x_i^{-\beta} + c_1$$

**Proof: -** The risk function of the estimator $\lambda$ under the precautionary loss function $l(\hat{\lambda},\lambda) = \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}$, is given by the formula

$$R(\hat{\lambda},\lambda) = \int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \pi_3(\lambda/\underline{x})\, d\lambda \tag{7.13}$$

On substituting (4.3) in (7.13), we have

$$R(\hat{\lambda},\lambda) = \int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \frac{T^{n\delta+1}}{\Gamma(n\delta+1)} \lambda^{n\delta}\, e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda},\lambda) = \frac{T^{n\delta+1}}{\hat{\lambda}\,\Gamma(n\delta+1)} \left[ \hat{\lambda}^2 \int_0^\infty \lambda^{n\delta+1-1}\, e^{-\lambda T}\, d\lambda + \int_0^\infty \lambda^{n\delta+3-1}\, e^{-\lambda T}\, d\lambda - 2\hat{\lambda}\int_0^\infty \lambda^{n\delta+2-1}\, e^{-\lambda T}\, d\lambda \right] \tag{7.14}$$

On solving (7.14), we get

$$R(\hat{\lambda},\lambda) = \hat{\lambda} + \frac{(n\delta+2)(n\delta+1)}{\hat{\lambda}T^2} - 2\frac{(n\delta+1)}{T} \tag{7.15}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_P = \frac{\left[(n\delta+2)(n\delta+1)\right]^{1/2}}{T} \tag{7.16}$$

## 8 Bayesian Estimation of λ under the Assumption of Gamma Prior

**Theorem 8.1**:- Assuming the loss function $l_s(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_S = \frac{(n\delta+a)}{T}, \quad T = \sum_{i=1}^{n} x_i^{-\beta} + b$$

**Proof: -** The risk function of the estimator $\lambda$ under the squared error loss function $l(\hat{\lambda},\lambda) = c(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda) = \int_0^\infty c(\hat{\lambda}-\lambda)^2 \pi_4(\lambda/\underline{x})\, d\lambda \tag{8.1}$$

On substituting (4.4) in (8.1), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty c(\hat{\lambda}-\lambda)^2\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\lambda^{n\delta+a-1}e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=c\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\left[\hat{\lambda}^2\int_0^\infty\lambda^{n\delta+a-1}e^{-\lambda T}\,d\lambda+\int_0^\infty\lambda^{n\delta+a+2-1}e^{-\lambda T}\,d\lambda-2\hat{\lambda}\int_0^\infty\lambda^{n\delta+a+1-1}e^{-\lambda T}\,d\lambda\right] \tag{8.2}$$

On solving (8.2), we get

$$R(\hat{\lambda},\lambda)=c\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\left[\hat{\lambda}^2\frac{\Gamma(n\delta+a)}{T^{n\delta+a}}+\frac{\Gamma(n\delta+a+2)}{T^{n\delta+a+2}}-2\hat{\lambda}\frac{\Gamma(n\delta+a+1)}{T^{n\delta+a+1}}\right]$$

$$R(\hat{\lambda},\lambda)=c\left[\hat{\lambda}^2+\frac{(n\delta+a+1)(n\delta+a)}{T^2}-2\hat{\lambda}\frac{(n\delta+a)}{T}\right] \tag{8.3}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_S=\frac{(n\delta+a)}{T},\quad T=\sum_{i=1}^{n}x_i^{-\beta}+b \tag{8.4}$$

**Theorem 8.2**:- Assuming the loss function $l_A(\hat{\lambda},\lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_A=\frac{(n\delta+c_2+a)}{T},\quad T=\sum_{i=1}^{n}x_i^{-\beta}+b$$

**Proof: -** The risk function of the estimator $\lambda$ under the Al-Bayyati's loss function $l(\hat{\lambda},\lambda)=\lambda^{c_2}(\hat{\lambda}-\lambda)^2$ is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty\lambda^{c_2}(\hat{\lambda}-\lambda)^2\pi_4(\lambda/\underline{x})\,d\lambda \tag{8.5}$$

On substituting (4.4) in (8.5), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty\lambda^{c_2}(\hat{\lambda}-\lambda)^2\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\lambda^{n\delta+a-1}e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\left[\hat{\lambda}^2\int_0^\infty\lambda^{n\delta+c_2+a-1}e^{-\lambda T}\,d\lambda+\int_0^\infty\lambda^{n\delta+c_2+a+2-1}e^{-\lambda T}\,d\lambda-2\hat{\lambda}\int_0^\infty\lambda^{n\delta+c_2+a+1-1}e^{-\lambda T}\,d\lambda\right] \tag{8.6}$$

On solving (8.6), we get

$$R(\hat{\lambda},\lambda)=\frac{1}{\Gamma(n\delta+a)}\left[\hat{\lambda}^2\frac{\Gamma(n\delta+c_2+a)}{T^{c_2}}+\frac{\Gamma(n\delta+c_2+a+2)}{T^{c_2+2}}-2\hat{\lambda}\frac{\Gamma(n\delta+c_2+a+1)}{T^{c_2+1}}\right] \tag{8.7}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_A = \frac{(n\delta + c_2 + a)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} + b \tag{8.8}$$

**Theorem 8.3**:- Assuming the loss function $l_E(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_E = \frac{(n\delta + a - 1)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} + b$$

**Proof: -** The risk function of the estimator $\lambda$ under the Entropy loss function $l(\hat{\lambda}, \lambda) = b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right)$ is given by the formula

$$R(\hat{\lambda}, \lambda) = \int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right)\pi_4(\lambda \mid \underline{x})\, d\lambda \tag{8.9}$$

On substituting (4.3) in (8.9), we have

$$R(\hat{\lambda}, \lambda) = \int_0^\infty b_1\left(\frac{\hat{\lambda}}{\lambda} - \log\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right)\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\lambda^{n\delta+a-1}\, e^{-\lambda T}\, d\lambda$$

$$R(\hat{\lambda}, \lambda) = b_1 \frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\begin{bmatrix} \hat{\lambda}\int_0^\infty \lambda^{n\delta+a-1-1}\, e^{-\lambda T}\, d\lambda - \log(\hat{\lambda})\int_0^\infty \lambda^{n\delta+a-1}\, e^{-\lambda T} d\lambda \\ +\int_0^\infty \log(\lambda)\, \lambda^{n\delta+a-1}\, e^{-\lambda T} d\lambda - \int_0^\infty \lambda^{n\delta+a-1}\, e^{-\lambda T} d\lambda \end{bmatrix} \tag{8.10}$$

On solving (8.10), we get

$$R(\hat{\lambda}, \lambda) = b_1\left[\hat{\lambda}\frac{T}{(n\delta + a - 1)} - \log(\hat{\lambda}) + \frac{\Gamma'(n\delta+a)}{\Gamma(n\delta+a)} - 1\right] \tag{8.11}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_E = \frac{(n\delta + a - 1)}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} + b \tag{8.12}$$

**Theorem 8.4**:- Assuming the loss function $l_p(\hat{\lambda}, \lambda)$, the Bayesian estimator of the parameter $\lambda$, if the parameters $\delta$ & $\beta$ are known, is of the form

$$\hat{\lambda}_P = \frac{\left[(n\delta + a + 1)(n\delta + a)\right]^{1/2}}{T} \quad , T = \sum_{i=1}^{n} x_i^{-\beta} + b$$

**Proof: -** The risk function of the estimator $\lambda$ under the precautionary loss function $l(\hat{\lambda}, \lambda) = \frac{(\hat{\lambda} - \lambda)^2}{\hat{\lambda}}$, is given by the formula

$$R(\hat{\lambda},\lambda)=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}\pi_4(\lambda/\underline{x})\,d\lambda \tag{8.13}$$

On substituting (4.4) in (8.13), we have

$$R(\hat{\lambda},\lambda)=\int_0^\infty \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}}\frac{T^{n\delta+a}}{\Gamma(n\delta+a)}\lambda^{n\delta+a-1}\,e^{-\lambda T}\,d\lambda$$

$$R(\hat{\lambda},\lambda)=\frac{T^{n\delta+a}}{\hat{\lambda}\Gamma(n\delta+a)}\left[\hat{\lambda}^2\int_0^\infty \lambda^{n\delta+a-1}\,e^{-\lambda T}\,d\lambda+\int_0^\infty \lambda^{n\delta+a+2-1}\,e^{-\lambda T}\,d\lambda -2\hat{\lambda}\int_0^\infty \lambda^{n\delta+a+1-1}\,e^{-\lambda T}\,d\lambda\right] \tag{8.14}$$

On solving (8.15), we get

$$R(\hat{\lambda},\lambda)=\hat{\lambda}+\frac{(n\delta+a+1)(n\delta+a)}{\hat{\lambda}T^2}-2\frac{(n\delta+a)}{T} \tag{8.15}$$

Minimization of the risk with respect to $\hat{\lambda}$ gives us the optimal estimator

$$\hat{\lambda}_P=\frac{\left[(n\delta+a+1)(n\delta+a)\right]^{1/2}}{T} \tag{8.16}$$

## 9 Real Life Data

The data set is given by Lee and Wang (2003) which represent remission times (in months) of a random sample of 128 bladder cancer patients. The data are as follows: 0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

We estimate the unknown parameters of each distribution by the maximum-likelihood method, and the Bayes Estimates are obtained to compare the candidate distributions.

**Table 1:** Bayes Estimates of λ under Uniform Prior

| $\delta$ | $\beta$ | MLE | SELF | ABLF | | ELF | PLF |
|---|---|---|---|---|---|---|---|
| | | | | $C_2$=0.5 | $C_2$=1.0 | | |
| 0.5 | 0.5 | 1.012679 | 1.028502 | 1.036414 | 1.036384 | 1.012679 | 1.036384 |
| 0.5 | 1.0 | 1.242364 | 1.261776 | 1.271482 | 1.281188 | 1.242364 | 1.271445 |
| 1.0 | 0.5 | 2.025358 | 2.041181 | 2.049093 | 2.057004 | 2.025358 | 2.049078 |
| 1.0 | 1.0 | 2.484729 | 2.504141 | 2.513847 | 2.523553 | 2.484729 | 2.513828 |

MLE=Maximum Likelihood, SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

**Table 2:** Bayes Estimates of λ under Jeffreys Prior

| $\delta$ | $\beta$ | MLE | SELF | ABLF | ELF | PLF |
|---|---|---|---|---|---|---|

| | | | | $C_2$=0.5 | $C_2$=1.0 | | |
|---|---|---|---|---|---|---|---|
| 0.5 | 0.5 | 1.012679 | 1.012679 | 1.020591 | 1.028502 | 0.99686 | 1.02056 |
| 0.5 | 1.0 | 1.242364 | 1.242364 | 1.252070 | 1.261776 | 1.22295 | 1.252033 |
| 1.0 | 0.5 | 2.025358 | 2.025358 | 2.033270 | 2.041181 | 2.009535 | 2.033254 |
| 1.0 | 1.0 | 2.484729 | 2.484729 | 2.494435 | 2.504141 | 2.465317 | 2.494416 |

MLE=Maximum Likelihood, SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function

**Table 3:** Bayes Estimates of λ under exponential prior

| $\delta$ | $\beta$ | $c_1$ | MLE | SELF | ABLF | | ELF | PLF |
|---|---|---|---|---|---|---|---|---|
| | | | | | $C_2$=0.5 | $C_2$=1.0 | | |
| 0.5 | 0.5 | 0.4 | 1.012679 | 1.022033 | 1.029895 | 1.037757 | 1.00631 | 1.029865 |
| 0.5 | 1.0 | 0.4 | 1.242364 | 1.252054 | 1.261686 | 1.271317 | 1.232792 | 1.261649 |
| 1.0 | 0.5 | 0.4 | 2.025358 | 2.028343 | 2.036205 | 2.044067 | 2.01262 | 2.03619 |
| 1.0 | 1.0 | 0.4 | 2.484729 | 2.484846 | 2.494478 | 2.504109 | 2.465584 | 2.494459 |

MLE=Maximum Likelihood, SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function

**Table 4:** Bayes Estimates of λ under gamma prior

| $\delta$ | $\beta$ | $a$ | $b$ | MLE | SELF | ABLF | | ELF | PLF |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | $C_2$=0.5 | $C_2$=1.0 | | |
| 0.5 | 0.5 | 1.4 | 0.4 | 1.012679 | 1.028323 | 1.036185 | 1.044046 | 1.012599 | 1.036155 |
| 0.5 | 1.0 | 1.4 | 0.4 | 1.242364 | 1.259759 | 1.269391 | 1.279022 | 1.240497 | 1.269354 |
| 1.0 | 0.5 | 1.4 | 0.4 | 2.025358 | 2.034633 | 2.042495 | 2.050356 | 2.018909 | 2.042479 |
| 1.0 | 1.0 | 1.4 | 0.4 | 2.484729 | 2.492551 | 2.502183 | 2.511814 | 2.473289 | 2.502164 |

MLE=Maximum Likelihood, SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

Bayes risk is computed in the following tables:

**Table 5:** Bayes Risk of λ under Uniform Prior

| $\delta$ | $\beta$ | SELF | | ABLF | | ELF | | PLF |
|---|---|---|---|---|---|---|---|---|
| | | C=0.5 | C=1.0 | $C_2$=0.5 | $C_2$=1.0 | b1=0.5 | b1=1.0 | |
| 0.5 | 0.5 | 0.00814 | 0.01627 | 0.01660 | 0.01706 | 2.077038 | 4.154076 | 0.01576 |
| 0.5 | 1.0 | 0.01225 | 0.02449 | 0.02767 | 0.03138 | 1.97483 | 3.94966 | 0.01934 |
| 1.0 | 0.5 | 0.01615 | 0.03229 | 0.04628 | 0.06644 | 2.07509 | 4.15018 | 0.01579 |
| 1.0 | 1.0 | 0.02431 | 0.04861 | 0.07715 | 0.12267 | 1.97288 | 3.94577 | 0.01937 |

SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

**Table 6:** Bayes Risk of λ under Jeffrey's Prior

| $\delta$ | $\beta$ | SELF | | ABLF | | ELF | | PLF |
|---|---|---|---|---|---|---|---|---|
| | | C=0.5 | C=1.0 | $C_2$=0.5 | $C_2$=1.0 | b1=0.5 | b1=1.0 | |
| 0.5 | 0.5 | 0.00801 | 0.01602 | 0.01622 | 0.01648 | 2.07710 | 4.15420 | 0.01576 |
| 0.5 | 1.0 | 0.01206 | 0.02412 | 0.02704 | 0.03043 | 1.97489 | 3.94978 | 0.01934 |
| 1.0 | 0.5 | 0.01602 | 0.03205 | 0.04574 | 0.06541 | 2.07511 | 4.15021 | 0.01579 |
| 1.0 | 1.0 | 0.02412 | 0.04823 | 0.07625 | 0.12078 | 1.97289 | 3.94579 | 0.01937 |

SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

**Table 7:** Bayes Risk of λ under exponential Prior

| $\delta$ | $\beta$ | $c_1$ | SELF | | ABLF | | ELF | | PLF |
|---|---|---|---|---|---|---|---|---|---|
| | | | C=0.5 | C=1.0 | $C_2$=0.5 | $C_2$=1.0 | b1=0.5 | b1=1.0 | |
| 0.5 | 0.5 | 0.4 | 0.00803 | 0.01607 | 0.01634 | 0.01668 | 2.08019 | 4.16038 | 0.01566 |
| 0.5 | 1.0 | 0.4 | 0.01206 | 0.02412 | 0.02714 | 0.03066 | 1.97869 | 3.95739 | 0.01919 |
| 1.0 | 0.5 | 0.4 | 0.01595 | 0.03189 | 0.04555 | 0.06519 | 2.07825 | 4.15649 | 0.01569 |
| 1.0 | 1.0 | 0.4 | 0.02393 | 0.04786 | 0.07567 | 0.11986 | 1.97675 | 3.95350 | 0.01923 |

SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

**Table 8:** Bayes Risk of λ under gamma prior

| $\delta$ | $\beta$ | $a$ | $b$ | SELF | | ABLF | | ELF | | PLF |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | C=0.5 | C=1.0 | $C_2$=0.5 | $C_2$=1.0 | b1= | b1= | |
| 0.5 | 0.5 | 1.4 | 0.4 | 0.00808 | 0.01617 | 0.01649 | 0.01688 | 2.08017 | 4.16034 | 0.01566 |
| 0.5 | 1.0 | 1.4 | 0.4 | 0.01213 | 0.02427 | 0.02739 | 0.03104 | 1.97867 | 3.95734 | 0.01919 |
| 1.0 | 0.5 | 1.4 | 0.4 | 0.01599 | 0.03199 | 0.04576 | 0.06559 | 2.07824 | 4.15648 | 0.01569 |
| 1.0 | 1.0 | 1.4 | 0.4 | 0.02401 | 0.04801 | 0.07602 | 0.12059 | 1.97674 | 3.95349 | 0.01922 |

SELF=squared error loss function, ABLF=Albayyti's loss function, ELF=Entropy loss function, PLF=precautionary loss function.

## 10 Conclusion

On comparing the Bayes posterior risk of different loss functions, it is observed that for smaller values of δ SELF has less Bayes posterior risk and for higher values of δ Precautionary loss function gives less posterior risk than other loss functions in both non informative and informative priors than other loss functions. According to the decision rule of less Bayes posterior risk we conclude that SELF is more preferable loss function for smaller values of δ and precautionary loss function is preferable for higher values of δ.

It is clear from Table 5 to Table 8, the comparison of Bayes posterior risk under different loss function using non-informative as well as informative priors has been made through which we conclude that within each loss function informative exponential prior provides less Bayes posterior risk than gamma prior so it is more suitable for the generalized inverse Weibull distribution and among non informative priors Jeffrey's prior provides less posterior risk than uniform prior.

## References

[1] Afaq Ahmad, S.P Ahmad and A.Ahmed., (2015), "Bayesian Estimation of Exponentiated Inverted Weibull Distribution under Asymmetric Loss Functions", Journal of Statistics Applications & Probability, vol.4, no.1, 183-192.

[2] Calabria, R. and Pulcini, G., (1994), "Bayes 2-Sample Prediction for the Inverse Weibull Distribution", Communications in Statistics-Theory and Methods, 23(6), 1811-1824.

[3] Calabria, R. and Pulcini, G., (1990), "On the Maximum Likelihood and Least Squares Estimation in Inverse Weibull Distribution", Statistica Applicata, 2, 53-66.

[4] D. Kundu, (2008), "Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring", *Technometrics*, vol. 50, no. 2, pp. 144–154.

[5] F. R. S. De Gusmao, E. M. M. Ortega, G.M.` Cardeiro. , (2011), "The Generalized Inverse Weibull Distribution", Stat. papers springer- verlag 52: 591-619.

[6] Jeffery's, H. (1964), "An invariant form for the Prior Probability in estimation problems", Proceedings of the Royal Society of London, Series. A., 186: 453-461.

[7] Johnson, N.L., Kotz, S., and Balakrishnan, N (1995), "Continous Univariate Distributions", Vol 2. John Wiley & Sons New York.

[8] Keller, A. Z., Giblin, M. T., and Farnworth, N. R., (1985), "Reliability Analysis ofCommercial Vehicle Engines", Reliability Engineering, 10, 15-25.

[9] Lee ET, Wang JW., (2003), "Statistical methods for survival data analysis. (3rd edn)", John Wiley and Sons, New York, USA.

[10] M.Pararai, G. Warahena-Liyanage and Broderick O. Oluyede., (2014), "A New Class of Generalized Inverse Weibull Distribution with Apllications", journal *of Applied Matematics & Bioinformatics,* vol. 4, no.2, 17-35.

[11] Mudholkar GS, Kollia GD., (1994), “Generalized Weibull family: a structural analysis”, Commun Stat Ser A, 23:1149–1171.

[12] Nelson, W., (1982), “Applied Life Data Analysis”, J.Wiley, N.Y.USA.

[13] Oluyede, B. O., and Yang, T., (2014), “Generalizations of the Inverse Weibull and Related Distributions with Applications”, Electronic Journal of Applied Statistical Analysis, Vol. 7, Issue 1, 94-116.

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